Problem 1

Consider a fluid (of density $\rho$) in incompressible, laminar flow in a plane narrow slit of length $L$ and width $W$ formed by two flat parallel walls that are a distance $2B$ apart. End effects may be neglected because $B \ll W \ll L$. The fluid flows under the influence of both a pressure difference $\Delta p$ and gravity.

a) Using a differential shell momentum balance, determine expressions for the steady-state shear stress distribution and the velocity profile for a Newtonian fluid (of viscosity $\mu$).

b) Obtain expressions for the maximum velocity, average velocity and the mass flow rate for slit flow.

Solution

1.a Differential Shell Momentum Balance in Rectangular Cartesian Coordinates.

A shell momentum balance is used below to derive a general differential equation that can be then employed to solve several fluid flow problems in rectangular Cartesian coordinates. For this purpose, consider an incompressible fluid in laminar flow under the effects of both pressure and gravity in a system of length $L$ and width $W$, which is at an angle $\beta$ to the vertical. End effects are neglected assuming the dimension of the system in the $x$-direction is relatively very small compared to those in the $y$-direction ($W$) and the $z$-direction ($L$).

Figure 1.2: Differential rectangular slab (shell) of fluid of thickness $\Delta x$ used in $z$-momentum balance for flow in rectangular Cartesian coordinates. The $y$-axis is pointing outward from the plane of the computer screen.
Since the fluid flow is in the \( z \)-direction, \( u_x = 0 \), \( u_y = 0 \), and only \( u_z \) exists. For small flow rates, the viscous forces prevent continual acceleration of the fluid. So, \( u_z \) is independent of \( z \) and it is meaningful to postulate that the velocity \( u_z = u_z(x) \) and the pressure \( p = p(z) \). The only nonvanishing components of the stress tensor are \( \tau_{xz} = \tau_{zx} \), which depend only on \( x \).

Consider now a thin rectangular slab (shell) perpendicular to the \( x \)-direction extending a distance \( W \) in the \( y \)-direction and a distance \( L \) in the \( z \)-direction. A 'rate of \( z \)-momentum' balance over this thin shell of thickness \( \Delta x \) in the fluid is of the form:

\[
\text{Rate of } z \text{-momentum} = \text{In} \ + \ \text{Out} \ + \ \text{Generation} \ - \ \text{Accumulation}
\]

At steady-state, the accumulation term is zero. Momentum can go 'in' and 'out' of the shell by both the convective and molecular mechanisms. Since \( u_z(x) \) is the same at both ends of the system, the convective terms cancel out because \( (\rho u_z^2 W \Delta x)|_{z=0} = (\rho u_z^2 W \Delta x)|_{z=L} \). Only the molecular term \( (L \Delta W \tau_{xz}) \) remains to be considered, whose 'in' and 'out' directions are taken in the positive direction of the \( x \)-axis. Generation of \( z \)-momentum occurs by the pressure force acting on the surface \( (pW \Delta x) \) and the gravity force acting on the volume \( [(\rho g \cos \beta)LW \Delta x] \).

The different contributions may be listed as follows:

- rate of \( z \)-momentum in by viscous transfer across surface at \( x \) is \( LW \tau_{xz}(x) \)
- rate of \( z \)-momentum out by viscous transfer across surface at \( x + \Delta x \) is \( LW \tau_{xz}(x + \Delta x) \)
- rate of \( z \)-momentum in by overall bulk fluid motion across surface at \( z = 0 \) is \( \rho W \Delta x u_z^2(z = 0) \)
- rate of \( z \)-momentum out by overall bulk fluid motion across surface at \( z = L \) is \( \rho W \Delta x u_z^2(z = L) \)
- pressure force acting on surface at \( z = 0 \) is \( p_0 W \Delta x \)
- pressure force acting on surface at \( z = L \) is \( p_L W \Delta x \)
- gravity force acting in \( z \)-direction on volume of rectangular slab is \( (\rho g \cos \beta)LW \Delta x \)

On substituting these contributions into the \( z \)-momentum balance, we get

\[
LW \tau_{xz}(x) - LW \tau_{xz}(x + \Delta x) + (p_0 - p_L)W \Delta x + (\rho g \cos \beta)LW \Delta x = 0 \quad (1.a.1)
\]

Dividing equation (1.a.1) by \( LW \Delta x \) yields

\[
\frac{\tau_{xz}(x + \Delta x) - \tau_{xz}(x)}{\Delta x} = \frac{p_0 - p_L + \rho g L \cos \beta}{L} \quad (1.a.2)
\]

On taking the limit as \( \Delta x \to 0 \), the left-hand side of the above equation is exactly the definition of the derivative. The right-hand side may be written in a compact and convenient way by introducing the modified pressure \( P \), which is the sum of the pressure and gravitational terms. The general definition of the modified pressure is \( P = p + \rho g h \), where \( h \) is the distance upward (in the direction opposed to gravity) from a reference plane of choice. The advantages of using the modified pressure \( P \) are that (i) the components of the gravity vector \( g \) need not be calculated; (ii) the solution holds for any flow orientation; and (iii) the fluid may flow as a result of a pressure difference, gravity or both. Here, \( h = z \cos \beta \) and therefore \( P = p_0 z \cos \beta \). Thus, \( p_0 = p_0 \) at \( z = 0 \) and \( P_L = p_L \rho g L \cos \beta \) at \( z = L \) giving \( p_0 p_L + \rho g L \cos \beta = P_0 P_L \equiv \Delta P \). Thus, equation (1.a.2) yields

\[
\frac{d\tau_{xz}}{dx} = \frac{\Delta P}{L} \quad (1.a.3)
\]

