Arithmetic

A Teacher’s Introduction
To The Cuisenaire-Gattegno Methods
Of Teaching Arithmetic

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Educational Solutions Worldwide Inc.
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Introduction

Although four general explanatory books and a series of textbooks have already been published to introduce the new method of teaching arithmetic, there has been a widespread demand for a simpler introduction. The somewhat technical character of the existing books seems to have created difficulties for the primary school teacher who wishes to understand and use the method. A more accessible introduction, avoiding unfamiliar terms and ideas was clearly needed, and the present work sets out to meet that need.

Its approach will, thus, take full account of the point of view of the teacher who is working within the usual curriculum according to the familiar methods but who would like, nevertheless, to take advantage of this new method and its materials. It is hoped that, once the reader has studied this introduction, he or she will tackle the more advanced works already published and, equipped with the insights and experience gained, will master the all-important basic ideas they present. Only in this way can the true subject-matter of the method be grasped so as to lead to a reform of the general
approach to the teaching of arithmetic, and this is most important if we are to give the coming generations their due.

M. Cuisenaire, whose insight led to the development of the extraordinary materials now universally known as NUMBERS IN COLOR, began putting his ideas into practice some thirty years ago. Only in December 1952 did his first book appear. In it he showed the world how his new approach could be used to teach the early stages of arithmetic, and we are greatly in his debt for what he taught us. The central inspiration of his outlook lay in his recognition that the child must learn by action and will, thereby, acquire confidence. If he manipulates materials, sees how bonds are formed, can correct himself and write down what he now sees and knows, it is clear that he will feel very differently from the child who merely repeats sounds he hears which, however meaningful they may be, mean nothing to him. Cuisenaire found to his amazement that normal children of 6 or 7, using the materials which will later be described, became remarkably able at arithmetic. His observations were to be confirmed by many thousands of teachers in more than sixty countries.

Before describing the material, we must deal with a few matters that will enable the reader to appreciate the novelty of the new approach.

For children, experience is an undivided whole. They do not say to themselves: "Now I am learning a word, now I am acquiring a skill; now I am playing, or sleeping." They simply live and the complex fact of living remains a unity for them; it is analyzed into its parts only under the suggestions of the adult and
through being introduced to definite distinctions. Thus, language for the child is not something separate from living, and solving mathematical problems is not separate from meeting life’s other challenges as he goes on his way. Moreover, in this the child is quite right.

It is our adult attitudes and influences that draw the attention of children to the separate aspects of life. If we say, “How many?”, in a situation that evokes that question, we do not stop to notice that we are embarking upon the operation of counting or that we are entering the realm of arithmetic. If we can count we simply use that skill, finish the job, and go on to the next task, whatever it may be. Counting is the appropriate response to the situation to which the question; “How many?” applies, just as looking into the larder or the refrigerator is the answer to the situation to which the question: “What is there to eat?” applies. This is important. Life is not split into pieces; our interests and needs draw out of life’s wholeness the various answers that produce fulfillment, but the separate aspects turn out to be chapters in the single book of experience. We can make an index to this book by noting the types of question we put and the activities that result from putting them. How much? How many? How long? What? What for? Who? When? Why?—these questions direct us to the chapters that meet our need, and steer us away from the parts that are useless to us at that particular moment.

We can provisionally say that counting is the skill required to answer the question: How many? and that various measurements answer the question: How much? or, How long? If we think of our children as meeting life’s diverse and distinct questions in this way, we can name the skills they need to find
the answers. If they wish to know how many buns are needed for the party or how many marbles they have, they must learn to count. But counting does not answer the question, How much?, and measurement does. So they must learn to measure too. These two, counting and measuring, are needed to answer a host of questions presented to us in the modern world, and in this lies their immense importance to the child entering that world. Most teachers believe they are the basis of all mathematics and try to impart these skills.

There is a persistent tendency to regard counting as the obvious entry into arithmetic. It is adopted as the gateway to addition. If I receive two sets of coins, I can find out how many I have in all by first putting them together (which is not counting) and then counting up from 1. Or I can count each pile separately, again starting each time from 1, and then, by a new awareness, I can discover that out of the two labels (or numerals) I can get a third. This construction of the third label we call addition and, if we are consistent with our labels, the sounds “two” and “three” will suggest “five”. Counting simply enables us to settle the labels and then we learn to add the numbers. By such means, the teacher tries to lead the child from counting to addition and they suppose that only through counting can addition be achieved.

This serves our purpose well enough so long as we are dealing with beads, pens, chairs, and marbles (i.e. discrete quantities) but it breaks down when we consider continuous quantities such as time, flour, gas, and the like. These we cannot count. The teaching tradition, built up over centuries, consists in large measure of a combination of counting skill and memorization
and seldom do teachers stop to ask themselves whether memorization is not called in to conceal our ignorance of the proper solution to the situation. We have already seen that a new awareness is needed to pass from counting to addition and similarly, new phases of awareness will be needed to pass from addition to subtraction, to multiplication, to division and so on. If counting is restricted to answering the question, “How many?” then we shall need new mental operations to meet the challenges of life.

Thus, our purpose in the new method is not to do away with counting but to give it its proper place beside the other activities needed to answer the other questions to which mathematics offers its answers.

In the notes that follow these activities will be described. We shall explain why we expect them to work well with children in their mathematical development, knowing already from experience the incomparably better progress they make when led along this new road.

When teachers, generally, discover the advantages of this novel approach, and come to grips with the deeper principles that underlie it, the results will be truly remarkable.
I The Contribution Of Cuisenaire

Let us begin by describing the materials invented by Cuisenaire, and presenting briefly his concept of teaching arithmetic by activity.

1 The Materials

The Colored Rods
Cuisenaire constructed sets of rods containing two hundred and ninety-one pieces. These are cut from lengths of wood 1 sq. cm. in cross-section, in lengths ranging in centimeters from 1 cm. to 10 cm. Thus the smallest unit is a centimeter cube, and the longest is a rod 10 cm. long. Each of the ten rods has a characteristic color according to its length, and experiments in varying the color-system in an attempt to improve upon it have led to no advance. They fall into three families based upon the primary colors, yellow, red and blue, together with white and black. The smallest rod, namely, the 1-cm. cube, is a sub-multiple of all the numbers, and is white. The 7 cm. rod is black. The 5 cm. and 10 cm. rods are respectively yellow and orange. The 3 cm. 6 cm. and 9 cm. rods are respectively light-green,
dark-green and blue. The series 2, 4 and 8, are red, purple and brown.

It will be seen that the colors in these family groups deepen as the lengths increase, and this, together with other features of the rods, derives from the fact that Cuisenaire was inspired by musical analogies and thought in terms of chords and dischords, and varying depths in pitch. The rods when formed into a staircase are reminiscent of the pipes of Pan and seem to form a keyboard.

The fact that all the rods are cut from lengths of wood uniformly 1 cm. square, being related to one another in their respective lengths, is invaluable, for when laid end-to-end they can form new lengths of any reasonable size.

Quite apart from the value of the colors—for rapid identification of the rod required, for secondary and parallel association with number value and size value, and for the feeling they give for number and size relationships— the gay appearance they offer, whether piled in haphazard heaps, or distributed in patterns, proves most attractive to children. They enjoy touching and handling them.

It may not be out of place to notice here something that will become abundantly clear later on. These rods are *numbers in color*; whatever we do with them they *behave as numbers behave*, so that what is being done with the hands faithfully reflects in semi-abstract form what is going on in the mind.
The Cardboard Materials
As an adjunct to the rods—and a very useful one—Cuisenaire devised three cardboard aids which can be used to gain practice in rapid mental calculation and to ensure that products and factors become second-nature to the child. They have one feature in common, namely that they represent *products*. No numerals appear upon them and the colors, which tally with those of the rods, are used instead.

The products and their factors are suggested by white squares flanked by rectangles (or by circles flanked by ‘crescent moons’) of appropriate colors in pairs. Sometimes, when the product has two pairs of factors, there are four rectangles (or crescents) around the white central portion.

\[
\begin{align*}
2 \times 2 &= 4 \\
2 \times 10 &= 20 \\
4 \times 5 &= 20 \\
7 \times 7 &= 49
\end{align*}
\]
The three simple sets of equipment that make use of this device are as follows:

1. A Wall Chart containing 37 products arranged by color families and based upon duplication. Thus the first line suggests, or represents,

\[
\begin{array}{cccc}
4 \times 4 \\
2 \times 2 & 2 \times 4 & 2 \times 8 & 4 \times 8 & 8 \times 8 \\
\end{array}
\]

The second runs,

\[
\begin{array}{cccc}
2 \times 10 & 4 \times 10 \\
2 \times 5 & 4 \times 5 & 8 \times 5 & 8 \times 10 \\
\end{array}
\]

And so on . . .

2. Product Cards. These contain the same 37 products set out separately, one product on each card, with either one pair or two pairs of factors as the case may be. The numbers represented on the product cards consist of the products that break down into two factors neither of which is greater than 10. Thus, for example, 26 will not be found because its factors are 2 and 13, and 13 has no color of its own (being usually represented by a ten centimeter rod plus a 3 centimeter rod, i.e., orange plus light green).

3. The Lotto Game consists of a set of trays to hold the product cards, a set of the product cards, and a set of 37 small counters printed with numerals from 4 to 100 which
I The Contribution Of Cuisenaire

correspond to the 37 products set out on the product cards described above. These counters will fit into the central portion of the product cards. Thus, the counter marked ‘4’ belongs to the product card bearing two red rectangles (or crescents).

These materials can be used, also, in conjunction with the chart illustrated on p. 32 of Numbers in Color, and these taken together are protected by world copyright.

2 Cuisenaire’s Philosophy Of Active Teaching

In his book Cuisenaire says:

"With this material, seeing is associated with doing, understanding, reckoning and checking.

1 Seeing.

• Numbers and their multiples are represented by related colors.

• The various lengths being of regular gradation allow active use of eyes and hands.

• Dimensions and colors constitute a double link between numbers. This classification facilitates the identification of numbers, their groupings and the discovery of relationships between them ensures that they are precisely and firmly fixed in the memory, and prepares the way for mental perception."
Doing. The child's need for action finds an outlet in the spontaneous construction of numerous combinations, freely produced by him and based only upon his awareness of relationships and groupings of numbers. These combinations allow of a great variety of decompositions.

Understanding. Seeing and doing lead to conviction and to ease in retaining results. The imagination is stimulated and reckoning becomes automatic.

Reckoning. Through manipulating the rods the child discovers new combinations which increase not only his skill in calculating, but also his interest, experience and knowledge.

Verification. This is an important phase of the child's experimental work, for he checks his own results and learns to rely on his own criteria for correcting his mistakes. Thus, by the use of the method of colored numbers:

- Each child starts from the beginning and is compelled to re-discover arithmetic for himself, at his own pace and according to his capacity.
- Visual, muscular and tactile images, clearly defined and durable, are created.
- Each number acquires and retains its individuality in the numerous combinations and decompositions in which it and its various multiples play a part.
The child is gradually brought to a certain level of abstraction through repeated practice in seeing mentally.

Since it is the child's own thought which takes material form through *his own* manipulations and with the active intervention of *his* senses, colors and dimensions thus being constructively associated, *his* analytic capacity is developed through *his own* calculation and *his own* experience. He acquires without strain mental flexibility and an attitude of objectivity.

Work becomes attractive and interesting, time is saved and the teacher's task is simplified.

The bridge is formed between the early experience gained through play and observation, and the stage of systematic work.

### 3 Cuisenaire’s Treatment Of Early Experience

The child takes the brown rod, say, which he knows as measuring 8 white ones end-to-end. Underneath and against it, he places rods known or unknown and sees that each is either too short or too long. If it is the latter he discards it, if the former, he looks for the complementary one. He then obtains a series of parallel equal rows forming *his pattern* of the brown rod that could look very different from that of his neighbor also working on the brown rod.
In 4 minutes a child of 6 made 12 lines, which he then read as 8,
4 + 4, 5 + 3, 3 + 5, 6 + 2, 1 + 7, 2 + 6, 4 + 3 + 1, 1 + 5 + 2, 1 +
3 + 4, 2 + 2 + 2 + 2, 2 + 4 + 2, 1 + 1 + 1 + 1 + 1 + 1 + 1.

Each child reads out his pattern (the correctness of which is left
to be checked by the other children in his group) and writes it
down in his class work book, using the figures for each number.
This idea of the pattern of a length is fundamental in the new
development of experience. As the rods are put back into the box
an exercise in counting finds its occasion. As this is being done,
the teacher asks the children to control the value of the various
rods beginning with the smallest.

At first they are collected by families (1; 2, 4, 8; 5, 10; 3, 6, 9; 7)
then by doubling (2, 4, 8; 5, 10; 3, 6) or trebling (2, 6; 3, 9).
These exercises increase the children’s speed and accuracy in
mental calculation and give a sense of order and tidiness. In
later calculations without the rods, it can be noticed that
children sometimes close their eyes to find the answer as if they
were looking for an image (a colored one presumably to help
them). In the first year’s work in Cuisenaire’s class and following
the official Belgian curriculum, the results set out below were
achieved by every child (6 years and 9 months by then):

1 Identification and precise and systematic knowledge of the
first ten numbers. The operations studied include 1 1/2, 1 1/3, 1 1/4,
1/3, 2/3, 2/4, 3/4, 2/5, 3/5, 4/5.

2 Location of the numbers between 10 and 20 either by
doubling 2, 4, 8, 16; 5, 10, 20; 9, 18; 7, 14; or by counting 11,
13, 15, 17, 19 etc. Location of numbers does not mean that there is identification.

3 Identification of the numbers located between 10 and 20. This takes place as before and the operations $\frac{1}{6}, \frac{1}{8}, \frac{2}{6}, \frac{2}{8}, \frac{3}{6}, \frac{3}{8}, \frac{4}{6}, \frac{4}{8}, \frac{5}{8}, \frac{5}{8}, \frac{6}{8}, \frac{6}{8}, \frac{7}{8}, \frac{7}{8}$ are added to the previous fractions.

For the numbers between 10 and 20, the difficulties encountered can be overcome by frequent use of the following type of exercise. Pupils form the pattern of 12, say. They will be shown, one after the other, the rods whose numbers are 4, 8, 5, 7, 3, 9 and 6 and will be asked to say quickly what would be required to make the value 12.

