Άσκηση

Έστω $M(t)$ η ανανεωτική συνάρτηση μιας απεριοδικής ανανεωτικής διαδικασίας $\{N(t), t \geq 0\}$ με $0 < E[X_n] = \tau < \infty$ και $Var[X_n] = \sigma^2 < \infty$.

Να βρεθεί ανανεωτική εξίσωση για την

$$H(t) = M(t) - \frac{t}{\tau}$$

και ν.δ.ο

$$\lim_{t \to \infty} \left(M(t) - \frac{t}{\tau} \right) = \frac{\sigma^2 - \tau^2}{2\tau^2}.$$

(Μπορείτε να χρησιμοποιήσετε τις σχέσεις

$$\int_0^\infty u(1 - G(u))\, du = \frac{\sigma^2 + \tau^2}{2} = \frac{E[X_n^2]}{2}$$

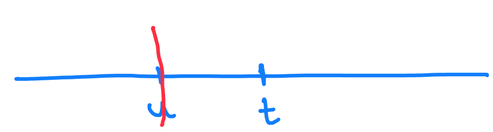
$$\int_t^\infty (u - t)\, dG(u) = \int_t^\infty (1 - G(u))\, du, \quad t \geq 0 \quad)$$

<u>Λύση:</u> $H(t) = M(t) - \frac{t}{\tau} = E[N(t)] - \frac{t}{\tau} = E\left[N(t) - \frac{t}{\tau}\right]$

Εφαρμόζω ανανεωτικό επιχείρημα (δεσμεύω ως προς S_1).

$$H(t) = \int_0^\infty E\left[N(t) - \frac{t}{\tau} \,\middle|\, S_1 = u\right] dG(u)$$

Αν $u \leq t$

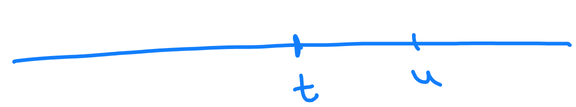


$$E\left[N(t) - \frac{t}{\tau} \,\middle|\, S_1 = u\right] = 1 + E[N(t-u)] - \frac{t}{\tau} = 1 + M(t-u) - \frac{t}{\tau}$$

$$\underline{\underline{H(t-u) = M(t-u) - \frac{t-u}{\tau}}} \quad 1 + H(t-u) + \frac{t-u}{\tau} - \frac{t}{\tau}$$

$$= 1 + H(t-u) - \frac{u}{\tau}$$

Αν $u > t$



$$E\left[N(t) - \frac{t}{\tau} \,\middle|\, S_1 = u\right] = 0 - \frac{t}{\tau} = -\frac{t}{\tau}$$

Οπότε $H(t) = \int_0^\infty E\left[N(t) - \frac{t}{\tau} \,\middle|\, S_1 = u\right] dG(u) =$

$$\int_0^t \left(1 + H(t-u) - \frac{u}{\tau}\right) dG(u) + \int_t^\infty \left(-\frac{t}{\tau}\right) dG(u) \Rightarrow$$

$$H(t) = \underbrace{\int_0^t \left(1-\frac{u}{\tau}\right) dG(u) - \int_t^\infty \frac{t}{\tau}\, dG(u)}_{D(t)} + \int_0^t H(t-u)\, dG(u)$$

$$D(t) = \int_0^t \left(1-\frac{u}{\tau}\right) dG(u) - \int_t^\infty \frac{t}{\tau}\, dG(u)$$

$$= \int_0^\infty \left(1-\frac{u}{\tau}\right) dG(u) - \int_t^\infty \left(1-\frac{u}{\tau}\right) dG(u) - \int_t^\infty \frac{t}{\tau}\, dG(u)$$

$$= \cancel{\underbrace{\int_0^\infty 1\, dG(u)}_{1}} - \cancel{\underbrace{\int_0^\infty \frac{u}{\tau}\, dG(u)}_{\frac{\tau}{\tau}=1}} - \int_t^\infty 1\, dG(u) + \int_t^\infty \frac{u}{\tau}\, dG(u) - \int_t^\infty \frac{t}{\tau}\, dG(u)$$

$$= -(1-G(t)) + \frac{1}{\tau} \underbrace{\int_t^\infty (u-t)\, dG(u)}_{\int_t^\infty (1-G(u))\, du}$$

$$= \underbrace{\frac{1}{\tau}\int_t^\infty (1-G(u))\, du}_{D_1(t)} - \underbrace{(1-G(t))}_{D_2(t)}$$

Οι $D_1(t)$ και $D_2(t)$ είναι φθίνουσες

$$0 \le D_1(t) = \frac{1}{\tau}\int_t^\infty (1-G(u))\, du \le D_1(0) = \frac{1}{\tau}\underbrace{\int_0^\infty (1-G(u))\, du}_{\tau} = \frac{\tau}{\tau} = 1$$

$$0 \le D_2(t) = 1 - G(t) \le 1$$

$$\int_0^\infty D_1(t)\, dt = \int_0^\infty \frac{1}{\tau}\int_t^\infty (1-G(u))\, du\, dt = \frac{1}{\tau}\int_0^\infty \int_t^\infty (1-G(u))\, du\, dt$$

$$\underline{\underline{0<t<u<\infty}}\ \frac{1}{\tau}\int_0^\infty \int_0^u (1-G(u))\, dt\, du = \frac{1}{\tau}\int_0^\infty (1-G(u)) \underbrace{\int_0^u dt}\, du =$$

$$\frac{1}{\tau}\int_0^\infty u(1-G(u))\, du = \frac{1}{\tau}\frac{E[X_n^2]}{2} = \frac{\sigma^2+\tau^2}{2\tau} < \infty$$

$$\int_0^\infty D_2(t)\, dt = \int_0^\infty (1-G(t))\, dt = \tau < \infty$$

Οπότε

$$\int_0^\infty |D(t)|\, dt = \int_0^\infty |D_1(t) - D_2(t)|\, dt \le \int_0^\infty (D_1(t)+D_2(t))\, dt$$

$$= \int_0^\infty D_1(t)\, dt + \int_0^\infty D_2(t)\, dt = \frac{\sigma^2+\tau^2}{2\tau} + \tau < \infty$$

Εφαρμόζοντας το Βασικό Ανανεωτικό Θεώρημα παίρνουμε

$$\lim_{t\to\infty} H(t) = \frac{1}{\tau}\int_0^\infty D(t)\,dt = \frac{1}{\tau}\int_0^\infty (D_1(t) - D_2(t))\,dt =$$

$$\frac{1}{\tau}\left[\int_0^\infty D_1(t)\,dt - \int_0^\infty D_2(t)\,dt\right] =$$

$$\frac{1}{\tau}\left(\frac{\sigma^2+\tau^2}{2\tau} - \tau\right) = \frac{\sigma^2-\tau^2}{2\tau}$$