This first-order differential equation may be simply integrated to give

\[
\tau_{xz}(x) = \frac{\Delta P}{L} x + C_1 \quad (1.a.4)
\]

Here, \( C_1 \) is an integration constant, which is determined using an appropriate boundary condition based on the flow problem. Equation (1.a.4) shows that the momentum flux (or shear stress) distribution is linear in systems in rectangular Cartesian coordinates. Since equations (1.a.3) and (1.a.4) have been derived without making any assumption about the type of fluid, they are applicable to both Newtonian and non-Newtonian fluids. Some of the axial flow problems in rectangular Cartesian coordinates where these equations may be used as starting points are given below.
1.b Determination of solution.

Since the problem is a laminar flow in a plane narrow slit, the obvious choice for a coordinate system is Cartesian coordinates. The fluid flow is in the $z-$direction, therefore the velocity vector is of the form $\vec{v} = (0, 0, v_z)$. Further, $v_z$ is independent of $z$ and it is meaningful to postulate that $v_z$ is a function only for $x$, $v_z = v_z(x)$, and also for the pressure $p = p(z)$. From these, and from the definition of the stress tensor, its nonvanishing components are $\tau_{xz} = \tau_{zx}$, which depend only on $x$.

Using the differential shell momentum balance method, consider now a thin rectangular slab (shell) perpendicular to the $x$-direction extending a distance $W$ in the $y$-direction and a distance $L$ in the $z$-direction. A ‘rate of $z$-momentum’ balance over this thin shell of thickness $\Delta x$ in the fluid is of the form:

$$\text{Rate of } z\text{-momentum} = \text{In} + \text{Out} + \text{Generation} = \text{Accumulation}$$

At steady-state, the accumulation term is zero. Momentum can go ‘in’ and ‘out’ of the shell by both the convective and molecular mechanisms. Since $v_z$ is a function only of $x$, it is the same at both ends of the slit, and the convective terms cancel out because $(p v_z v_z W \Delta x)|_{z=0} = (p v_z v_z W \Delta x)|_{z=L}$. Only the molecular term $(L W \tau_{xz})$ remains to be considered. Generation of $z$-momentum occurs by the pressure force ($p W \Delta x$) and gravity force ($p g W L \Delta x$). On substituting these contributions into the $z$-momentum balance, we get

$$(L W \tau_{xz})|_x - (L W \tau_{xz})|_{x+\Delta x} + (p_0 - p_L) W \Delta x + p g W L \Delta x = 0 \quad (1.b.1)$$

Dividing the above equation by $L W \Delta x$ yields

$$\frac{\tau_{xz}|_{x+\Delta x} - \tau_{xz}|_x}{\Delta x} = \frac{(p_0 - p_L) + p g L}{L} \quad (1.b.2)$$

On taking the limit as $\Delta x \to 0$ the left term of the above equation is the definition of the derivative $\frac{d\tau_{xz}}{dx}$. The right-hand side may be compactly and conveniently written by introducing the modified pressure $P$, which is the sum of the pressure and gravitational terms. The general definition of the modified pressure is $P = p + p g h$, where $h$ is the distance upward (in the direction opposed to gravity) from a reference plane of choice. Since the $z$-axis points downward in this problem, $h = z$ and therefore $P = p g z$. Thus, $P_0 = p_0$ at $z = 0$ and $P_L = p_L p g L$ at $z = L$, giving $p_0 p L + p g L = P_0 P_L \equiv P$. Then equation (1.b.2) becomes

$$\frac{d\tau_{xz}}{dx} = \frac{\Delta P}{L} \quad (1.b.3)$$

Equation (1.b.3) on integration leads to the following expression for the shear stress distribution tensor:

$$\tau_{xz} = \frac{\Delta P}{L} x + C_1 \quad (1.b.4)$$

To find the velocity, we substitute $\tau_{xz}$ from (1.b.4) into Newton’s law of viscosity, and get:

$$-\mu \frac{dv_z}{dx} = \frac{\Delta P}{L} x + C_1 \quad (1.b.5)$$

The above first-order differential equation is simply integrated to obtain the following velocity profile:

$$v_z = -\frac{\Delta P}{2 \mu L} x^2 - \frac{C_1}{\mu} + C_2 \quad (1.b.6)$$

It is worth noting that equations (1.b.3) and (1.b.4) apply to both Newtonian and non-Newtonian fluids, and provide starting points for many fluid flow problems in rectangular Cartesian coordinates.

Using the no-slip boundary conditions at the two fixed walls, $v_z(x)|_{x=B} = v_z(x)|_{x=-B} = 0$, the integration constants are $C_1 = 0$ and $C_2 = \frac{\Delta P B^2}{2 \mu L}$.