4 Location of numbers between 20 and 100 by:

- **Doubling**

  11 (22, 44, 88)
  12 (24, 48)
  13 (26)
  14 (28)
  15 (30, 60)
  20 (40, 80)

- **Counting and number-patterns**

<table>
<thead>
<tr>
<th></th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
<th>70</th>
<th>80</th>
<th>90</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>21</td>
<td>31</td>
<td>41</td>
<td>51</td>
<td>61</td>
<td>71</td>
<td>81</td>
<td>91</td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>22</td>
<td>32</td>
<td>42</td>
<td>52</td>
<td>62</td>
<td>72</td>
<td>82</td>
<td>92</td>
<td></td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>23</td>
<td>33</td>
<td>43</td>
<td>53</td>
<td>63</td>
<td>73</td>
<td>83</td>
<td>93</td>
<td></td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>24</td>
<td>34</td>
<td>44</td>
<td>54</td>
<td>64</td>
<td>74</td>
<td>84</td>
<td>94</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Identification of all these numbers is not to be expected at this stage. At the end of the first year the knowledge acquired should be reorganized in terms of the families of numbers \((2, 4, 8; 5, 10; 3, 6, 9, 7)\).

In *Numbers in Color* the complete treatment by Cuisenaire of the Belgian syllabus for the first years \((6—9)\) can be found. Here we only need stress that the original work of Cuisenaire has been taken much further. Much earlier now, in schools, pupils are performing what he found appropriate to the years 6 to 9, and in my books Cuisenaire’s ideas have been assimilated and recast so that they meet my experience of children, of learning, and of mathematics.
II The Use of The Material

The number of sets required for classroom efficiency depends upon the stage reached by the children. Each child must have enough rods at his disposal to do what is required and there must be sufficient left for the other children in his group to be able to carry out, each for himself, the same operation.

The beginners, concerned with numbers up to ten, must clearly have enough 10 cm. rods. A box will therefore hardly serve more than two children working together. When the higher numbers are studied a box will suffice for four children but, when proportions are introduced, more children can use the same set. The ideal is for every child to have his own box. Sets can be pooled for special studies requiring large numbers of particular rods. This question of sufficiency of equipment is not a mere matter of practical need, for children respond to the sense of plenitude. They enjoy having masses of rods at their disposal, whether they will need to use them all or not, and are frustrated if they run out of what they need to complete a project upon which they are bent.
It is worth taking care of the material. The rods and their boxes are durable and, to ensure that they are not lost it is worth instituting a drill for replacing them in their compartments. It can then be seen at a glance whether any are missing and each set will be complete and ready for use again.

The rods should be kept clean so that the colors remain bright and attractive. No harm will result from their being sucked, and their only danger lies in the possibility of the small rods being swallowed by a very young child. Actually there has been no report of such an occurrence. The rods should not be left in the sun as sunlight makes the colors fade in time.

Many teachers suppose that the smaller the child, the bigger should be the objects he handles. They urge that the rods should be much larger. This is a mistake, for several reasons. The child must be able to hold a number of rods in his hand, and this would be impossible with larger blocks of wood. Size would also defeat the child when wishing to use a large number of rods for some elaborate pattern and, again, the metric system upon which the rods are based is familiar in nearly all the countries of the world, even where other systems are also used. Finally, it is a complete mistake to suppose that the children themselves would prefer larger rods. To the small child, the rods seem bigger than they do to the adult, just as the rooms we lived in as children seemed immense to us then and appear to have shrunk when we see them again in later life.

It is a mistake, too, for teachers to make large models of the rods for demonstration. The children have the rods before them and, by reference to their named colors and sizes, can easily follow
and reproduce what the teacher is doing without needing to see the teacher's set of rods. If children need to be shown what to do it is best to give individual attention, using the rods the child has before him. If lectures are to be given, the film strip referred to in the Bibliography is far better than unwieldy models, and it has been specially prepared for this purpose with explanatory notes for the lecturer. Teachers who use Books A and B in the series of pupils' textbooks listed in the Bibliography* will see how much can be done by children using only the rods and the exercises set out in them.

The disposition of rods in the various color-groups is as follows:

<table>
<thead>
<tr>
<th>Color</th>
<th>Length</th>
<th>Number of rods</th>
</tr>
</thead>
<tbody>
<tr>
<td>ORANGE</td>
<td>10 cms.</td>
<td>10</td>
</tr>
<tr>
<td>YELLOW</td>
<td>5 &quot;</td>
<td>20</td>
</tr>
<tr>
<td>PURPLE</td>
<td>4 &quot;</td>
<td>25</td>
</tr>
<tr>
<td>RED</td>
<td>2 &quot;</td>
<td>50</td>
</tr>
<tr>
<td>WHITE</td>
<td>1 &quot;</td>
<td>100</td>
</tr>
</tbody>
</table>

Each set of these colors is thus a meter in total length. The rest are slightly less than a meter each. They are disposed as follows:

<table>
<thead>
<tr>
<th>Color</th>
<th>Length</th>
<th>Number of rods</th>
</tr>
</thead>
<tbody>
<tr>
<td>BLUE</td>
<td>9 cms.</td>
<td>11</td>
</tr>
<tr>
<td>BROWN</td>
<td>8 &quot;</td>
<td>12</td>
</tr>
<tr>
<td>BLACK</td>
<td>7 &quot;</td>
<td>14</td>
</tr>
</tbody>
</table>

* Reference to Books A, B, C, D, 7, 8, 9 and 10, throughout this work denote the textbooks so listed in the Bibliography.
This arrangement has been found very satisfactory for classwork, but teachers who wish to arrange them differently can make up boxes to their own requirements.

Classroom arrangements. Though every teacher will wish to suit his or her own convenience, some suggestions may be helpful.

In kindergartens, children can work in small groups at tables (placed together if necessary) with one box among four children. The rods should lie in a heap before them, their haphazard profusion stirring the young imagination. If the rods are left in the box there is a tendency for children to take them from one compartment only. It is this profusion that makes the Cuisenaire rods so popular, because is suggests an infinite variety of games and patterns, and teachers should watch to ensure that frustration is not occasioned by insufficiency of material.

The advantages of orderliness can be obtained by directing, and if necessary helping in, the packing up of the rods in their boxes when the play is finished.

In the first two grades, the best arrangement is to have a long table at the side of the classroom or at the back where the pupils can work with sufficient space to use notebook and manipulate rods up to lengths of 35 cm. There must, in addition, be sufficient space for as many sets as possible.
Beyond these elementary stages, individual tables are best. Sloping desks are, of course, inadvisable though these can, if necessary, be used by working on blotting paper or some similar substance that prevents the rods slipping, or, where ‘towers’ are to be made, the lid can be propped open to give a level surface. Ideally, each pupil should have his own box, but, if this is not possible, he should have personal access to a sufficient assortment of rods kept in a box or bag so that he can take them home. Each pupil must be made responsible for losses. Since replacement rods can be obtained, such diminished personal sets can be made up to any required number at modest cost, but an undue parsimony will prove frustrating and the general balance of the assortment should be substantially retained.

The Wall Charts are hung or fastened where they can readily be seen by all. The usual arrangement is to have one in front of the class and one on each side wall. The white centers should be left blank. If very light writing in pencil is allowed at some stage it should be borne in mind that a visible figure which cannot be erased, however faint, will destroy the value of the Chart.

Beyond the kindergarten, the display of the colors on large pieces of paper or cloth is not recommended, and it is unwise to call the color by its figure name (e.g. 2 for red). This creates confusion and is a waste of time.

At first the product cards may be used for recognition purposes, so that the pupil has a chance to find the card corresponding to a product he has learned. The pack of product cards is used both for games played usually by only two pupils, and with the Lotto
trays and counters for games that may be played by larger
groups of children.

*The Lotto Game.* When sufficient products have been learned, 
the counters corresponding to these products are selected by the 
teacher and put into a bag. Each of the players gets one or more 
plastic trays, and the product cards to be used are distributed 
among the players. One child acts as the caller of the counters or 
‘banker’. One by one the counters are drawn out by the banker, 
who calls out the number. The players scan their cards. If they 
spot the card with the factors of that number they call out ‘me’, 
and, if this is correct, the child who called places the counter on 
the appropriate card upon his tray. If the child whose card 
contains the factors fails to call, the banker, if he spots it 
himself, puts the counter upside down on the appropriate card 
upside down. If he cannot locate the card, the counter is placed 
to one side. The game can be played with as many of the 
products as have been learned, and the number in use can be 
steadily increased. At first more than one child may scan the 
same set of cards, but as more proficiency is developed this will 
be reduced to one child for each set. The banker, however, is 
always needed and is responsible for checking claims, counting 
scores and so forth. The trays are exchanged or the cards 
reshuffled for each fresh game, and turns are taken at being 
banker.

Variations on the form of the game can be devised by teachers 
and children, provided the game decided upon involves the 
recognition of factors in color for the numbers called.
Product Cards. Games with this pack of cards are usually played by two pupils only. The cards with the products being studied are selected by the teacher and they are shuffled and dealt. They are then played alternately. The products are called out by the players who recognize them first, the score each time being the value of the product. The player with the larger score wins, so that luck and skill each play their part in the result. Again, variations can be introduced as desired so long as the purpose of the game is kept in view. For example, the cards can be played one by one and compared. The player who has the larger product takes over the one with the smaller product (or vice versa if desired). The successful player scores the value of the two products. The scores are recorded and are added up when all the cards have been played. The final score will look something like this:

<table>
<thead>
<tr>
<th>1st Player</th>
<th>2nd Player</th>
</tr>
</thead>
<tbody>
<tr>
<td>9 + 4 = 13</td>
<td>6 + 14 = 20</td>
</tr>
<tr>
<td>15 + 12 = 27</td>
<td>10 + 16 = 26</td>
</tr>
<tr>
<td>20 + 8 = 28</td>
<td>18 + 24 = 42</td>
</tr>
<tr>
<td><strong>68</strong></td>
<td><strong>88</strong></td>
</tr>
</tbody>
</table>

The second player wins.

The number of cards is increased until all thirty-seven are in play. Both Lotto and the Card Game are very popular with children and they spell the final disappearance of the child who does not know his products. In some schools, the games are produced as a treat for the break period so that a keenness to acquire speed in reproducing products is linked with play-time to the great benefit of all.
Recording and Checking. The four stages in the passage from first contact to full mastery are,

1 Experience is gathered by manipulation and by trial and error so that the pupil learns not only what the situation actually contains but the limitations upon that situation.

2 This experience is then translated into any of the ‘languages’ the learner uses, which may be words, drawings, or notations. When it is words, we speak of reading the situation; when it is diagrams or drawings, we speak of a ‘translation’ in which lines, dots, or colors represent the agreed notation. This will often be the case in geometrical situations. The ‘language’ used will depend not only upon the situation faced but upon the gifts of the learner: the blind will rely upon words and the deaf upon notation. It is for the teacher to recognize the equivalence of these ‘languages’, the proof of which is found in the way children move from one to the other as occasion demands.

3 The third stage consists of the welding of experience and translation into one whole, the form of translation now being as dynamic as the experience and capable of suggesting new steps to be taken both in the material itself and in notation.

4 Only when this third step has taken place can we speak of mastery of the situation in the sense that the pupil can rely upon that mastery as a firm basis for further developments.
If we now apply this knowledge of the learning process to our present purpose, we can see that premature recording may be an obstacle. Writing should be introduced only when it helps the learner and then only for its proper purpose. To use recording because it helps the teacher to check progress is self-defeating, for progress is hampered when notation, as a substitute for experience, is called for to provide evidence of conformity with the steps in the teacher's own reasoning process. For example, to replace 5 + 7 by 12 demands that addition be understood, that one knows it is the result one is after and nothing else; otherwise the equality of 5 + 7 = 12 is pure magic, an esoteric play with signs as meaningless to the child as this text would be to the teacher if it were printed in Japanese characters.

Recording is a step in the learning process. It succeeds trial and error and represents a step forward into the world in which experience is reduced into signs that are neutral and general in their application. So, let the children first work out their problems with the rods, or with the other material, and then let them translate the situations so worked out into writing or figures recorded upon the page of the notebook.

Here are examples from my Book A to show how in the use of numbers in color, these four steps are explicitly made into a teaching technique.
Since we can all read the pattern, we shall now try to write it, using the following signs: 1 for the white rod ... 10 for the orange. Try to write down your own pattern.

If you started with the same rod as someone else, change places and see if the pattern has been written properly.

Change place with a neighbor who has a different pattern.

Take a different rod and make a different pattern. Write it down. Do it again and again until you can write down any pattern.

Now we shall all take the red rod and form its pattern. What do you find? How many white ones form the red one? If we take one of them away, what is left against the red rod? If we write down 2 for the red rod, and 1 + 1 for the white line, when we take away one of the white ones what we have left is 1, the white rod.

* Already done in the 50 questions in the Appendix of this book.
The Use of The Material

\[ r = w + w \]
\[ 2 = 1 + 1 \]
\[ 1 = \frac{1}{2} + \frac{1}{2} \]

Take the light-green rod and form its pattern. What do you find? Write it down, using the sign +. The white line is . . . . . . The white and red is . . . . . . Can you put these two rods end-to-end another way? How will you write it?

Let us write \( 3 - 1 \) for the pattern made by a green rod with a white rod covering part of it at one end. Which rod will cover the part that is left? Give it its sign.
Now we can write $3 - 1 = 2$ and read it "three minus one equals two". Can you read $3 - 2 = 1$, $2 - 1 = 1$?

Can you complete these patterns in writing:

\[
\begin{align*}
1 + 1 &= \Box \\
\Box - 2 &= 1 \\
3 &= 1 + \Box \\
1 + \Box + \Box &= 3 \\
2 + 1 &= \Box
\end{align*}
\]

\[
\begin{align*}
2 - 1 &= \Box \\
3 - \Box &= 1 \\
\Box - 1 &= 2 \\
3 - \Box &= 2 \\
3 &= 2 + \Box
\end{align*}
\]

\[
\begin{align*}
2 &= 1 + \Box \\
1 + 1 + 1 &= \Box \\
2 + \Box &= 3 \\
1 + \Box &= 2 \\
3 - 2 &= \Box
\end{align*}
\]

This example shows how carefully we must move from first contact towards mastery. The existence of mastery will be proved when the child does complete, using only his pencil, the pattern above. If he cannot do it, then he must resort to the rods to gain more experience. But it is expected that, if he does again and again what was asked of him earlier, he will not find it difficult to translate his experience into notation and use the notation dynamically.

Indeed the number of questions put in writing above are similar to the dynamics of action. Addition allows the reversal of the order of the terms, equality is symmetrical. Hence all these various questions.

Because we move so carefully in introducing the various phases of the work, we find that much time is being saved all along. Repetition here is 'biological' and is continued until mastery is attained—not for memory to be fixed. Once we have mastered a skill or an image or a thought, we do not need to remember it
consciously; it comes back by itself as speech does to all of us who have mastered it. Do we say that we remember the words of our mother-tongue? They are there and we feel them coming at the right time, in the right form, to express adequately what we have in mind. So it is with the answer to each of these notational questions. Recording will be seen by pupils as a saving of time upon manipulation. They will like it because it is easier, shorter, and can be done even if the materials are not there. This is a new attribute of writing, far removed from the drill of exercises given at present. Checking ultimately is used in order to ensure that we can pass on to a new investigation. If mastery is there, it is clear that we can move ahead, do something else, or apply our knowledge. When children learn to appreciate that only correct answers are acceptable—and in the pages above we show that pupils can check each other when the topic is suited to their level of attainment—they are careful and wish to check all their work. Recourse to the teacher is natural if the child is puzzled by what he meets and cannot, by himself or with his school mates, find why things turn out to be wrong. If teachers only intervene when asked to and watch their classes at work, going round to see what is happening to everyone, their job will be much more rewarding and they will be more useful in tackling true problems. If checking is wanted for marking, to satisfy parents or the school authorities, then daily or weekly tests can be given on prepared forms which children are asked to complete. If this is done, it is advisable not to return the test to the pupils, but to learn from them what is to be done to help x or y in his work. These tests should serve to guide the teachers and not as a means of creating competition between pupils. On the whole, if we do our work properly, the vast majority of the pupils will score very high marks. My books contain pages of exercises that
can serve as models for tests and they are, quite frequently, used for that purpose.

Here is an example from Book A:

**Complete**

\[
\begin{align*}
6 \times 9 &= \Box \\
9 \times 6 &= \Box \\
54 - 27 &= \Box \\
54 - (\frac{1}{2} \times 54)2 &= \Box \\
2 \times 8 &= \Box \\
27 - 9 &= \Box \\
3 \times 18 &= \Box \\
\frac{1}{7} \times (54 - (2 \times 20)) &= \Box \\
\frac{7}{9} \times 54 &= \Box \\
(\frac{4}{5} \times 54) + (3 \times 10) &= \Box \\
\frac{3}{16} \times (54 - 14) &= \Box \\
54 - 12 &= \Box \\
54 \div 11 &= \Box \\
54 \div 4 &= \Box \\
1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 &= \Box \\
\frac{1}{5} \times 54) + (\frac{7}{8} \times 54) - (\frac{3}{5} \times 54) &= \Box \\
54 + 36 &= 5 \times \Box \\
36 + 54 &= 9 \times \Box \\
18 &= \frac{1}{2} \times \Box \\
36 &= \frac{3}{4} \times \Box \\
36 + 54 &= 9 \times \Box
\end{align*}
\]

If teachers prefer it, they can have large sheets of white paper prepared with such written questions to put in front of the class, asking every one to write down, in order, the answers as he finds them, sometimes using the rods, sometimes (as in the text) only using the written form, i.e., pencil and paper. Pupils can take their papers to the teacher when completed and he can note down the time taken by each. Teachers who use this method, find it an eye-opener as far as pupils' ability is concerned, and it is a very satisfactory approach to their class, for pupils now look
II The Use of The Material

for hurdles to jump rather than trying to evade them. But it is recommended not to use the tests for competitive ends since discouragement of some pupils may follow. That every child does his best is all we should require and indeed some who are slow at first become fast workers in due time, when they have gathered the necessary background. Such pupils would be discouraged and would lag behind if entered too early in a race.

Other uses of the Cuisenaire rods and other materials. It has been found that the Cuisenaire rods can be used in language lessons, for their uniformity, their colors and their sizes, permit us to create language situations in the same way as we provide mathematical ones. Here we shall only give one or two examples for the mother tongue. For foreign languages the principles can easily be extended as has been done by Mrs. F. M. Hodgson and myself.*

1 Let us imagine pupils who have to distinguish the words that describe spatial arrangements. Taking 3 rods we can put them one on top of the other and this gives two examples: $a$ is on top of $b$ and $b$ on top of $c$, but also $b$ is under $a$ and $c$ under $b$. If more rods are used we have more opportunities to contrast on top of or above with underneath and under. But by just moving the rods we find two new situations that can be described using the words $a$ is on the left of $b$, $b$ is between $a$ and $c$, $c$ on the right of $b$ or if we take a new position, $a$ is in front of $b$, $b$ behind $a$ etc. (In English, the

---

words 'between', 'behind', 'below', do not require the preposition of, while 'in front', 'on top', 'on the right', do. This can be clearly established by the lesson and many opportunities given to use the expressions.

2 Let us imagine that we decide to teach the comparative and superlative adjectives. We then use the rods to introduce 'bigger', 'smaller', 'the bigger one', 'the smaller one', 'the biggest', 'the smallest', 'the heavier', 'the lighter', 'the darker', 'the prettier', placing them on the floor, on chairs, or on tables, we can speak of 'the higher', 'the lower one', and their superlatives.

Then we can try to transfer that knowledge to other adjectives. In English that will introduce the rule about one or two syllable adjectives and the others. Naturally the rods can serve to introduce more and fewer.

3 Finally one more example on language. To introduce possessive pronouns, we give various rods to different pupils and ask: What is the color of your rod? (blue) and yours? and yours? They can answer one by one: mine is . . . or together: ours are . . . . And theirs? and his? or hers? We can also compare 'my rod is . . . ' with 'mine is' or 'their rods are' with 'theirs are . . . '.

After this digression on language let us return to mathematical materials. Teachers will feel that the Cuisenaire rods emphasize the idea of length mainly. This is true but it is also clear that these rods are discrete and can be
counted like any other objects. They are prisms and have area and volume, so they can serve to introduce these experiences also. As their lengths vary in an abrupt manner *they are not a continuous material* and they do not suffice to give fully the experience of measurement. We shall need materials that are truly continuous, and there are many at hand: liquids in vessels, a tape measurer, materials of cotton or silk—all can serve.

Let it be well understood: with the Cuisenaire materials we can give all the mathematical experience required, but our physical or social experiences require other materials and, if need be, the whole world is at hand.

In the next section we shall show how mathematical experience can be extracted from the rods and in the last section of this book we shall consider applications of mathematics in the physical and social environment.
III The Stages of Mathematical Study

In this chapter we shall offer a broad picture of the progress children can be expected to make with the use of the rods, taking into account that the reader may have to conform to the requirements of a settled curriculum. Happily, children learning their arithmetic by the methods described can be expected, not simply to keep up with what is required of them, but to move confidently and swiftly to an attainment that will astonish the most critical inspector. Hence, the teacher can banish any lingering anxiety and can resolve that he will be unhurried in his methods and will not harry the children under his or her care to provide a justification for the decision to adopt the new method. The best pace, and the swiftest, is the child's own pace.

Kindergarten Stage

It is generally agreed that learning is achieved through both free and directed play at this early stage of development. The
Cuisenaire material can thus be used as a toy to provide all kinds of games.

Indeed, the wise teacher will use the rods for long stretches of free play and will be in no hurry to introduce any trace of direction. Free play is, precisely, free. Organization or direction destroys its essential character. It may need faith to stand by watching children building an Eiffel Tower or making doorways, pillars, pyramids, houses, patterns, trains and the like, especially when the construction seem to have no particular mathematical significance, but six to eight weeks spent in this free play will pay handsome dividends and, even when formal activities have been introduced, ample time for this completely free activity should be allowed.

While thus engaged children meet, without words or other intermediary between their own spirits and reality, a rich variety of relationships inherent in the rods. As the teacher discovers the rods for himself he will come to discern just how manifold and important is the mathematical experience acquired in this phase of exploration.

The colors are distinguished from the beginning, even if the names are not yet known. Color-blind children are not at such a disadvantage as might be supposed. They distinguish the rods (as do their companions) by length, but they are able to label the rods by color-names also, and no report has been received of a child encountering difficulty in this. Their lack of appreciation of the actual color is a misfortune, but they appear to make up for this disadvantage in a ready recognition of identity by a natural
process of compensation, just as the truly blind child compensates by his sensitivity of touch.

When children show that they have perceived these two attributes of the rods we find they use them simultaneously, or substitute one for the other, according to the need of the moment. For example, if they want a rod of some particular length to complete a pattern or construction upon which they are working, they will unhesitatingly pick it out of a heap even though the greater part of its length is concealed and, conversely, they are able to tell the color of a rod held behind their backs by reference to its size. In most situations the two attributes support each other. This is so with children of all ages and not only in the kindergarten but, for the purpose of suggesting continuous development, we can assume that it generally happens at the kindergarten stage.

This apprenticeship to arithmetic by way of a profound familiarity with a concrete number-symbol which is, in turn, dissolved into a color-symbol, produces immensely valuable consequences. This cannot be over-emphasized but, clearly, free activity and unfettered exploration cannot be discussed at length as though it were part of a formal curriculum. Exercises in free play cannot be devised, nor can this element of the child's mathematical education be treated formally in a chapter in a textbook. But make no mistake, something is happening, and that 'something' matters enormously. Without it the value of much that follows would be largely lost. Building on this foundation. I introduce some organized games, to achieve (a) relative knowledge of the rods and (b) absolute knowledge of each. The first means that they identify the rods by comparison,
and the second, that any one rod is known in its own identity by feeling it and without resort to comparison. Much the same is true of music, in which notes may be known by comparison (higher, lower, louder, and softer) but the acute ear of a few musicians fastens upon absolute pitch. In music, this achievement is rare whereas, in number, it lies within the reach of all.

These organized games are described in detail in the Appendix II, and some examples are given here:

1 The children take the rods and compare them to find out which are equal and which are not. They then place any two end-to-end and look for a single rod that is equal to the length so formed.

2 Patterns are made by forming as many equal rows as possible using two or more rods end-to-end. They compare the patterns they have made with those of their neighbors.

3 They learn to ‘read’ the patterns, using color-names only. Later, they shut their eyes and listen to a pattern being read to see whether it is possible or correct.

4 Sizes are compared, labeling the rods as smaller, bigger, smallest and biggest.

5 They take four rods that will form two pairs giving the same length when put end-to-end, then mix them up and find which pair up and which do not.
6 'Staircases' are formed with one rod of each color and, with eyes shut, they give the color-names of the steps in order,

\[
\begin{align*}
&y \\
&\text{r + g} \\
&\text{g + r} \\
&w + p \\
p + w \\
r + r + w \\
w + w + w + w + r \\
w + w + w + w + w + w \\
\end{align*}
\]

A pattern of yellow

A staircase on dark green

from the smallest to the largest and vice versa.

7 They find complementary rods for given lengths, first with their hands, then by using their eyes, and then only by thinking of them with eyes shut.

8 If we extend this play by selecting two rods which together are longer than the orange rod we introduce the children to a much richer game-situation with, of course, much more to learn.

9 Lengths can be made of rods of one color. This is called 'playing at trains'. These trains provide many important concepts which can be evoked by questions. Can we make two trains of equal length but different color, using for each
train rods of the same color? What is the smallest train we can make with any two rods? With any three?

A red train and a green train equal in length.

\[ 12 \times 2 = 24 \]
\[ 8 \times 3 = 24 \]

Two trains with different colors.

\[ 2 + 5 + 1 + 3 + 7 + 2 = 20 \]
\[ 3 + 2 + 4 + 5 + 6 = 20 \]

10 Trains of different colors lead to new games and new discoveries.

11 Absolute knowledge of the rods can be gained by having a selection of rods on a tray placed on the head, and feeling
them, taking care not to handle more than one at a time. The lid of the box makes a convenient tray for this game.

Most children succeed at these games very well and, in those found in the text-books, most of the basic concepts of arithmetic find their place; but, even when the children have achieved mastery through these games, I do not advocate an immediate stride forward into the realm of numbers. Teachers may or may not be prepared to accept the view now put forward, but they are asked to give it careful consideration, for experience has amply confirmed its value. We move, not to numbers immediately, but to operations. Kindergarten children take to it very well though it involves an approach to mathematics through algebra instead of through counting and arithmetic.

Now that the color-names are familiar they can as easily be called by their initial letters. The problem of the repetition of the letter 'b' is overcome by calling black, brown and blue, k, n, and u, so that we get the series,

\[ w, r, g, p, y, d, k, n, u, o \]

The letters are easily remembered because they are related to the now-familiar color-names, and this new letter game has an incidental value in supplementing what is being taught of spelling, reading, and writing*.

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* Some teachers prefer to use other means of solving the problem of the three b's. Black, tan-brown and navy blue, give b, t and n. Or b, B and b. What matters is that a convention should be settled and adhered to.
Teachers who try this new suggestion see at once that there is nothing frightening in it for the child. My own experience shows that they find fun in playing their games in this way.

Three rods, say, y, r, k can be disposed in different arrangement: y and r end-to-end fitting against k, or r and y against k. Two further arrangements are found with r and y (or y and r) on top of k, or k on top of them. When the children see the rods positioned in these ways, they enjoy reproducing what they see in the new writing:

\[
\begin{align*}
y + r &= k \\
\text{or} & \quad r + y = k \\
\text{or} & \quad k = y + r \\
\text{or} & \quad k = r + y
\end{align*}
\]

If we remove one rod from the triplet we can have twelve new ways of writing what is seen and we introduce problems that can be readily solved:

\[
\begin{align*}
y + □ &= k & \quad & r + □ &= k \\
\text{or} & \quad k = □ + r & \quad & k = □ + y \\
y + r &= □ & \quad & r + y = □ \\
\text{or} & \quad k = y + □ & \quad & k = r + □ \\
□ + r &= k & \quad & □ + y &= k \\
\text{or} & \quad □ = y + r & \quad & □ = r + y
\end{align*}
\]

In this game we look for the one letter needed to re-form the triplet; and it will dawn upon the child as he plays this game that the smaller rod against the larger one leaves a part of the larger one uncovered. In terms of the letter game this can be expressed:

\[
\begin{align*}
k - y &= □ \\
k - r &= □
\end{align*}
\]
The missing letter completes the statements just as the missing rod completes the visible pattern.

The + and - signs are read as *plus* and *minus*, for children should learn the correct words and symbols from the beginning and they find no difficulty in accepting them as they accept any word in common use. ('Plus' is as easy to learn as Veronica, and minus as easy as Geoffrey—and both are easier to spell!)

These exercises in the translation of patterns are most valuable when numbers are finally introduced. We shall deal with this new stage in the next section pausing only to remark that, when the approach outlined is adopted, children in the kindergarten can master a surprising amount of actual arithmetic if the teacher wishes to introduce it.