Then the final expressions for the shear tensor and the velocity are:

$$\tau_{xz} = \frac{\Delta P}{L} x \quad (1.b.7)$$

$$v_z = \frac{\Delta P B^2}{2 \mu L} \left(1 - \frac{x^2}{B^2}\right) \quad (1.b.8)$$
It is observed that the velocity distribution for laminar, incompressible flow of a Newtonian fluid in a plane narrow slit is parabolic.

From the velocity profile, various useful quantities may be derived. The maximum velocity occurs at \( x = 0 \) (where \( \frac{dv_z}{dx} = 0, \frac{d^2v_z}{dx^2} < 0 \)). Therefore,

\[
v_z \max = v_z|_{x=0} = \frac{\Delta PB^2}{2\mu L} \tag{1.b.9}
\]

The average velocity is obtained by dividing the volumetric flow rate by the cross-sectional area,

\[
v_z \avg = \frac{\int_{-B}^{B} v_z W dx}{\int_{-B}^{B} W dx} = \frac{1}{2B} \int_{-B}^{B} v_z dx = \frac{\Delta PB^2}{3\mu L} = \frac{2}{3}v_z \max \tag{1.b.10}
\]

Thus, the ratio of the average velocity to the maximum velocity for Newtonian fluid flow in a narrow slit is \( \frac{2}{3} \).

The mass rate of flow is obtained by integrating the velocity profile over the cross section of the slit as follows:

\[
w = \int_{-B}^{B} \rho v_z W dx = 2\rho WBv_z \avg \tag{1.b.11}
\]

and substituting \( v_z \avg \) from (1.b.10) we have the final expresion for the mass rate of flow.

\[
w = \frac{2\Delta PB^3\rho W}{3\mu L} \tag{1.b.12}
\]

The flow rate vs. pressure drop (\( w \) vs. \( \Delta P \)) expression above is the slit analog of the Hagen-Poiseuille equation (originally for circular tubes). It is a result worth noting because it provides the starting point for creeping flow in many systems (e.g., radial flow between two parallel circular disks; and flow between two stationary concentric spheres). Finally, it may be noted that the above analysis is valid when \( B \ll W \). If the slit thickness \( B \) is of the same order of magnitude as the slit width \( W \), then \( v_z = v_z(x, y) \), i.e., \( v_z \) is not a function of only \( x \). If \( W = 2B \), then a solution can be obtained for flow in a square duct.
Problem 2

Steady, laminar flow occurs in the space between two fixed parallel, circular disks separated by a small gap $2b$. The fluid flows radially outward owing to a pressure difference $(P_1 - P_2)$ between the inner and outer radii $r_1$ and $r_2$, respectively. Neglect end effects and consider the region $r_1 \leq r \leq r_2$ only. Such a flow occurs when a lubricant flows in certain lubrication systems.

Figure 2.1: Radial flow between two parallel disks.

a) Simplify the equation of continuity to show that $ru_r$ is a function of only $z$.
b) Simplify the equation of motion for incompressible flow of a Newtonian fluid of viscosity $\mu$ and density $\rho$.
c) Obtain the velocity profile assuming creeping flow.
d) Sketch the velocity profile $u_r(r, z)$ and the pressure profile $P(r)$.
e) Determine an expression for the mass flow rate by integrating the velocity profile.
f) Derive the mass flow rate expression in e) using an alternative short-cut method by adapting the plane narrow slit solution.

Solution

2.a  Simplification of continuity equation.

Since the steady laminar flow is directed radially outward, only the radial velocity component $u_r$ exists. The tangential and axial components of velocity are zero, so the velocity vector becomes $\vec{u} = (u_r, 0, 0)$.

For an incompressible flow, the continuity equation gives $\nabla \cdot \vec{u} = 0$. In cylindrical coordinates, this is expanded as follows:

$$\frac{\partial}{\partial r} (ru_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \Rightarrow$$

$$\frac{\partial}{\partial r} (ru_r) = 0 \Rightarrow$$

$$ru_r = f(\theta, z)$$

We are expecting the solution to be symmetric around the $z$-axis, therefore the solution must be independent of $\theta$, hence

$$ru_r = f(z) \quad (2.a.1)$$

This proves that $ru_r$ is a function of $z$ only.

2.b  Simplification of the equation of motion for a Newtonian fluid.

For a Newtonian fluid, its equation of motion is the Navier-Stokes equation,

$$\rho \frac{D\vec{u}}{Dt} = -\nabla P + \mu \nabla^2 \vec{u} \quad (2.b.1)$$

in which $P$ includes both the pressure and gravitational terms. On noting that $u_r = u_r(r, z)$, equation (2.b.1) can be separated into its coordinate components and for steady flow they may be simplified as given below.
\[ r \text{- component: } \rho \left( u_r \frac{\partial u_r}{\partial r} \right) = -\frac{\partial P}{\partial r} + \mu \frac{\partial^2 u_r}{\partial z^2} \quad (2.b.2) \]

\[ \theta \text{- component: } 0 = \frac{\partial P}{\partial \theta} \quad (2.b.3) \]

\[ z \text{- component: } 0 = \frac{\partial P}{\partial z} \quad (2.b.4) \]

From (2.b.3) and (2.b.4) it is easy to prove that \( P = P(r) \). Recall that \( ru_r = f(z) \) from the continuity equation. Substituting \( u_r = f/r \) and \( P = P(r) \) in equation (2.b.2) then gives

\[-\rho \frac{f^2(z)}{r^3} = -\frac{dP(r)}{dr} + \frac{\mu}{r} \frac{d^2 f(z)}{dz^2} \quad (2.b.5)\]

This is the equation of motion for an incompressible, steady laminar flow of a Newtonian fluid of viscosity \( \mu \) and density \( \rho \).