**After the Kindergarten—The first three years**

We are now ready to go forward, if the step has not already been taken, to a new notation—the familiar numerals of arithmetic. We have seen that experience is separated from notation, which is only the language of some signs written as figures or numerals and pronounced as numbers. We can now concentrate upon written and mental arithmetic. When we use speech we shall employ words to denote numbers and, when we use writing, we shall express them in figures.

Some children will be familiar with numbers through having learned to count but, for those who have not yet acquired this accomplishment, we shall introduce numerals by way of
measurement. It is best, again, to proceed slowly as I show in Book A. Experience is gained with 1, 2, 3, 4 and 5 which is then transferred to 6, 7, 8, 9 and 10. The rods are successively measured by the white rod. A red rod and green rod, end-to-end, are measured against a yellow rod and what has hitherto been red plus green equals yellow (or r + g = y and its 31 variations) becomes 2 + 3 = 5 with its similar variations. The appearance of these familiar numerals should not be allowed to make teachers wish to use them as their starting point: the algebraic formulation is so much more powerful.

Those who use the approach here described will soon be aware that they are avoiding the artificialities and the difficulties that dog the footsteps of the child brought up to rely upon counting and memorization.

Now let us see what, in these first three years of formal schooling, children will be learning.

1 They will be mastering the four operations of adding, subtracting, multiplying and dividing simultaneously.

2 Though they write their arithmetic, experience of number perpetually dominates the notation and the notation does not become a substitute for arithmetical insight and understanding. Instead of being introduced to units, tens, and hundreds, the children work 'horizontally' and gain a wealth of experience of number behavior before adopting the 'vertical' notation, by which time it is seen to be only a variation of what they have been doing with complete and
The Stages of Mathematical Study

immediate comprehension. This approach does away with
the teaching of the various methods of subtraction taught 'to
make sums come right'.

3 The children gain as much experience as possible with small
numbers (up to 100 or 1000) but extend the operations
naturally to larger numbers if no new principle is involved.

4 Symbolic situations are created by the use of rods that
represent quantities relevant in their everyday experience, so
that they acquire a social arithmetic.

The steps by which these achievements are attained are set
out in detail in the textbook series.

As we are here attempting only to introduce the approach and
the material we shall not pursue these points further, but some
further indications of the novelty of our approach ought to be
given. We shall, therefore, make specific reference in the
sections that follow to fractions as operators, the learning of
products, and the treatment of subtraction.

Fractions

It is commonly supposed that fractions are difficult and should
be reserved until the pupil has reached a relatively advanced
stage. The difficulties are created by treating fractions as bits
and operations upon bits do prove difficult. If, however, we see
that \(3 \times 1 = 3\) can be read as \(1 = \frac{1}{3}\) of 3 we acquire the habit of
reading each product in this dual fashion, so that fractions are
conceived as operators. If 3 of one kind make a certain quantity, then the unit, when viewed in relation to that quantity, is a fraction. Thus, \( \frac{1}{3} \) is seen every time we see three identical (or, in some sense, equivalent) elements forming a fourth.

\[
6 = 3 \times 2, \text{ so } 2 = \frac{1}{3} \text{ of } 6 \\
9 = 3 \times 3, \text{ so } 3 = \frac{1}{3} \text{ of } 9
\]

"\( \frac{1}{3} \) of" is the fraction operator, that is to say, the operator that replaces 6 by 2 or 9 by 3. "\( \frac{1}{3} \)" does something to 3, 6, 9 etc., and we come to think of it as an operator until, by sheer familiarity with the concept, it becomes second nature.

Such conceptions as \( \frac{1}{2}, \frac{1}{4}, \frac{2}{3} \) are met in the first year and, indeed, in the kindergarten. Hence they appear in Book A.

\[
6 = 3 \times 2, \text{ so } 2 = \frac{1}{3} \text{ of } 6
\]
Many thousands of teachers, using Book A, and introducing fractions at this age, have found how amazingly the children rise to the opportunity to tackle fractions. The response is immediate and natural, and is observed within about three months of their introduction to numerical notation.

Here is an example of a test successfully completed at this stage by a majority of the children tested:

\[
\begin{align*}
(\frac{1}{2} \times 6) + (\frac{3}{5} \times 5) + (\frac{3}{4} \times 4) &= \square \\
\frac{1}{2} \times (9 - 7) + \frac{1}{4} \times (9 - 5) &= \square
\end{align*}
\]
There is another way of introducing fractions, but this is for the best pupils in the third year, and it must be left for treatment at a later stage. It is dealt with in Book D.

**Products**

Reference has already been made to the Product Cards and other materials. Their value is such that we can affirm that the learning of tables can be dispensed with altogether. By the age of seven, a child can produce his own tables if he wishes because these simply set out systematically the multiplication bonds he already knows.

The trains made of rods of one color can be changed into rectangles by placing the rods side-by-side instead of end-to-end. If trains of equal length, but each of different color, are now converted into such rectangles, it will be seen that these are sometimes congruent. Thus, a 12 cm. train may be composed of six reds, four greens, three purples or two dark-greens. We observe that the four rectangles derived from these trains (when the rods are placed, as described, side-by-side instead of end-to-end) fall into two sets of congruent pairs. The congruence can be readily demonstrated by superimposing the one upon the other. If, however, one rectangle is taken from each pair we find that these lack congruence and cannot be superimposed as they are so as to fit one another. The congruent rectangles are, respectively, the red and the dark-green, the purple and the light-green. If we next experiment with a 6 cm. train we find it
yields only one pair—red and light-green, while 9 cm. gives only a light-green square, and 16 cm. a purple square and two congruent rectangles, respectively red and brown.

Now, if instead of the superimposed rectangles we content ourselves with a single rod of one color set across the other we have a new symbol (a cross) for the product. Take, for example, the 12 cm. trains which were converted into two pairs of congruent rectangles. These pairs would be reduced to two pairs of single rods forming a red and dark-green cross, and a purple and light-green cross. The rods in each cross are interchangeable just as, in terms of figures we can write down $2 \times 6$ or $6 \times 2 = 12$ and $4 \times 3$ or $3 \times 4 = 12$.

The Product Cards and other cardboard materials retain the colors (and nothing more) of the pairs seen in the crosses. Thus, the movement towards abstraction has been carried from the visible and tangible construction of rods in trains and then in rectangles to the symbolic cross and on to a sign that is devoid of immediate meaning but which is linked by color association with the concrete situation from which the child set out.
Four trains of equal length but different colors.

The rectangles are symbolically represented by a red and dark-green cross side.

Red and dark green are congruent. So are purple and green.

\[
\begin{align*}
2 \times 6 &= 12 \\
6 \times 2 &= 12 \\
3 \times 4 &= 12 \\
4 \times 3 &= 12
\end{align*}
\]
The rectangles are symbolically represented by a red and dark-green cross and a green and purple cross.

We can visualize the paired rectangles which we could complete if we wished.

Here only color is left to bridge the gulf between number and the concrete situation displayed in the rods as trains and rectangles.

The counter that belongs to the two pairs of factors has been placed in position on the white central circle.

For further uses of the Cuisenaire product cards and wall charts see Appendix I.

From this point, teaching progresses according to the steps shown in Book A.

1 An orange rod and a brown rod end-to-end form a length we call *eighteen* and write as 18. Make a pattern for 18 and write it
down. (Note, the pupil is not set at once and deliberately to the
discovery of the multiplication bonds that may be found in a
number but proceeds as he has previously done in forming a
pattern.) With the beginnings of familiarity established a further
step will be taken:

Complete the following patterns in writing:

\[
\begin{array}{ccc}
9 + \color{red}{\square} &=& 18 \\
18 - \color{red}{\square} &=& 6 \\
2 \times 9 &=& \color{red}{\square} \\
18 - (2 \times 7) &=& \color{red}{\square} \\
18 - 16 + (\frac{1}{9} \times 16) &=& \color{red}{\square} \\
10 + 3 + \color{red}{\square} &=& 18 \\
(\frac{1}{3} \times 12) + (\frac{2}{7} \times 14) + (\frac{5}{6} \times 12) &=& \color{red}{\square} \\
\frac{2}{3} \times (18 - 6) &=& \color{red}{\square} \\
\end{array}
\]

(2 \times \color{red}{\square}) + 6 = 18

(2 \times 8) + \color{red}{\square} = 18

(2 \times 3) + (2 \times \color{red}{\square}) = 18

(2 \times \color{red}{\square}) = 2 \times \color{red}{\square}

(2 \times 3) + (2 \times \color{red}{\square}) = 18

(2 \times \color{red}{\square}) = 2 \times \color{red}{\square}

If you cannot do it, use the rods.

How many nines are there in eighteen? and how much left?

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52
What is half of eighteen or $\frac{1}{2} \times 18$?

What is a third of eighteen or $\frac{1}{3} \times 18$?

What is a sixth of eighteen or $\frac{1}{6} \times 18$?

What is one-ninth of eighteen or $\frac{1}{9} \times 18$?

What is $\frac{2}{9} \times 18$? and $\frac{3}{18}$? and $\frac{4}{6} \times 18$?

Compare $\frac{3}{9} \times 18$ and $\frac{1}{3} \times 18$.

Compare $\frac{7}{6} \times 18$ and $\frac{1}{2} \times 18$ and $\frac{3}{6} \times 18$.

Since eighteen white rods cover the length called eighteen, each white rod is one-eighth of that length and we write it $\frac{1}{18} \times 18$.

What is $\frac{2}{18} \times 18$? $\frac{1}{18} \times 18$? $\frac{5}{18} \times 18$?

" " $\frac{7}{9} \times 18$? $\frac{11}{18} \times 18$?

Compare $\frac{2}{18} \times 18$ and $\frac{1}{9} \times 18$

" " $\frac{3}{18} \times 18$ and $\frac{1}{6} \times 18$

" " $\frac{4}{18} \times 18$ and $\frac{2}{6} \times 18$

" " $\frac{5}{18} \times 18$ and $\frac{3}{9} \times 18$ and $\frac{1}{3} \times 18$

" " $\frac{8}{18} \times 18$ and $\frac{4}{9} \times 18$

" " $\frac{10}{18} \times 18$ and $\frac{5}{9} \times 18$

" " $\frac{12}{18} \times 18$ and $\frac{8}{9} \times 18$ and $\frac{2}{3} \times 18$
What are the factors of 18?

Can you find in the wall-chart the sign that shows the colors for these factors? and in the pack of cards?

By how much is 18 bigger than 13? and than 11?

By how much is 18 bigger than 3 × 5? and 2 × 6? and 7 × 2?

Which is the biggest of the following:

\[ 3 \times 6; \ 2 \times 7; \ 7 + 9; \ 11 + 6; \ 3 \times 2 + 4 \times 3; \ 10 + \frac{1}{2} \times 16? \]

\[ 3 \times 10 = 6 \times 5 = 30 \quad 4 \times 10 = 8 \times 5 = 40 \]

Form the crosses for these two numbers and find their signs in the wall-chart.

Find in the wall-chart the signs for the following: —

4, 8, 16; 6, 12; 10, 20, 40, 80; 50, 100; 9, 18; 15, 30, 60, 70; 90; 14.

Find them in the pack of cards.
III The Stages of Mathematical Study

Double 12, double again and again. Write down your answers. \(2 \times 12 = \square, 2 \times (2 \times 12) = \square, 2 \times (2 \times 2 \times 12) = \square\)

What is \(\frac{1}{2} \times 24? \frac{1}{4} \times 48? \frac{1}{8} \times 96?\)

What are the factors of 12, 24, 48?

Can you find their signs in the wall-chart? (in the pack of cards?)

What is \(\frac{1}{2} \times 96?\) Double that number.

What is \(\frac{1}{4} \times 96?\) Multiply that number by 4.

What is \(\frac{1}{8} \times 96?\) Multiply that number by 8.

Start with 96, halve it, halve it again, and again and again. Write down your answers, to check them, double the last one, double again and again and again. Were you right?

What is \(\frac{1}{3} \times 24? \frac{1}{3} \times 48? \frac{2}{3} \times 24? \frac{1}{8} \times 24? \frac{1}{8} \times 48? \frac{3}{8} \times 24? \frac{3}{8} \times 48?\)


Find the factors of 36 and 72 etc.

Write down all the products you have found so far. Make a table of those numbers and compare it with the following:
(Here is set out the double-entry table of products from $2 \times 2$ to $10 \times 10$).

Are they the same? Where do they differ?

Find the factors smaller than 10, of 25, 27, 32, 18, 28, 72, 56, 12, 64, 81, 35.

Find all the factors of: 33, 36, 72, 66, 99, 96, 84, 81.

Which numbers up to 100 are multiples of 2?

Which number up to 100 are multiples of 3 . . . . of 12?

Thus the multiplication tables are reached as an end result of the exploration of numbers and not by sheer memorization. Factors are remembered because they are known by experience to be factors and not upon the basis of the teacher’s say-so or the authoritative printed matter in a book of tables. The successful construction of tables by the child is a proof that he knows all the products and the prime and composite numbers up to 100. This is intended to be mastered by the age of 7, and the knowledge thus firmly grasped and readily retained will be used in much more advanced work as will be seen in Book B, where numbers up to 1,000 are explored, but operations are performed on any numbers that can be written even if they are not yet named.
Subtraction

In Book A, subtraction is met in its horizontal form, though vertical subtraction appears towards the end of Book A. By ‘horizontal’ is meant that the numbers are thought of in themselves and not by reference to units, tens and hundreds. Subtraction has been encountered in the comparison of larger and smaller rods (or lengths of rods) laid side-by-side to reveal the complement needed to achieve equality. The brown rod is less than the orange rod by a red rod, or the orange rod is greater than the brown rod by a red rod.

Since operations of this sort have already been performed in considerable numbers, the children will have an immense stock of information at their finger-tips and will be able to pass from \(2 - 1\) via \(3 - 1\) and \(3 - 2\) right on to \(20 - 1\) with no difficulty.

Here a further important insight may be introduced.

Every number is equal to an infinite number of differences of pairs of numbers: \(1 = 2 - 1 = 3 - 2 = 4 - 3\) etc. \(17 = 18 - 1 = 19 - 2 = 20 - 3\) etc. So, if we are given a subtraction, two attitudes are possible: we can perform the subtraction by whatever method we have learned, or we can find pairs of numbers in the same family of equivalent differences that are easier to subtract at once. Thus, \(43 - 29\) is equivalent to \(44 - 30\) and also to \(23 - 9\) and, so, to \(24 - 10\). When children grasp this they find they can do work in a form that comes easily to them and can avoid meeting what otherwise might, for the time being, prove difficult.
Since this process of transforming differences is introduced from the early stages, it never occurs to the child that 43 minus 29 is harder than 63 - 21 or 76 - 45. Teachers suppose that it is harder, but do not appreciate that the problem is created by the notation of our number system and the belief that we must treat 43 - 29 as we would treat 43 - 22, i.e., by taking away units from units and tens from tens. Since this cannot be done it supposed that a method must be devised to make it possible, and hence flow the various complications and tricks by means of which children are helped to overcome the imaginary problem. That problem is imposed upon them by a confusion of thought that arises out of the dominance of notation over experience.

Let us take, for example, the 'very hard' subtraction, 10,001 - 6,748. We hear the names of the numbers and 'drop' six thousand from each to obtain 4,001 - 748. This is equivalent to 4,003 - 750 which is equivalent to 4,253 - 1,000 from which the answer 3,253 is at once found.

This method of finding equivalents is much easier than the usual subtraction methods, of which there are about twenty, the most popular being borrowing, equi-addition, and by complementarity. In fact, no other method of subtraction need be taught in view of this observation about equivalent differences. By making use of this, we not only save our pupils
many months of effort but make them much more efficient than at present.

We need not set out here at length what has already been fully discussed in my other textbooks for teachers. We have gone far enough to introduce the approach and its notable advantages over orthodox and other methods, but readers may wish to know whether, having covered so much ground in the first three years, there will still remain enough of sufficient novelty and simplicity in the succeeding three years. Many of the matters taught in earlier stages are, in the orthodox methods, re-taught later on but, if the approach here described in outline is adopted, such going over of old ground becomes unnecessary, and I have discovered many ways of broadening and enlivening the later stages of the primary school curriculum.

Book D deals with fractions as families of ordered pairs, and this can be tackled in the third year and carried forward into the fourth. Decimals and percentages then become very simple because they represent only variations in notation, and all this can be mastered in about three months at the age of 9.

In Books, 7, 8, 9 and 10, we introduce a wealth of concepts which can readily be assimilated by pupils of nine to eleven and which have either not been previously taught at school-level at all (such as the study of the set of whole numbers) or only late in the school curriculum (such as square roots, permutations and combinations, equations, and the elements of geometry).
These books are based upon practical experience with children of these ages, and their use leads to an amazingly successful response over and over again. This is not simply a matter of my own experience, for some hundreds of teachers throughout the world, who have assimilated and adopted the approach here introduced, have made the discovery that it is convention, prejudice, and a perpetual bias towards teaching arithmetic via counting and memorization, that has prevented their giving their pupils the opportunity to achieve what lies within their grasp. Once the new horizons were opened to the teachers, and they began to go forward boldly and experimentally, they found the same eager response and confirmed the same astonishing results.

The unconvinced reader is urged to resist the impulse to discount this large claim, and to spend time studying the various textbooks and experimental lessons noted in the bibliography. Above all, he is urged to make a trial of the approach, for he will learn more from the response of the children themselves than can ever be conveyed by the printed page. Teachers need not, and should not, accept what is said on the authority of the author; they should critically, but sincerely, put to the test the claims he has made.
Experience in teaching mathematics according to the principles outlined in this book has revealed that the standards we have hitherto applied to our pupils often fail to reflect their true aptitudes, and all-too-frequently operate to turn a bright child into a dull pupil. When new standards are applied we often find such pupils are not only equal in mathematical ability to the ‘clever children’ at the top of the class, but superior.

The main reason for this reversal of evaluation is not far to seek. Teaching is traditionally based upon words. Admittedly, much has been done to mobilize the child’s capacity for learning by visual means and, in arithmetic, the use of various material aids is quite widespread but words remain the pivot of the teaching process, and the materials and devices used in arithmetic have depended upon verbal explanation. The transition to writing in terms of mathematical symbol is no escape from the verbal net, for this is itself a language and, when errors occur, it is to words the teacher resorts to show where and why the mistake was made.
Clearly, a child whose skill in the verbal realm is weak or ill developed may be ‘bad at mathematics’ not because of any native mathematical ineptitude but because the verbal mode of teaching presents him with an obstacle at the very threshold of the subject. A child who is a born football player but who never gets access to the playing field because the latch on the gate baffles him, will never have a chance to show us his skill, and so it is with many a child in the sphere of mathematics.

Thus, performance at school is often not a reflection of aptitude but of the method of teaching. If this suits a particular child, he is ‘bright’. If it fails to meet his need, he is ‘dull’. A similar situation arises with the child with unsuspected deafness or myopia, who does badly until it is discovered that he is missing part of what is being said in class, or cannot read everything that is put on the blackboard. If we resolve the problem of communication—however it arises—and ensure that the child can use the means at his disposal to ‘answer back’, we find the real child, and, only then are we in a position to assess his ability. A philosopher who discovered that he fared very poorly in a non-verbal test of intelligence for ten-year-olds was both dismayed and illuminated. His belief in his own intellectual superiority received a rude shock, but he was fascinated because he had learned that intelligence functions differently at different levels, sometimes with little apparent connection between those levels.

If children are ‘dull’ at any subject we must resist the temptation to write them off by declaring that Johnny is bad at geography and Jane weak at arithmetic. The appropriate response is to ask ourselves whether the way we are teaching Johnny geography is
the way that enables him to lay hold upon it with the intellectual powers he has. Are we speaking his mental language? Or is it a ‘foreign’ one? Clearly if Jane has not only to learn arithmetic but to learn it in a ‘language’ she cannot follow, she is as badly off as a Chinese boy learning arithmetic in English before he can speak that language properly. Both will be unlikely to shine, and for the same reason.

I have had a very wide experience of teaching ‘slow learners’ in many countries and have been able to establish that slowness is not an absolute concept. In one lesson with children of high intelligence quotient, the performance was distinctly disappointing to their teachers because the pupils were unable to meet non-verbal challenges as readily as the verbal ones that suited their style of intelligence. Their natural aptitude for verbalizing had been well polished by continual use but, since it did not serve their purpose in the novel situation of manipulating rods, they had to fall back upon undeveloped (and possibly inherently weak) levels of intelligence. They thus appeared to be dull. On another occasion the top boy, who was sitting next to a ‘slow learner’, was observed desperately trying to keep up by copying his neighbor’s actions. What he found difficult proved to be the natural ‘language’ of the supposed dullard.

If we could learn to look at our pupils not as we wish them to be, but as they are, the achievement of many, both in school and in life, would be very different.

It is often said that teachers are born and not made. Long experience in many parts of the world compels one to admit that
the majority are not born teachers. They find their work frustrating, and its results disheartening. Sometimes they conceal, even from themselves, a guilty conscience and, feeling their own inadequacy, look anxiously for help that rarely comes and, in the end, convince themselves that the problem lies in the dullness of the children they are trying to teach or in the natural limitations of particular age-groups. There is, also, the danger that the child who is baffled will be told he is not trying and the child who forgets what he has been taught will be blamed for laziness.

Many teachers, experiencing this sense of frustration have, on adopting the materials and the approach described in this book, discovered that they are good teachers. Often those with the least adequate academic preparation and the least teaching experience were the first to give their pupils the opportunities they had previously lacked for making swift and confident progress. Perhaps this very lack of qualification resulted in their having fewer obstacles to overcome in themselves in taking to the new line of approach. I have been driven to conclude that standards of achievement in schools are, generally speaking, a reflection of the beliefs the teachers hold about themselves, about their background, and about the value of their own methods. In less than a week, a teacher who is quite inexperienced in the approach here advocated can discover that his or her teaching is improving day by day through learning from his pupils and their mistakes. In a series of world-tours I have observed this metamorphosis taking place again and again as teachers have come under the spell of the material and the principles implicit in it. I can claim that there are already many more teachers of mathematics who are distinctly better at their
job, and more deeply satisfied by it, as a result of encountering the Cuisenaire material and responding to its challenge. They have found themselves progressively stimulated to question their beliefs about themselves, about the children they teach, about education in general and their own subject in particular. What a revolution there would be in the standard of teaching generally, and the resultant standard of achievement of pupils, if teachers everywhere could be brought to accept and respond to this far-reaching challenge. Perhaps the readers of this book will be led by these comments to look at themselves and their pupils with fresh eyes, seeing their subject as a daily challenge to themselves rather than as a test of intelligence, memory, and industry for the children.

Having discussed standards in these broad terms, let us now go on to indicate systematically the standards of attainment in arithmetic that can consistently be achieved when the Cuisenaire material is used in the sequence, and according to the principles set out which meet so much better the real mind of a child and its functioning.

At the age of six or thereabouts, children can perform without mistake, and in about 10 to 15 minutes, each of the following tests:
1. 
\[
\begin{align*}
2 + 7 &= 9 - 6 &= 3 + 2 + 4 &= 10 - 3 - 2 = \\
4 + 3 &= 8 - 4 &= 5 + 2 + 3 &= 9 - 4 - 3 = \\
5 + 3 &= 7 - 5 &= 4 + 2 + 3 &= 8 - 5 - 2 = \\
2 + 8 &= 10 - 5 &= 3 + 4 + 3 &= 11 - 6 - 5 = \\
8 + 3 &= 10 - 8 &= 2 + 7 + 2 &= 7 - 2 - 3 = \\
7 + 4 &= 10 - 9 &= 6 + 2 + 3 &= 11 - 2 - 7 = \\
5 + 6 &= 11 - 5 &= 4 + 2 + 4 &= 11 - 3 - 6 = \\
9 + 2 &= 11 - 7 &= 2 + 3 + 4 &= 11 - 5 - 5 = \\
6 + 4 &= 11 - 8 &= 2 + 5 + 2 &= 10 - 3 - 7 = \\
7 + 3 &= 11 - 9 &= 4 + 3 + 4 &= 11 - 4 - 5 = \\
\end{align*}
\]

2. 
\[
\begin{align*}
10 + 1 - 4 &= 3 \times 2 &= 8 \div 2 &= (2 \times 4) - 6 = \\
11 - 7 + 2 &= 4 \times 2 &= 10 \div 2 &= (3 \times 3) + 2 = \\
11 - 3 + 2 &= 3 \times 3 &= 6 \div 6 &= (4 \times 2) - 5 = \\
11 - 8 + 4 &= 1 \times 4 &= 8 \div 4 &= (5 \times 2) + 1 = \\
9 + 2 - 5 &= 5 \times 2 &= 8 \div 8 &= (3 \times 2) - 5 = \\
11 - 2 - 2 + 4 &= 2 \times 3 &= 9 \div 3 &= (2 \times 4) + 3 = \\
11 - 3 + 2 - 7 &= 2 \times 5 &= 6 \div 2 &= (7 \times 1) - 3 = \\
10 - 5 + 6 - 8 &= 2 \times 4 &= 10 \div 5 &= (2 \times 5) - 5 = \\
11 - 9 + 2 + 5 &= 2 \times 2 &= 6 \div 3 &= (2 \times 2) + 7 = \\
8 + 3 - 7 + 4 &= 1 \times 5 &= 4 \div 2 &= (2 \times 4) + 6 = \\
\end{align*}
\]
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3. \[(10 \div 2) - 3 = \quad (10 \div 5) \times 4 = \quad (8 \times 2) - (3 \times 2) = \]
\[(7 \div 7) + 8 = \quad (8 \div 4) \times 2 = \quad (10 \div 5) + (3 \times 3) = \]
\[(9 \div 9) = 10 - \Box \quad (10 \div 2) \times 2 = \quad (8 \div 2) \times (8 \div 4) = \]
\[(10 \div 5) + 9 = \quad (9 \div 3) \times 2 = \quad (9 \div 3) \times (10 \div 5) = \]
\[(4 \div 2) + 7 = \quad (6 \div 3) \times 3 = \quad (4 \div 2) \div (8 \div 4) = \]
\[(6 \div 3) + 5 = \quad (7 \div 7) \times 8 = \quad (10 \div 2) \times (6 \div 3) = \]
\[(8 \div 2) - 3 = \quad (9 \div 3) \times 3 = \quad (8 \div 4) \times (9 \div 3) = \]
\[(6 \div 2) \div 2 = \quad (9 \div 9) \times 10 = \quad (9 \div 3) \times (8 \div 4) = \]
\[(8 \div 4) - 2 = \quad (8 \div 2) \times 2 = \quad (10 \div 2) \times (8 \div 4) = \]
\[(9 \div 3) + 8 = \quad (4 \div 2) \times 5 = \quad (5 \div 2) \div (10 \div 2) = \]

4. \[\frac{1}{2} \text{ of } 10 = \quad \text{Starting with } 0 \text{ count by twos.} \]
\[\frac{1}{5} \text{ of } 10 = \quad " \quad " \quad 0 \quad " \quad " \quad \text{tens.} \]
\[\frac{1}{4} \text{ of } 8 = \quad " \quad " \quad 0 \quad " \quad " \quad \text{fives.} \]
\[\frac{1}{3} \text{ of } 9 = \quad " \quad " \quad 0 \quad " \quad " \quad \text{threes.} \]
\[\frac{1}{2} \text{ of } 8 = \quad " \quad " \quad 0 \quad " \quad " \quad \text{fours.} \]
\[\frac{1}{2} \text{ of } 6 = \quad " \quad " \quad 2 \text{ double again and again.} \]
\[\frac{1}{3} \text{ of } 6 = \quad " \quad " \quad 10 \quad " \quad \]
\[\frac{2}{4} \text{ of } 8 = \quad " \quad " \quad 5 \quad " \quad \]
\[\frac{2}{3} \text{ of } 9 = \quad " \quad " \quad 4 \quad " \quad \]
\[\frac{2}{5} \text{ of } 10 = \quad " \quad " \quad 3 \quad " \quad \]

The children are only required to write down the answers.

My Book A being ordinarily used by children of six or seven, contains many examples that can serve as tests. Once teachers are sure their pupils can work out such tests they are advised to
provide for themselves their own tests since it is quite likely that however daring Cuisenaire and myself have been, we have still fallen short of children’s possibilities.