2.c Determination of the velocity profile assuming creeping flow.

The equation of motion (2.b.5) is an inhomogeneous and generally non-linear differential equation (it depends on the form of \( f(z) \)). It has no solution unless the nonlinear term (that is, the \( f^2 \) term on the left-hand side) is neglected. This is the ‘creeping flow’ assumption, and under this assumption, equation (2.b.5) becomes an homogeneous equation. The latter can be written as

\[ r \frac{dP(r)}{dr} = \mu \frac{d^2 f(z)}{dz^2} \quad (2.c.1) \]

The left term of (2.c.1) is a function of only \( r \), while the right term is a function of only \( z \). The equality can be true only if both terms are equal with a constant \( (a_0 \in \mathbb{R}) \), so that

\[ r \frac{dP(r)}{dr} = \mu \frac{d^2 f(z)}{dz^2} = a_0 \]

Thus the original partial differential equation has been decoupled into two ordinary differential equations

\[ \frac{dP(r)}{dr} = \frac{a_0}{r} \quad (2.c.2) \]

\[ \mu \frac{d^2 f(z)}{dz^2} = a_0 \quad (2.c.3) \]

To solve this system, all we need to do is integrate each equation (equation (2.c.3) needs to be integrated twice). We thus find

\[ P(r_2) - P(r_1) = a_0 \ln \left( \frac{r_2}{r_1} \right) \quad (2.c.4) \]

\[ f(z) = \frac{a_0}{2\mu} z^2 + a_1 z + a_2 \quad (2.c.5) \]

where \( a_1 \) and \( a_2 \) are integration constants.

By substituting \( a_0 \) from equation (2.c.4) into equation (2.c.5), we finally get that

\[ f(z) = \frac{P(r_2) - P(r_1)}{2\mu \ln \left( \frac{r_2}{r_1} \right)} z^2 + a_1 z + a_2 \quad (2.c.6) \]

The no-slip boundary conditions of our problem dictate that
\[ u_r (r, z = +b) = u_r (r, z = -b) = 0 \]

If we insert these into (2.c.6), the result is that
\[
 f (z) = \frac{P (r_2) - P (r_1)}{2 \mu} b^2 \left( \frac{z^2}{b^2} - 1 \right) \tag{2.c.7}
\]
therefore as we have assumed creeping flow, the velocity profile is:
\[
 u_r (r, z) = \frac{P (r_2) - P (r_1)}{2 \mu r} b^2 \left( \frac{z^2}{b^2} - 1 \right) \tag{2.c.8}
\]

Furthermore the expression for the pressure \( P (r) \) is given from the solution of (2.c.2) and is
\[
 P (r) = P (r_1) + \frac{[P (r_2) - P (r_1)] \ln \left( \frac{r_2}{r_1} \right)}{\ln \left( \frac{r_2}{r_1} \right)} \tag{2.c.9}
\]
where we have substituted \( a_0 \) from (2.c.4).

2.d Sketch of the velocity and pressure profiles.

The velocity profile is described by equation (2.c.8) and the pressure profile by equation (2.c.9). We produce a sketch for each using gnuplot, with the following numerical values:

\[
\begin{align*}
b &= 0.1, \quad \mu = 1 \\
r_2 &= 5, \quad r_1 = 1 \\
P (r_2) &= 1, \quad P (r_1) = 5
\end{align*}
\]

Regardless of the precise numerical values, the graphs are representative of the fluid’s behavior when the pressure is greater at the center. The fluid begins to flow outwards rapidly, but both the velocity and pressure drop as the fluid approaches the outer perimeter. The fluid’s velocity also diminishes as one approaches either plate, and remains highest at the central area between the plates.
2.e  Mass flow rate by integration of the velocity profile.

The mass flow rate $w$ is rigorously obtained by integrating the velocity profile using

$$ w = \int \hat{n} \cdot \rho \vec{u} dS $$

where $\hat{n}$ is the unit normal to the element of surface area $dS$ and $\vec{u}$ is the fluid velocity vector. For the radial flow between parallel disks, $\hat{n} = \hat{r}$, $\vec{u} = u_r \hat{r}$, and $dS = 2\pi r dz$. Then, substituting the velocity profile and integrating gives

$$ w = \int_{-b}^{b} \rho (\hat{n} \cdot \vec{u}) dS = \int_{-b}^{b} \rho u_r dS = 2\pi \int_{-b}^{b} \rho u_r(r,z) r dz $$

and using (2.a.1) and (2.c.7) for $ru_r$ we finally have for the mass flow rate that

$$ w = \frac{4\pi}{3\mu} \frac{P_1 - P_2}{b^3 \rho} \ln \left( \frac{r_2}{r_1} \right) $$

(2.e.1)

**Mass flow rate using short-cut method by adapting narrow slit solution**

The plane narrow slit solution may be applied locally by recognizing that at all points between the disks, the flow resembles the flow between parallel plates provided $u_r$ is small (that is, the creeping flow is valid).