At the age of seven, or thereabouts, children should complete the following tests without mistake in about 5, 8, 12 and 15 minutes respectively:

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<td></td>
<td>(3 \times 8) \times 2 =</td>
<td>(30 - 6) \div 4 =</td>
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<td>(5 \times 5) \times 2 =</td>
<td>\frac{1}{5} \text{ of } 25 =</td>
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<td>(4 \times 7) \div 2 =</td>
<td>(6 \times 5) \div 2 =</td>
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<td>\frac{1}{4} \text{ of } 32 =</td>
<td>(32 \div 8) \times 3 =</td>
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<td>\frac{1}{5} \text{ of } 35 =</td>
<td>(7 \times 5) \times 2 =</td>
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<td></td>
<td>(6 \times 6) \div 3 =</td>
<td>\frac{1}{4} \text{ of } 36 =</td>
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<td>(5 \times 8) - (6 \times 2) =</td>
<td>\frac{1}{5} \text{ of } 40 =</td>
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<td>40 \times 5 =</td>
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<td>2</td>
<td>\frac{3}{4} \text{ of } 21 =</td>
<td>\frac{1}{2} \text{ of } 24 =</td>
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<td>\frac{1}{2} \text{ of } 26 =</td>
<td>\frac{3}{4} \text{ of } 27 =</td>
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<td>\frac{7}{10} \text{ of } 30 =</td>
<td>\frac{1}{2} \text{ of } 30 =</td>
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<td>\frac{4}{5} \text{ of } 35 =</td>
<td>\frac{5}{7} \text{ of } 35 =</td>
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\[
\frac{2}{3} \text{ of } 36 = \quad \frac{1}{2} \text{ of } 38 = \quad \frac{2}{3} \text{ of } 39 = \quad \frac{1}{3} \text{ of } 39 = \quad \frac{4}{5} \text{ of } 40 = \quad \frac{1}{2} \text{ of } 40 =
\]

\[
3 \quad 21 \times 2 = \quad 42 \times 2 = \quad 44 \div 2 = \quad 23 \times 2 = \\
23 \times 3 = \quad 24 \times 2 = \quad 48 \div 2 = \quad 50 \div 2 = \\
25 \times 3 = \quad 52 \div 2 = \quad 30 \times 2 = \quad 30 \times 4 = \\
40 \times 2 = \quad 80 \div 2 = \quad 80 \div 2 = \quad 82 \div 2 = \\
22 \times 4 = \quad 44 \times 2 = \quad 60 \div 3 = \quad 30 \times 3 = \\
25 \times 4 = \quad 100 \div 2 = \quad 100 \div 4 = \quad 100 \div 5 = \\
100 \div 10 =
\]

\[
4 \quad 21 + 2 + 2 + 2 + 2 + 2 = \quad 48 - 2 - 2 - 2 - 2 - 2 = \\
20 + 4 + 4 + 4 + 4 + 4 = \quad 31 + 4 + 4 + 4 + 4 + 4 = \\
53 + 4 + 4 + 4 + 4 = \quad 58 - 4 - 4 - 4 - 4 - 4 = \\
25 + 5 + 5 + 5 + 5 + 5 = \quad 33 + 5 + 5 + 5 + 5 + 5 = \\
53 - 5 - 5 - 5 - 5 - 5 = \quad 29 + 10 + 10 + 10 + 10 + 10 = \\
94 - 10 - 10 - 10 - 10 - 10 = \quad 20 + 3 + 3 + 3 + 3 + 3 = \\
85 - 3 - 3 - 3 - 3 - 3 = \quad 30 + 6 + 6 + 6 + 6 + 6 =
\]

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\[
80 - 6 - 6 - 6 - 6 - 6 = \quad 91 - 6 - 6 - 6 - 6 - 6 = \\
78 - 6 - 6 - 6 - 6 - 6 =
\]

69
In answers to IV, pupils must indicate the total of the numbers to be added to or subtracted from the first number in order to indicate that they know their products; only answers should be required.

At the age of eight or thereabouts, questions like the following extracted from my Book B are easily answered in a short time:

1 Read the following numbers,
731, 601, 756, 987, 814, 513, 247, 303, 888, 111.

Write in figures the following numbers,
Seven hundred and eighty-eight,
Six hundred and thirty-seven,
Nine hundred and eleven,
Five hundred and eight,
Three hundred and twenty-one,
Four hundred and one.

What are the numbers between,
320 and 330, 560 and 570, 690 and 700.

2 Give all the factors of 960 and find what are:
Can you write other fractions of 960 from your knowledge of the factors?

3  Find the answers to:

\[
\begin{array}{cccccc}
45 & 56 & 73 & 69 & 38 \\
\times 6 & \times 4 & \times 6 & \times 3 & \times 7 \\
43 & 27 & 54 & 65 & 47 & 96 & 87 \\
\times 9 & \times 8 & \times 6 & \times 6 & \times 4 & \times 3 & \times 7 \\
34 & 72 & 45 & 56 & 73 & 69 & 38 \\
\times 9 & \times 8 & \times 50 & \times 40 & \times 30 & \times 60 & \times 70 \\
43 & 27 & 54 & 65 & 37 & 96 & 83 \\
\times 90 & \times 80 & \times 50 & \times 60 & \times 40 & \times 30 & \times 70 \\
34 & 72 \\
\times 90 & \times 80
\end{array}
\]

4  Work out the following operations:
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(a) 

\[
\begin{array}{cccccc}
372 & 349 & 254 & 254 & 269 \\
+389 & +237 & +268 & +378 & +359 \\
+215 & +176 & +375 & +169 & +409 \\
\end{array}
\]

(b) 

\[
\begin{array}{cccccc}
529 & 847 & 784 & 635 & 547 \\
-217 & -639 & -369 & -447 & -39 \\
\end{array}
\]

(d) \(4819 + 23 = 7732 + 35\)

Only answers are required.

The test for age 8 contains at the end a test of mastery of procedures and, though all the rest of the text is restricted to numbers up to 1,000, the last section goes beyond, as does Book B.

At the age of nine or thereabouts, questions are mainly on fractions, decimals and percentages. Here are a few examples:

1. Find the first few terms of the families of fractions equivalent to:

\[
\begin{array}{ccc}
\frac{1}{3} = & \frac{2}{5} = & \frac{3}{7} = \\
\end{array}
\]

Give the reciprocals of \(\frac{2}{3}, \frac{5}{7}, \frac{6}{11}, \frac{7}{13}, \frac{21}{5}, \frac{12}{19}, \frac{4}{15}\).
2 \textbf{Find the answers to:}

\[
\begin{align*}
\frac{12}{15} + \frac{7}{60} &= \quad \frac{14}{55} + \frac{1}{11} &= \quad \frac{31}{57} + \frac{2}{19} = \\
\frac{9}{11} + \frac{23}{33} &= \quad \frac{1}{15} + \frac{2}{5} = \quad \frac{7}{45} + \frac{13}{15} = \\
\frac{7}{12} - \frac{5}{36} &= \quad \frac{3}{8} - \frac{3}{16} = \quad \frac{14}{55} - \frac{1}{11} = \\
\frac{1}{15} + \frac{7}{30} - \frac{1}{5} &= \quad \frac{3}{5} - \frac{3}{10} + \frac{7}{20} = \\
\frac{1}{3} + \frac{2}{3} &= \quad \frac{3}{7} + \frac{2}{11} = \quad \frac{4}{9} + \frac{3}{10} = \quad \frac{5}{12} + \frac{7}{8} = \\
\frac{2}{9} - \frac{1}{13} &= \quad \frac{3}{7} - \frac{1}{11} = \quad \frac{7}{8} - \frac{5}{9} = \quad \frac{5}{6} - \frac{4}{7} = \\
\frac{5}{24} + \frac{7}{36} &= \quad \frac{1}{11} + \frac{3}{22} + \frac{5}{33} = \quad \frac{8}{35} + \frac{5}{70} = \\
\end{align*}
\]

3 \textbf{Can you say whether the following fractions are equivalent?}

\[
\begin{align*}
\frac{3}{17} \text{ and } \frac{15}{85} &= \quad \frac{35}{55} \text{ and } \frac{40}{77} &= \quad \frac{57}{51} \text{ and } \frac{38}{54} \\
\frac{33}{88} \text{ and } \frac{39}{104} &= \quad \frac{45}{63} \text{ and } \frac{53}{99} &= \quad \frac{117}{104} \text{ and } \frac{81}{122} \\
\frac{44}{133} \text{ and } \frac{48}{51} &= \quad \frac{21}{75} \text{ and } \frac{27}{100} &= \quad \frac{34}{26} \text{ and } \frac{51}{39} \\
\end{align*}
\]

\textbf{Compare, i.e., tell, which is the bigger or smaller of:}

\[
\frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{15}{13}, \frac{7}{8}, \frac{9}{11}, \frac{11}{12}, \frac{7}{11}, \frac{8}{19}
\]

\textbf{Find what are: } \frac{1}{9} \text{ of } \frac{3}{5}, \frac{2}{3} \text{ of } \frac{1}{3}, \frac{4}{5} \text{ of } \frac{5}{7}, \frac{3}{4} \text{ of } \frac{4}{5} \text{ of } \frac{7}{11}, \frac{2}{3} \times \frac{4}{5} \times \frac{5}{7} \times \frac{2}{5}
\]

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4 Write down the decimal notations for:

\[
\frac{21}{10} = 2.1 \quad \frac{68}{10} = 6.8 \quad \frac{121}{100} = 1.21 \quad \frac{8}{100} = 0.08 \quad \frac{89}{100} = 0.89 \quad \frac{606}{100} = 6.06
\]

Find the fractions equivalent to:

0.3 2.4 4.12 13.1 17.24

and the answers to:

\[
\begin{align*}
0.6 \times 0.4 &= 0.24 \\
1.8 + 1.5 &= 3.3 \\
6.621 \times 4 &= 26.484 \\
2.101 - 5 &= -2.899 \\
3.65 \div 1.50 &= 2.433
\end{align*}
\]

Which fractions give a % form that is smaller than 1%? Bigger than 1%? Bigger than 10%? Smaller than 100%?

Which is bigger, $\frac{1}{2}$ or $\frac{1}{3}$%? 5% or 5?

Only answers should be required.
Quite different questions could have been asked and problems on the operations studied up to this age (like those in the next section) could be included in the tests.

At the age of ten and eleven, the experience of mathematics children have acquired is such that it permits us to ask questions on the following topics, and tests can readily be devised.

1 Operations on large numbers.

2 Comparison of numbers to know whether they are prime, squares, cubes or multiples of 5 or 11.

3 Finding the Highest Common Factor (H.C.F.) or Lowest Common Multiple (L.C.M.) of two or more whole numbers.

4 Finding the nearest square root of a number to a unit, a 10th or a 100th.

5 Squaring sums or differences for quick calculations.
   Ex. $29^2 = (30 - 1)^2$ or $54^2 = (50 + 4)^2$

   Differences of squares for quick calculations.
   Ex. Find $989 \times 1011$ [(1000 - 11) \times (1000 + 11) not given]
   $75 \times 125$ [(100 - 25) \times (100 + 25) not given]

In my Books 7, 8, 9 and 10 many examples may be found of the type of question pupils of ten or eleven can now attempt successfully. I have not given any timing for the performance of
pupils after the age of seven because the tests have been derived from examples which only appeared in 1958. In the case of the first two sets of tests the experience of Cuisenaire, who used them over several years, is available and the later ones have, until now, been used only in classes taken by myself or my students, and no standard times are available. Still, we know that it is very much shorter than can be expected from experience with traditionally taught classes. If pupils take at eight and nine or ten, say 20 to 30 minutes to complete each section of the tests proposed for those ages, it will suggest that much progress in the teaching of elementary mathematics has already been achieved.

In a later edition of this work we shall include the times taken by average pupils in these tests, which have been included here to give some idea of what a few teachers (including myself) have been able to teach pupils of these ages, thanks to the use of the Cuisenaire rods.

The concluding pages of this chapter are devoted to figures illustrating some of the many possibilities that are to be found in the material. A study of these will suggest many more. The discoveries open to both teacher and pupil are, indeed, endless.

The textbooks show how at a surprisingly early age children can master in a fundamental way Highest Common Factor and Lowest Common Multiple, permutations and combinations, the binary system, and a wide range of algebraic and geometrical knowledge.

1 Towers, Powers, and an introduction to Logarithms.
IV Standards of Achievement

\[
\frac{3^4}{2187} \div \frac{3^3}{81} = \frac{3^3}{27}
\]

\[
\frac{4^4}{64} \times \frac{4^4}{256} = \frac{4^3}{16384}
\]
$10^{10}$ or 10,000,000,000
IV Standards of Achievement

Areas, Volume, Capacity.
What is the area of the surface of each of these figures? And what is the volume of each?
To find $x^2 - y^2$, the purple square is placed on the black square and the difference in their areas is then seen. By rearranging the black rods a black rectangle is formed the area of which can be recorded in terms of $x$ and $y$. 

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Arithmetic is, for most teachers, an applied subject, that is to say, its inclusion in the curriculum results from its usefulness rather than from its intrinsic intellectual interest. The children learn it, not to have fun with figures, but to equip themselves for a world in which number and measurement enter into everyone's life all the time. We are perfectly clear that arithmetic and daily life are closely intermingled, but we are not so clear about the role this subject plays in the unfolding mental life of the pupil.

In this book we began by considering how the ability to calculate swiftly and correctly can be evoked and developed. There is nothing novel about that. The young child soon has occasion to make use of the ability to count, reckon, measure, and deal with spatial problems, and he can only make use of what he knows to deal satisfactorily with situations requiring those skills. Though he may place great value on an orange, and none upon a dollar, he will change his mind when the significance of the dollar is realized. So, too, he will come to value arithmetical skills as their usefulness is revealed to him by experience. But is this to be the only source of his interest in mathematics?
Two aspects of arithmetic have their claims upon the developing
mind. It is, or can be, a source of intellectual delight; and it is
assuredly an essential tool for living. The latter remains,
however, the aspect that most teachers regard as the best means
of exciting the child's interest and, accordingly, attempts are
made from earliest stages to relate arithmetical processes to
actual situations. Sometimes this is done intelligently, but
sometimes situations are pressed into use that mean nothing to
the pupil. The pupil is expected to set sail upon the uncharted
waters of discount, banking, or stocks and shares before these
mean anything to him, and he finds that the dreariness of a dull
grown-up world is added to the native dreariness of arithmetic.

The fact is that this whole approach is misconceived. If the child
is to enjoy his arithmetic he must have the tools for the solution
of life's challenges and problems before they arise, and he must
know how to use those tools upon whatever comes into the
sphere of his expanding experience. It is certainly not a mistake
to regard arithmetic as an applied subject, but it remains an
important truth that is must be acquired for its intrinsic interest
in preparation for its ready application to actual situations.

Confusion over this matter has led to the belief that
mathematical concepts are more easily extracted from life's
situations than understood directly. For example, money is
commonly regarded as a valuable starting point for teaching
number, and scores of textbooks use that introduction. The
underlying principle doubtless is that children very quickly gain
an interest in money, and this interest ought to make it easier
for them to master counting, adding, and subtracting. This is
fallacious and produces confusion. If coins are to be used to
learn counting they can be replaced by any type of counters. If they are to be used in accordance with their token value, the child is brought face to face with an irrelevant complication. The additivity of numbers is concealed if, when a certain number have been collected, they must be exchanged for a coin of another denomination. A pile of ten pennies cannot be directly correlated with a nickel or a quarter. Here actuality proves to be an obstacle to comprehension instead of an aid.