The mass flow rate for a Newtonian fluid in a plane narrow slit of width $W$, length $L$ and thickness $2B$ is given by

$$ w = \frac{2\Delta P B^3 W \rho}{3\mu L} . $$

In this expression, $\Delta P/L$ is replaced by $-dP/dr$, $B$ by $b$, and $W$ by $2\pi r$. Note that mass is conserved, so $w$ is constant. Then, integrating from $r_1$ to $r_2$ gives the same mass flow rate expression [equation (2.e.1)], as shown below.

$$ w \int_{r_1}^{r_2} \frac{dr}{r} = 4\pi \frac{b^3 r}{3\mu} \int_{P_i}^{P_2} (-dP) \Rightarrow \quad w = \frac{4\pi}{3\mu} \frac{P_1 - P_2}{b^3 \rho} \ln \left( \frac{r_2}{r_1} \right) $$

(2.e.2)

This alternative short-cut method for determining the mass flow rate starting from the narrow slit solution is very powerful because the approach may be used for non-Newtonian fluids where analytical solutions are difficult to obtain.
Problem 3

A wire-coating die essentially consists of a cylindrical wire of radius $\kappa R$ ($\kappa < 1$) moving horizontally at a constant velocity $u$ along the axis of a cylindrical die of radius $R$. If the pressure in the die is uniform, then the polymer melt (which may be considered a non-Newtonian fluid described by the power law model and of constant density $\rho$) flows through the narrow annular region solely by the drag due to the axial motion of the wire (which is referred to as ‘axial annular Couette flow’). Neglect end effects and assume an isothermal system.

![Figure 3.1: Fluid flow in wire-coating die.](image)

a) Establish the expression for the steady-state velocity profile in the annular region of the die. Simplify the expression for $n = 1$ (Newtonian fluid).

b) Obtain the expression for the mass flow rate through the annular die region. Simplify the expression for $n = 1$ and $n = 1/3$.

c) Estimate the coating thickness $\delta$ some distance downstream of the die exit.

d) Find the force that must be applied per unit length of the wire.

e) Write the force expression in d) as a ‘plane slit’ formula with a ‘curvature correction’.

Solution

3.a Shear stress and velocity distribution.

The fluid flows in the $z$-direction shearing constant-$r$ surfaces. Therefore, the velocity component that exists is $u_z(r)$ and the shear stress component is $\tau_{rz}(r)$. A momentum balance in cylindrical coordinates gives

$$\frac{d(r\tau_{rz})}{dr} = \frac{\Delta P}{L}r \tag{3.a.1}$$

Since there is no pressure gradient and gravity does not play any role in the motion of the fluid, $\Delta P = 0$. Equation (3.a.1) with the right-hand side set to zero leads to the following expression for the shear stress distribution on integration:

$$\tau_{rz} = \frac{C_1}{r} \tag{3.a.2}$$

The constant of integration $C_1$ is determined later using boundary conditions.

Since the velocity $u_z$ decreases with increasing radial distance $r$, the velocity gradient $du_z/dr$ is negative in the annular die region and the power law model may be substituted in the following form for $\tau_{rz}$ in equation (3.a.2).

$$m \left( -\frac{du_z}{dr} \right)^n = \frac{C_1}{r} \tag{3.a.3}$$

Here, $m$ and $n$ are the consistency index and exponent in the power law model, respectively. (The Newtonian fluid model has $m = \mu$ and $n = 1$, where $\mu$ is the fluid’s dynamic viscosity.) The above differential equation is simply integrated to obtain the following velocity profile (for $n \neq 1$):

$$u_z = -\left( \frac{C_1}{m} \right)^{\frac{1}{n}} \frac{r^q}{q} + C_2 \tag{3.a.4}$$

where $q = 1 - \frac{1}{n}$. The integration constants $C_1$ and $C_2$ are evaluated from the following no-slip boundary conditions:
\[ u_z(r = R) = 0, \quad u_z(r = \kappa R) = V \]  
(3.a.5)  
(3.a.6)

From (3.a.5),

\[ C_2 = \left( \frac{C_1}{m} \right)^{\frac{1}{q}} \frac{R^q}{q} \]

So,

\[ u_z = \left( \frac{C_1}{m} \right)^{\frac{1}{q}} \frac{R^q}{q} \left[ 1 - \left( \frac{r}{R} \right)^q \right] \]

and (3.a.6) yields

\[ V = \left( \frac{C_1}{m} \right)^{\frac{1}{q}} \frac{R^q}{q} (1 - \kappa^q) \]

On substituting the integration constants into equations (3.a.2) and (3.a.4), the final expressions for the shear stress and velocity distribution (for \( n \neq 1 \)) are

\[ \tau_{rz} = \frac{m}{r} \left[ \frac{qV}{R^q(1 - \kappa^q)} \right]^n \]  
(3.a.7)  
\[ \frac{u_z}{V} = \frac{(r/R)^q - 1}{\kappa^q - 1} \]  
(3.a.8)