If the various everyday-life situations used in the teaching of arithmetic are examined we find that little of true mathematical value remains for the pupils. The teachers may themselves find such a value in the method, through their adult appreciation of the connection between mathematics and the situation they are introducing, but for the child they are trying to teach, it means that two things must be learned at the same time which do not seem to fit. A poor basis for teaching!

Let us be quite clear about it. Arithmetic is undoubtedly an applied subject and there can be no objection to teaching it because of the importance of its applications; but if this is the motive for teaching it, our first task is to see that the children achieve mastery of the arithmetic tools. When they find that mastery, the applications at once gain interest and importance and the pupil can be shown how to set to work upon the problems that lie around him in the world of experience.

This prior mastery of the tools can now be tackled, no longer as an initial drudgery leading to a useful achievement in the future, but as an intellectually stimulating and satisfying game with number and measurement. The Cuisenaire material provides
the child with the means for this play-activity, and provides all
sorts of interesting games (in the ordinarily accepted sense of
the word) through which the skills he will need can be acquired.
When he faces actuality—money, pints and quarts, distances,
speeds, time and the rest—he will already have the competence
he needs to cope with them. This is the truly fruitful approach—
not the approach in which these elements of experience are
made to serve as the means for learning arithmetic.

With this introduction we can now proceed to explore the ways
in which we can help our pupils who have already mastered the
tools to meet practical situations with confidence and
competence. These examples are taken from Book B.

**Grouping**

We begin, according to our established principle, with the rods,
moving on from the semi-abstract situations they offer to the
idea of grouping as it appears in the social context. Thus:

1. Take a handful of rods. Find out what length they would
   make if you put them end-to-end. Can you do it by just
   looking at them? Try several times with several handfuls.

   (Note: We remain at the game level and the purpose of the
game being played is not yet apparent.)
2. Take a handful of rods and arrange them in groups that add up to ten. Is it easier to find the total this way than by adding the lengths one to another? (See figure below).

Take another handful and group them by color. Can you find the length that would be made by all the rods laid end-to-end?

Now take another handful and find its value (i.e. the length if laid end-to-end) by:

- making groups that add up to ten;
- making groups that add up to seven;
- making groups that add up to twelve.
Calculation by grouping rods mentally in lengths of ten. The markings are introduced into the figure to show how the rods might be grouped in this case and the rods themselves are, of course, not marked. The child can find five tens and an eight and gives the answer as 58. In early stages the rods can be physically sorted into sets of ten, but the value of the game lies in sorting them mentally.

3 When you group by twelves you form dozens, and groups of six are called half-dozens.

How many dozens are there in: 12, 24, 35, 100?

Sticks of chalk, pencils, eggs and some other things are sold by dozens. Twelve dozen is called a gross.
A farm produces 150 eggs a day. How many one-dozen boxes are needed to dispatch these eggs? If there is only a one-gross crate, how many eggs will be left unpacked?

A school has a gross of pieces of white chalk. For twenty days the first class used a half a piece every day, the second class one piece, and the third and fourth two pieces each. How many were left in the box?

Every day a man buys twenty gumdrops. He eats ten and gives away half-a-dozen. How many gumdrops has he got left at the end of the week?

Find the number of dozens and the remainder in 13, 21, 29... 200. Write in figures 5 dozen, 10 dozen... 17 $\frac{1}{2}$ dozen.

4 A score is twenty.

How many score in 100?

Which numbers can be counted both in dozens and scores?

How many dozen in three score?

How many score in ten dozen?

5 One hour is sixty minutes and one minute is sixty seconds.
How many hours and minutes are there in 63, 75, 90 . . . 180 minutes?

How many minutes and seconds in 74, 84, 95 . . . 191 seconds?

I waited half an hour for my bus when I went home, and thirty minutes when I returned. On which journey did I wait the longer?

Every quarter of an hour there is a bus in each direction outside my home. How many buses will pass my door in each direction in 75 minutes? In 90? In 108? In two hours? And how many buses altogether will pass my door in those times?

Buses on three different routes stop at my bus stop. One is a three-minute service, one a six-minute service, and one a ten-minute service. How many buses stop there every hour?

Do I need more scores, dozens, or tens to form 120?

I have over 100 nuts and several bags. Do I use more bags when I put a few nuts in each or when I put a lot in each?

(Observe, these exercises show the close relationship between arithmetical and social experience. In each case the arithmetic is mastered before it is used as the key to increasingly complicated social situations. But these must be carefully scrutinized to ensure that they fall within the
V Problems From Everyday Life

general experience and interest of the pupil so that they do not introduce irrelevant difficulties.

Commercial questions for six seven and eight year-olds

The following questions are for children from six to eight:

1 The man who sells me candy buys it at one price and sells at another. Can you see why? Which is the smaller of the two prices?

(Note: The teacher begins by introducing to the class the broad principles of trading in terms intelligible and interesting to the children. This is not arithmetic; it is social experience upon which our arithmetic will be used as a tool.)

The cost price is what we pay to obtain something. Give cost price of the candy, ice cream and other things you buy.

2 Everything we buy is sold by someone. (Give examples. What about the slot machine?) For him the price we pay is called the selling price. What is the selling price of the candy, toys and other things you see in the shop-windows? Do you think the shopkeeper paid that price himself to buy these things? Why do you think there should be a difference?

3 We call the difference between the two prices the profit. Who makes the profit?
Nearly everyone who sells something he has bought wants to make a profit. Tell me the kind of people who sell things and make a profit. Tell me the kind of people who sell things and do not make a profit. Who are the people who take money but do not give any object in return?

(Note, again, all this is important social experience and, before arithmetic is applied to it, teachers must clarify the social context fully. Store owners, their employees, bus-conductors, movie-ticket sellers are all met by the children. So, too, are those who sell their services, doctors, amusement park proprietors, and their employees, and radio and TV repairmen. Time spent on the ramifications of money relationships in trade, provision of services, profit-making, charity, social services paid for by the community, and so on, is not wasted time. Only when the context of the applied arithmetic is clear and immediate for the child, will the operations he is called upon to perform provide the stimulus of interest.)

4 If I pay $3 for an object and sell it for $4, how much profit do I make?

What fraction of the cost price is the profit and what fraction of the selling price? Which is the smaller?

(This represents the mathematical awareness that one quantity may have different appearances according to the standard of comparison, and this has social significance, for
it shows how profit can be made to look smaller by referring it to one price rather than to the other.

“If I pay $3 for an object and sell it for $5, what profit do I make?”

“What fraction of the cost price is the profit, and what fraction of the selling price? Which is the smaller?”

5 Taking the white rod to represent one dollar (or other appropriate unit of currency) find the price of an article if you know that you pay:

$5. for it in the country abroad;
$1. for its transport to you,
$2. for customs duty in your country;  
$1. for transport in your country.

What should your selling price be if you want to make a profit of one third on the cost price to you?

(This is social and arithmetical experience blended to meet problems commonly met in trade. It also gives the pupil a chance to put himself in the shoes of the trader.)

**Time**

This last set of examples will show once more how we should proceed first to illuminate the context for our social arithmetic, and then to blend our experience and our arithmetic so that nothing is left to chance. If we are to deal with time, we must be sure that the pupils are completely familiar with what the hands of the clock show. In Book A, a method is given by which, in a couple of lessons, a child of six can become confident in the telling of the time. When this is completely mastered, and only then, we can go forward. Here are some simple calculations:

1. *Within the hour*

   If you start at 9:10 and move the hands so that the clock shows 9:40 how many minutes have passed according to the clock?
Do it once more from 7:15 to 7:50. How many minutes have passed?

From 6:05 to 6:55?

From 10:10 to 10:50?—and so on.

2 Beyond the hour

How many minutes will have gone by if the hands move from 9:05 to 10:50? From 9:15 to 10:20? From 8:10 to 9:35? From 6:00 to 8:10?

If it is now 9:15 how long have we got before the bell rings at 11:30? Or if it is now 9:40 before the bell rings at 12:20?

3 How many minutes are there in one hour? In two? In three? How many minutes make up an hour and a half? Two and a quarter hours? Three and three quarters?

How many hours in 120, 180, 240, 150 minutes?

Change the following into hours and minutes: 130 mins., 75 mins., 360 mins.

Which is longer, 2 hours 15 mins. or 125 mins.?

1 hour 17 mins. or 78 mins.?

173 mins. or 2 ½ hours?

4 hours or 250 mins.?
4 Trains start, travel and arrive, so there are three simple questions we can ask about them.

A train leaves at 8:12 a.m. and reaches its destination at 9:45 a.m. Another train leaves at 10:20 a.m. and reaches its destination at 12:12 p.m. How long does each train take for its journey? or,

Trains set off from a station every day at 10:15 a.m. and travel for one hour 15 minutes. When do they arrive at their destination? or,

Trains arrive at a station every day at 9:00 a.m. after travelling for two hours. When do they start their journey?

In Book C other temporal situations are treated as the basis of arithmetical questions, and the problems that arise out of the calendar, which is so important a factor in the lives of us all, are also explored. In Book 9 the subject is taken to new levels by the study of the perpetual calendar, which introduces factors of greater mathematical complexity.

Life is full of complex problems of which only some are mathematical in character. In Book 9 a stage is reached at which pupils are required to find for themselves the social material out of which the arithmetical problems arise. For example, if we wish to study the cost of public works, we must know a great deal about labor, transport, insurance, contracts, fluctuations in prices, and so forth, and, with the arithmetical apparatus at his command, the pupil is ready to apply what he knows to matters
as complex as his experience enables him to explore. I felt that it was my responsibility to open up these wider vistas of applied mathematics within the social context and have done so in these later textbooks.

Mathematical techniques may be acquired for the intellectual pleasure they give, but they open the door to broader fields of interest where new reasons for the enjoyment of the subject are to be found. Without this social application, it would remain a luxury—though a luxury now available to the many—whereas previously it was only for the few. When we go forward to its application, the many are still further enriched in their experience and, in turn, can give of their skills to a world in which such skills are needed more than ever before. But let us remember that social utility is ultimately derived from the deep well of intellectual delight. The tendency to reverse the order, which results from our well-meant current confusion in this matter, robs the world of the social skills it needs and, at the same time, destroys that delight in the young mind.
Conclusion

In this small book I have tried to meet the needs of the elementary schoolteacher who wishes to examine and test for himself or herself the so-called Cuisenaire-Gattegno approach to the teaching of arithmetic.

My previous books for teachers did not altogether meet that need, for they were found by many to be too difficult. Now that a bridge has been thrown across the gap by this Introduction, I very much hope it will lead to an understanding of the others. In these, I have entered into a kind of dialogue with the teaching profession.

In my book, *Modern Mathematics*, I have offered a methodical and detailed study of the mathematics which can be found with the Cuisenaire material, and teachers who might have found the former work difficult without having reached it by way of this bridge will be able to explore the new territory, not with the immediate aim of teaching, but for deepening their own understanding and appreciation of the principles involved. The
new way of teaching will grow naturally out of a new way of understanding.

*Numbers in Color,* which incorporates the translation of Cuisenaire’s original and historic work, engendered an interest in the Cuisenaire material itself, and inspired in many teachers an enthusiasm, and even an affection, for it. This enthusiasm has proved contagious and has spread from teacher to teacher and from school to school.

In the present work is to be found, in simple and direct form, the fruits of nearly seven years of continuous experimentation, reflection, and discussion. The response of the children, the insights and the questions of many thousands of colleagues in the profession, and the long process of clarification that has resulted from this, have alone made it possible for me to write it. It has been more difficult to write than my ‘difficult’ books.

The use of the series of textbooks I have prepared has enabled me to include practical solutions to teaching problems actually encountered. They serve my purpose better than any commentary upon them. My masters have been the children I have met in so many lessons under the watchful eye of so many colleagues. I have been reluctant to put myself between the source of the insights I have gained (the children) and the learners (the teachers), and I have allowed myself to do so in a manner I never intended; but the demand for something of this sort has been so insistent that I have overcome my scruples and my doubts. If I can be sure that the effect will be to send my readers back to learn from the children they teach, I shall feel that this book was justified after all.
Conclusion

The three manuals and the pupils' textbooks to which I have referred, together with the film-strip with teachers' notes, represent all I think Cuisenaire and I can offer to the teaching profession, and this *Teachers' Introduction* is itself only a variation on the theme. It may, however, provide a point of entrance for many and facilitate the discovery of these new and exciting vistas of mathematical insight.
Bibliography

Mathematics with Numbers in Color

(Pupil’s Books)

Book A—Numbers to 100
Book B—Numbers to 1,000
Book C—Money and Measure
Book D—Fractions
Book 7—Study of Large Numbers
Book 8—Length, Area and Volume
Book 9—Numeration Bases, Proportions and Problems
Book 10—Algebra

(This is the series of textbooks referred to in this present work).
Modern Mathematics with Numbers in Color

This book offers a course of self-instruction for the teacher in the algebra, arithmetic and geometry of the Cuisenaire rods.

Numbers in Color


Film Strip (3rd Ed.) with new notes.

All the books mentioned are the work of Dr. C. Gattegno with the exception of Numbers in Color which is the joint work of G. Cuisenaire and C. Gattegno. They are obtainable from the Cuisenaire Company of America, Inc., 235 East 50 Street, New York 22, New York.
Appendix I

Charts and Cards

The product cards and wall charts were originally used by Cuisenaire for the purpose of teaching multiplication tables and making them completely articulated and automatic. This in itself would make them of considerable value, but we give here a detailed description of other purposes which they can serve.

Those interested in understanding why the Cuisenaire material proves so successful will find an important clue in work done with the charts. The value of the colors emerges most clearly if the charts are used without any previous acquaintance with the rods, as has sometimes been the case in experimental lessons given by the author. It then becomes obvious how arbitrary is the connection between number and color, since familiarity with the length of the rods is no longer there to support the connection of red with 2, black with 7, and so on.
A lesson with any group can take the form of the following experimental game. Exhibiting the chart containing the 37 colored designs, which sometimes differ in color only and sometimes also by the fact that there are four instead of two rectangles round the white center, we can first get recognition of the fact that only 9 colors are used and that each of these may appear in several designs. We then get the pupils to associate 2 with red, 4 with purple, 8 with brown, and having stated that we shall read the designs as being 2 by 2, 2 by 4, 2 by 8, we enquire what the reading of other designs will give. The children will find no difficulty in producing 4 by 4, 4 by 8 or 8 by 8, and we are not concerned as yet with obtaining the result of the operation “by”. We first establish the conditioned reflex of “reading” in the case of each design, and this presents no difficulty once the colors are known and the connection between them and 2, 3, ... 8, 9 is working in both directions. It may even happen, and this will depend on the age of the group, that a number of the color-number connections will lead to answers such as: 2 by 5 is 10, or 5 by 2 is 10, but this is a stage which we shall not pursue since for the moment our purpose is to see what the children can do with the chart as such rather than how it can serve the teacher as an aid.