When \( n = 1 \), the right-hand sides of equations (3.a.7) and (3.a.8) evaluate to 0/0; therefore, the evaluation may be done using De l’Hopital’s rule by differentiating the numerator and denominator with respect to \( n \) (or \( q \)) and then evaluating the limit as \( n \to 1 \) (or \( q \to 0 \)). On differentiating equation (3.a.7),

\[ \lim_{q \to 0} \left( \frac{q}{1 - \kappa^q} \right)^{\left(0/0\right)} \equiv \lim_{q \to 0} \left( \frac{1}{-\kappa^q \ln \kappa} \right) = \frac{1}{-\ln \kappa} \]

Similarly, on differentiating equation (3.a.8),

\[ \lim_{q \to 0} \left[ \frac{(r/R)^q - 1}{\kappa^q - 1} \right]^{\left(0/0\right)} \equiv \lim_{q \to 0} \left[ \frac{\ln (r/R)}{\kappa^q \ln \kappa} \right] = \frac{\ln (r/R)}{\ln \kappa} \]

Thus, on simplifying equations (3.a.7) and (3.a.8), the shear stress and velocity for a Newtonian fluid (\( n = 1 \) and \( m = \mu \)) are

\[ \tau_{rz} = \frac{\mu V}{r \ln (1/\kappa)} \]  
(3.a.9)  
\[ \frac{u_z}{V} = \frac{\ln (r/R)}{\ln \kappa} \]  
(3.a.10)

### 3.b Mass flow rate.

The mass rate of flow is obtained by integrating the velocity profile over the cross section of the annular region as follows.

\[ w = 2\pi \rho \int_{\kappa R}^{R} ru_z dr = 2\pi R^2 \rho \int_{\kappa}^{1} \frac{r}{R} \left[ \left( \frac{r}{R} \right)^q - 1 \right] d \left( \frac{r}{R} \right) \]

Integration then gives
\[ w = \frac{2\pi R^2 V \rho}{\kappa^q - 1} \left[ \frac{1}{2 + q} \left( \frac{r}{R} \right)^{2+q} - \frac{1}{2} \left( \frac{r}{R} \right)^2 \right] \]  

(3.b.1)

On evaluation at the limits of integration, the final expression for the mass rate of flow (for \( n \neq 1 \) and \( n \neq \frac{1}{3} \)) is

\[ w = \frac{2\pi R^2 V \rho}{\kappa^q - 1} \left[ \frac{1 - \kappa^{2+q}}{2 + q} - \frac{1}{2} \kappa^2 \right] \]  

(3.b.2)

When \( n = 1 \) (or \( q = 0 \)), the right-hand side of the above equation gives \( 0/0 \). Again, using De l’Hopital’s rule gives

\[ \lim_{q \to 0} \left[ \frac{1}{\kappa^q - 1} \left( \frac{1 - \kappa^{2+q}}{2 + q} - \frac{1}{2} \kappa^2 \right) \right] = \lim_{q \to 0} \left[ \frac{1}{\kappa^q \ln \kappa} \frac{(2 + q)(-\kappa^{2+q}) \ln \kappa - (1 - \kappa^{2+q})}{(2 + q)^2} \right] = -\frac{2\kappa^2 \ln \kappa + (1 - \kappa)^2}{4 \ln \kappa} \]

Substituting this in equation (3.b.2) gives the mass flow rate for a Newtonian fluid (\( n = 1 \)) as

\[ w = \frac{\pi R^2 V \rho}{2} \left[ \frac{1 - \kappa^2}{\ln(1/\kappa)} - 2\kappa^2 \right] \]  

(3.b.3)

When \( n = 1/3 \) (or \( q = 2 \)), the term \((1\kappa^{2+q})/(2 + q)\) becomes \( 0/0 \). Again using l’Hopital’s rule,

\[ \lim_{q \to -2} \left( \frac{1 - \kappa^{2+q}}{2 + q} \right) = \lim_{q \to -2} (-\kappa^{2+q} \ln \kappa) = -\ln \kappa \]

Substituting this last result in equation (3.b.2) gives the mass flow rate for a fluid with \( n = 1/3 \) \((q = 2)\) as

\[ w = \frac{2\pi R^2 V \rho}{\kappa^q - 1} \left[ \ln \frac{1}{\kappa} - \frac{1 - \kappa^2}{2} \right] \]  

(3.b.4)

3.c Coating thickness.

Some distance downstream of the die exit, the liquid (polymer melt) is transported as a coating of thickness \( \delta \) exposed to air. A momentum balance in the ‘air region’ yields the same shear stress expression [equation (3.a.2)]. The boundary condition at the liquid-gas interface is \( \tau_{rz} = 0 \) at \( r = \kappa R + \delta \), which now gives \( C_1 = 0 \) and \( \tau_{rz} = 0 \) throughout the coating. Furthermore, equations (3.a.4) and (3.a.6) are valid and give the velocity profile as \( u_z = V \) through the entire coating. As the velocity becomes uniform and identical to that of the wire, the liquid is transported as a rigid coating. The mass flow rate in the air region is given by

\[ w_{\text{air}} = 2\pi \rho V \int_{\kappa R}^{\kappa R + \delta} r dr = \pi R^2 V \rho \left[ \frac{1}{2} \left( \kappa + \frac{\delta}{R} \right)^2 - \kappa^2 \right] \]  

(3.c.1)