Now, concentrating on the designs containing only two colors, and covering one of the colors, we ask the pupils which one it is. They may remember by its position on the chart, but they may also find out by its connection with the next (or the previous) design since the chart is so arranged that there is an affinity between neighboring designs. For instance, if in the design next to the one that is partially
covered there is dark green in the corresponding position, the one covered will be light green. There is a third possibility of finding the answer; since each color is linked with all the others, by elimination of those that are visible the one that is covered can be found. The fact that there are three possible ways of finding the right answer means that all the pupils can very quickly produce it with ease.

We can now introduce new words to specify what we are doing. When for instance we cover a black rectangle, instead of saying "seven" we shall say "a seventh of this one" and then name the number of the remaining color. If the design in question is, for example, purple and black and we cover the black, a seventh will be the purple, 4.

This game provides practice (completely gratuitous, of course) in using the nine words half, third ... eighth, ninth, and in the acrobatics of reacting in terms of the color that is visible although the word uttered has an affinity with the one that is covered. In fact, this exercise frees us from the restriction of the conditioned reflex established earlier and makes it available for new uses.

This experiment only shows, of course, how easily these reflexes are established and how easily children can perform elaborate exercises when the situation is unambiguous and at their level. The swiftness of the learning process goes hand in hand with easy retention and ability to use sensori-active knowledge for the mental exercise that follows.
Let us assume now that we combine all the reflexes that have been acquired. If we look at the chart and say: 2 by 4 is 8, we have merged the colors into one entity and given it a name, we have a concept of 8 corresponding to the sound, and this sound is linked with red and purple rather than with brown. Pointing to one of these colors in the design we polarize the attention, and when we ask what is half or a quarter of 8 it is the other color which provides the answer, which is however uttered at once as 4, or 2.

There are of course many conditioned reflexes present in this type of activity and some teachers may object to it on this ground, but when I reflect on my mathematical activity I find it full of such mechanisms, which are an aid rather than a hindrance in any creative work. What is certain is that the reflexes have no restrictive effect since immediately they are established they are disarticulated, to become tools for awareness of the contribution they can make to further discoveries.

In addition, we have shown that what makes difficult mathematical processes simple and immediate is the fact that color is being used in mental patterns all of which are at the level of perception and do not yet require the constitution of the intellectual synthesis of perception and action, a synthesis in which these two are replaced by the idea. If we start with the idea, we fail more than we succeed, but if instead we begin by mastering the perceptive components, there is every hope that the child will be able to combine them with action and progress ever further.
Now that it has been understood why the charts make for quick retention, we shall develop lessons with them with a view to simplifying several difficult topics, all of them of course within the field of multiplication, factors and fractions.

Once the pupils know some of the products through their manipulation of the material and can recognize them in the decompositions by their colors, we present them with the chart or the cards and ask them to find the design containing the colors of the factors. This is quite easily done. They quickly come to know the designs symbolizing the products 4, 6, 8, 10, 12, 15 and their familiarity with the length of the rods enables them to establish an order between them, 4 less than 6 less than 8, etc., thinking of the rods but also of the factors. It is of the latter that the charts give experience. Seeing 12 as $2 \times 6$ and $3 \times 4$ at the same time in one design rather than as four rows equal to 12 will create a different attitude to numbers. Four experiences are synthesized in one perceptive grasp of the design: the experience that 2, 3, 4, and 6 are present in some way in 12, that these only are present, that 2 is paired with 6, 3 with 4, and conversely that 6 is paired with 2 and 4 with 3.

As however sight is a very mobile sense and its activity very near the intellectual, it is easy to transform the awareness of these relationships by asking the pupils to consider the product, which is evoked only by the colors of the selected factors, as given, and to relate each of the factors to it. As we said in § 1, this is done by establishing the reflex involving
the use of the words half, third, quarter . . . and indicating the appropriate factor.

This means that in addition to practice in multiplication the charts can be used for exercises with fractions. In this way, fractions will become mental relationships instead of representing the bit of the whole which cannot be used in further operations. This understanding of what a fraction is represents so great an advance that teachers will be amazed at the performance of their pupils.

Psychologically speaking, this is what occurs. Confronted with one of the colored designs, the mind is capable of noticing one of the colors and pushing the others into the background, or of thinking of one while looking at the other, and even of looking at one while imagining the other changed into a different one.

To verify that this is so, the reader has only to look at the chart and see that in any pair of designs containing, say, only two rectangles and in which one of the colors is common, the two designs can be mentally interchanged although their position on the chart is static. This simple exercise is important, for the relationship thus made evident can be used to link the various products. For example, in the case of 42 or $6 \times 7$ and 63 or $9 \times 7$ the dark green and the blue will be interchangeable. Hence if we wish to pass from one to the other we know how to proceed, and the expression of the process will be: 42 is $\frac{6}{9}$ or $\frac{2}{3}$ of 63 and 63 is $\frac{3}{2}$ of 42.

There are many products on the chart that are linked and we can find a multiplicity of fractions linking them, thus gaining
an experience with fractions involving fairly large numbers. For example, when there are no common factors and the fractions are irreducible, we get \( \frac{49}{81}, \frac{64}{81}, \frac{81}{36}, \) etc.

4 The other feature of the charts which is interesting is the doubling of numbers. Each row of designs, with one exception, is composed of a product and its double or the double of its double, etc. While this considerably reduces the effort needed for memorizing the products, it also means that we can halve the various doubles and obtain smaller products. Since we can see that half of half is a quarter, half of half of half or half of a quarter is an eighth, etc., we have the opportunity for further exercises on special fractions and on their products.

The chart does not contain all the products of the numbers concerned nor all the results of all possible operations with these numbers. For instance: half 28 is 14, half 14 is 7, and twice 32 is 64 and twice 64 is 128. These two facts can be known to the children but 7 and 128 do not appear in the table. We have here another opportunity for weaning the children from the teaching aid.

5 In conclusion, we would suggest that what we have said shows clearly that while the chart and cards can be used for the automatic mastery of the products of multiplication as advised by Cuisenaire, they also offer further varied opportunities. We have seen that it is primarily the colors in the chart which account for the swiftness of the calculations performed by pupils using this method. Moreover, through the systematic formation of certain conditioned reflexes the
mental structures of arithmetic are produced and maintained. Teachers will find that by taking advantage of them they can lead their pupils very far and establish permanently the mental connections that express mathematical experience.

This learning of arithmetic aims at providing the ability to work in this field rather than at the stereotyped reproduction of static wholes. It frees the mind by showing how things work, instead of making it dependent upon an accumulation of facts, and by so doing it is a valuable educational agent.
Appendix II

1 Take your colored rods and find the ones that are the same color. Mix them again. Now find the ones that are the same length.

2 Take two rods and put them end to end, like this.
   Take two others and do it again.
   Can you put three rods end to end? Do it
   Now change the rods and do it again.

3 Tell me which ones you’ve put end to end.

4 Can you put instead of this rod one that is equal to it?

5 Take a brown rod.
   Find two rods that are equal to the brown rod if you put them end to end.
   Can you do the same with a blue rod?
   With an orange rod?
   With a green rod?
You take a black rod, you a blue one, you an orange one and you a brown one.
Try to find all the rods that are equal to the one you have if you put them end to end.
See how I do it with the yellow rod:

The yellow
The purple and the white
The red and the light green
The white and the purple
The light green and the red
The white and two red ones
The red, the white and the red
The light green and two white ones
The white, the light green and the white
Two red and the white
Three white and a red
Five white ones.

Now see if you can do that with your black rod, with your blue rod, with your brown rod, with your orange rod.

Now let us all take the light green rod and try together. What have you found?

Let us try with the purple rod.
"Read" what you've got, but don't tell us.
Let's shut our eyes and listen to what—says he has got on each row.
Can it be right?
Let's all try and see.
Read what you've got on each line.
Can it be right?
Is it right?

9  Let us all take a yellow rod and do the same thing again.
Read what you've done but don't tell us.
We shall shut our eyes and listen to what—says. He will read two rows and we shall say if he can be right.
Is it right?

10 Let us take a dark-green rod and try to find which rods are equal to it if we put them end to end.
Read what you've got.
Let's shut our eyes and see whether we think he's right.

11 Let us all take a black rod and find the rods that, end to end, are equal to it.
Read what you've got.
Let's shut our eyes and see whether he's right.

12 Let us all take a brown rod.
(As before.)

13 Let us all take a blue rod.
(As before.)

14 Let us all take an orange rod.
(As before.)
15 If you put a red rod and an orange rod end to end, are they longer or shorter than the blue rod? Than the brown? Than the black? Than the orange? Than the orange and a white? Than the orange and a light-green?

16 If you take this rod, can you find a rod that is equal to it? One that is shorter? One that is longer? Two which, end to end, are equal to it? Or shorter? Or longer?

17 If you take this rod, and then that one, can you find a rod with which we can make this one the same length as the other? Take two others and try to find the missing rod.

18 Put two rods end to end. Now take another one that is shorter and find what you need to make it the same length as the others. Try again choosing other rods.

19 If we now mix four rods chosen as above, which must we put with which to make two equal lengths? Mix them again and put any two end to end. What happens then?
Appendix II

Try again exchanging your four rods for those of a neighbor who has rods.

20 Now let us take one rod of each color and make a staircase. Which rod is the longest? Which is the shortest? Which comes before the longest? Which comes after the shortest? Move up the staircase saying the colors of the rods you meet. Move down, saying the colors of the rods you meet.

21 Can you do the same thing with your eyes shut, starting with the white one? Starting with the orange one?

22 Let us look at our staircase. Which rods do we need if we want to make every step level with the orange rod? Find them and put them in their places.

23 Now take them away and leave the staircase as it was before. Can you tell which rod is needed for the blue to make it level with the orange? For the yellow? For the light-green?

24 Now take another orange rod and put a black rod against it. Which rod do you need to put with the black to make the length of the orange?
Which do you need to put with the dark green?
With the yellow?
With the blue?
With the red?
With the brown?
With the light-green?
With the purple?

25 Let us put a black rod and a light-green rod end to end. Which rod do we need to make the same length? Which do we need for the purple and dark-green? For the blue and white? For the red and brown? For the dark-green and purple? For the white and blue? For the brown and red?

26 Do the same thing again with the red and the black, with the white and the brown, with the light-green and the dark-green, with the yellow and the purple.

27 With the brown and the yellow end to end, find which other rods end to give the same length.

28 Take any pair of rods, put them end to end, and find as many pairs of rods as possible, which end to end, give the same length. Now put more than two rods end to end and then make equal lengths with other rods.
29 Take any rod, then another shorter one. Find which rod must be put with the shorter one to make the length of the longer one.

1 by using rods.
2 by looking only, and showing the rod you need.
3 by hearing only the names of the rods and saying which rod is needed.

30 Do this with rods of every color.
Now do it starting with two rods end to end.

31 Try to form the length of any one rod by using only red rods.
Can you always do it?
And with light-green rods only. Can you always do it?
And with purple rods only.
With yellow rods only.

32 Which rods can be covered by using only red rods?
By using only light-green ones?
By using only purple ones?
By using only yellow ones?

33 Which rod do you need to make up the length of the rod you started with when you can't do it with the red ones?
When you can't do it with the green ones?

34 Begin by making a train of red rods end to end, on a straight line. Take a few rods of each of the other colors and put
them, in turn, against the train, at one end. Which rods end at the end of a red rod in the train?

35 Make a train of light-green rods, and as before put in turn against the train at one end, a few rods of every other color. Which rods end at the end of a green rod in the train? Which do not?

36 Do the same with a train of purple rods. with a train of yellow rods.

37 Make trains with rods of the same color: all red, all light-green, all purple, and put them one against the other with one end level. Can you make trains of the same length if they are red and purple? If they are red and light-green? If they are light-green and purple? If they are red, light-green and purple?

38 Make trains using only dark-green rods and only black rods. Can they be the same length?

39 Make trains that are all brown and all orange. Can they be the same length?

40 Take any rod. Now take another of the same color. Are they the same length? Put them one against the other and move one a little to the right so that there is a part of each that is not covered. What happens to the lengths of the rods?
Push the rod you moved a little more to the right. Now can you find rods that will fill the spaces at the ends? What do you find when you start with red ones? With light-green ones? With black ones, etc.?

41 Can you do the same thing moving the rod to the left? Do it. Now move the rod you didn't move before, and fill in the spaces at the ends.

42 Make a train with any two rods. Now find two rods that give you the same length if you put them end to end. Push one train to the right or to the left, keeping the other where it is. What do you see? Push the train farther along, still without moving the other one. Can you find the rods that will fill the spaces?

43 Do the same thing with trains made of three rods end to end.

44 Do it with trains of more than three rods.

45 Let the brown rod be a train and put another train below it made of the red and two light-green rods. Are the trains the same length? Put the red between the two green ones. Take away the red one and push one green one up to the other. Which rod do
you need to make the second train the same length as the brown?

46 Let the orange rod be one train and make another with the yellow, the light-green and the red.
Is it the same length as the orange?
If you take away the green rod and push one of the rods that are left up to the other, either to the right or the left, which rod will you need to make the second train the same length as the orange?

47 Make these patterns:

1. take a blue rod; take two white rods and put them below the blue rod, one at each end.
   Which rod is needed to fill the space?
   Which rod is needed to fill the space if you use red rods instead of the white?
   If you use light-green?
   If you use purple?

2. take a blue rod, in the same place as before put any two rods which together are shorter than the blue.
   Which rod is needed to fill the space when you have used a white and a red?
   A white and a black, etc.?

48 Do this again with the black rod.
   with the brown rod.
   with the orange rod.
49  Can you say, without doing it, which rods will be needed to fill the space in these patterns?

1  a dark green rod with two white ones below it at the ends? with one white and one red?

2  a black rod with a red and a light green below it at the ends? with two red ones?
   with two light-green ones?
   • when you have the rods in front of you?
   • when you only hear the name of the rods?

50  Take any one of the three longest rods. Only listening to the names and using your eyes, find which rods would fill the spaces if we put a red rod at each end, a white at one end and a red at the other, a white at one end and a yellow at the other, a light-green at each end, etc.