Equating the above expression for the flow rate \( w_{\text{air}} \) in the air region to the flow rate \( w \) in the die region gives

\[ \frac{\delta}{R} = \sqrt{\kappa^2 + \frac{2w}{\pi R^2 V \rho}} - \kappa \]  

(3.c.2)

On substituting the appropriate expression for \( w \) from equations (3.b.4), (3.b.3), or (3.b.2), the final expression for the coating thickness is obtained. It may be noted that the coating thickness \( \delta \) depends only on \( R \), \( \kappa \) and \( n \), but not on \( V \) and \( m \).
3.d  **Force applied to the wire.**

The force applied to the wire is given by the shear stress integrated over the wetted surface area. Therefore, on using equation (3.a.7) or equation (3.a.9),

\[
\frac{F_z}{L} = 2\pi \kappa R [\tau_{rz}]_{r=\kappa R} \Rightarrow \frac{F_z}{L} = \begin{cases} 
2\pi m \left( \frac{qV}{\eta(1-\kappa\eta)} \right)^n, & n \neq 1 \\
\frac{2\pi \mu V}{\ln 1/\kappa}, & n = 1 
\end{cases} (3.d.1)
\]

3.e  **Plane slit formula for force.**

A very narrow annular region with outer radius \(R\) and inner radius \((1-\epsilon)R\), where \(\epsilon\) is very small, may be approximated by a plane slit with area \(2\pi(1\frac{\epsilon}{2})RL \approx 2\pi RL\) and gap \(\epsilon R\). Then, the velocity profile in the plane slit will be linear and the force applied to the ‘plane’ wire is given by

\[
F_{\text{plane}} = \frac{\mu V}{2\pi RL} \quad \text{or} \quad \frac{F_{\text{plane}}}{L} = \frac{2\pi \mu V}{\epsilon} (3.e.1)
\]

On substituting \(\kappa = 1 - \epsilon\) (since \(R - \kappa R = \epsilon R\) for the thin annular gap) in equation (3.d.1), the force applied to the wire is given by

\[
\frac{F_z}{L} = \frac{2\pi \mu V}{-\ln (1 - \epsilon)} (3.e.2)
\]

The Taylor series expansion given below is next used.

\[
\ln (1 - \epsilon) = -\epsilon - \frac{1}{2} \epsilon^2 - \frac{1}{3} \epsilon^3 - \frac{1}{4} \epsilon^4 - \frac{1}{5} \epsilon^5 - \ldots (3.e.3)
\]

Therefore, the denominator of equation (3.e.2) may be written as

\[
-\ln (1 - \epsilon) = \frac{1}{\epsilon} \left(1 + \frac{1}{2} \epsilon + \frac{1}{3} \epsilon^2 + \frac{1}{4} \epsilon^3 + \frac{1}{5} \epsilon^4 + \ldots \right)^{-1} = \frac{1}{\epsilon} \left(1 - \frac{1}{2} \epsilon - \frac{1}{12} \epsilon^2 - \frac{1}{24} \epsilon^3 - \frac{19}{720} \epsilon^4 - \ldots \right) (3.e.4)
\]

Equation (3.e.4) may be obtained by using \((1 + u)^{-1} = 1 - u + u^2 - u^3 + u^4 - \ldots\) (which is an infinite geometric progression with \(|u| < 1\)). On substituting in equation (3.e.2), the following force expression is finally obtained as a ‘plane slit’ formula with a ‘curvature correction’.

\[
\frac{F_z}{L} = \frac{2\pi \mu V}{\epsilon} \left[1 - \frac{1}{2} \epsilon - \frac{1}{12} \epsilon^2 - \frac{1}{24} \epsilon^3 - \frac{19}{720} \epsilon^4 - \ldots \right] (3.e.5)
\]

Problem 4

Consider an incompressible isothermal fluid in laminar flow between two coaxial cylinders, whose inner and outer wetted surfaces have radii of $\kappa R$ and $R$, respectively. The inner and outer cylinders are rotating at angular velocities $\Omega_i$ and $\Omega_o$, respectively. End effects may be neglected.

Figure 4.1: Tangential annular flow between two slowly rotating cylinders.

a) Determine the steady-state velocity distribution in the fluid (for small values of $\Omega_i$ and $\Omega_o$).
b) Find the torques acting on the two cylinders during the tangential annular flow of a Newtonian fluid.

Solution

4.a Steady-state velocity distribution.

Simplification of the continuity equation

In steady laminar flow, the fluid is expected to travel in a circular motion (for low values of $\Omega_i$ and $\Omega_o$). Only the tangential component of velocity exists. The radial and axial components of velocity are zero; so, $u_r = 0$ and $u_z = 0$. For an incompressible fluid, the continuity equation gives $\vec{\nabla} \cdot \vec{u} = 0$.

In cylindrical coordinates,

$$\frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \Rightarrow \frac{\partial u_\theta}{\partial \theta} = 0$$

So, $u_\theta = u_\theta(r,z)$.

If end effects are neglected, then $u_\theta$ does not depend on $z$. Thus, $u_\theta = u_\theta(r)$.

Simplification of the equation of motion

There is no pressure gradient in the $\theta$-direction. Therefore, the components of the equation of motion simplify to

- $r$-component:
  $$-\rho \frac{u_\theta^2}{r} = -\frac{\partial p}{\partial r}$$

- $\theta$-component:
  $$0 = \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} (ru_\theta) \right]$$

- $z$-component:
  $$0 = -\frac{\partial p}{\partial z} - \rho g \quad (z \text{ points upwards})$$

where in the equation for the $\theta$-component the partial derivatives have been replaced with ordinary ones, since $u_\theta$ is a function only of $r$ as was shown in the previous step. The $r$-component provides the radial pressure distribution due to centrifugal forces and the $z$-component gives the axial pressure distribution due to gravitational forces (the hydrostatic head effect).

Solution of the differential equation and the velocity profile

The $\theta$-component of the equation of motion can be integrated to get the velocity profile:
\[
\frac{1}{r} \frac{d}{dr} (ru_\theta) = C_1 \Rightarrow ru - \theta = C_1 \frac{r^2}{2} + C_2 \Rightarrow u_\theta (r) = \frac{C_1}{2} r + \frac{C_2}{r}
\] (4.a.1)

The no-slip boundary conditions at the two cylindrical surfaces are

\[
u_\theta (r = \kappa R) = \Omega_i \kappa R \iff \Omega_i \kappa R = \frac{C_1}{2} \kappa R + \frac{C_2}{\kappa R} \tag{4.a.2}
\]

\[
u_\theta (r = R) = \Omega_o R \iff \Omega_o R = \frac{C_1}{2} R + \frac{C_2}{R} \tag{4.a.3}
\]

Solving equations (4.a.2) and (4.a.3) for the integration constants we find that they are given by

\[
(\Omega_i - \Omega_o) \kappa R = C_2 \left( \frac{1}{\kappa R} - \frac{\kappa}{R} \right), \tag{4.a.4}
\]

\[
\left( \frac{\Omega_i \kappa R - \Omega_o R}{\kappa} \right) = C_1 \left( \frac{\kappa R}{R} - \frac{R}{\kappa} \right).
\]

Substituting the above values for \(C_1\) and \(C_2\) in equation (4.a.1), the velocity profile is obtained as

\[
u_\theta (r) = \frac{\Omega_o - \Omega_i \kappa}{1 - \kappa^2} r + \frac{\Omega_i - \Omega_o \kappa^2 R^2}{1 - \kappa^2} \frac{r}{r} = \frac{\kappa R}{1 - \kappa^2} \left[ (\Omega_o - \Omega_i \kappa^2) \frac{r}{\kappa R} + (\Omega_i - \Omega_o) \frac{\kappa R^3}{r} \right] \tag{4.a.4}
\]

The velocity distribution given by the above expression can be written in the following alternative form:

\[
u_\theta (r) = \frac{\Omega_o \kappa R}{1 - \kappa^2} \left( \frac{r}{\kappa R} - \frac{\kappa R}{r} \right) + \frac{\Omega_i \kappa^2 R^2}{1 - \kappa^2} \left( \frac{R}{R} - \frac{r}{r} \right) \tag{4.a.5}
\]

In the above form, the first term on the right-hand-side corresponds to the velocity profile when the inner cylinder is held stationary and the outer cylinder is rotating with an angular velocity \(\Omega_o\). This is essentially the configuration of the Couette - Hatschek viscometer for determining viscosity by measuring the rate of rotation of the outer cylinder under the application of a given torque (as discussed below). The second term on the right-hand-side of the above equation corresponds to the velocity profile when the outer cylinder is fixed and the inner cylinder is rotating with an angular velocity \(\Omega_i\). This is essentially the configuration of the Stormer viscometer for determining viscosity by measuring the rate of rotation of the inner cylinder under the application of a given torque (as discussed next).

4.b Torques acting on the cylinders.

**Determination of momentum flux**

From the velocity profile, the momentum flux (shear stress) is determined as

\[
\tau_{r\theta} = -\mu r \frac{d}{dr} \left( \frac{u_\theta}{r} \right) = 2 \mu \frac{\Omega_i - \Omega_o \kappa^2 R^2}{1 - \kappa^2} \frac{r}{r^2} \tag{4.b.1}
\]

**Determination of torque**

The torque is obtained as the product of the force and the lever arm. The force acting on the inner cylinder itself is the product of the inward momentum flux and the wetted surface area of the inner cylinder. Thus,

\[
T_z = (2\pi \kappa R L) (\kappa R) (-\tau_{r\theta} |_{r=R}) = 4\pi \mu (\Omega_i - \Omega_o) \frac{\kappa^2}{1 - \kappa^2} R^2 L \tag{4.b.2}
\]

The tangential annular flow system above provides the basic model for rotational viscometers (to determine fluid viscosities from measurements of torque and angular velocities) and for some kinds of friction bearings.
Problem 5

An incompressible isothermal liquid in laminar flow is open to the atmosphere at the top. Determine the shape of the free liquid surface \( z(r) \) at steady state (neglecting end effects, if any) for the following cases:

(a) - case.
(b) - case.
(c) - case.
(d) - case.

Figure 5.1: Free surface shape of liquid in four cases of tangential flow.

a) when the liquid is in the annular space between two vertical coaxial cylinders, whose inner and outer wetted surfaces have radii of \( \kappa R \) and \( R \), respectively. Here, the inner cylinder is rotating at a constant angular velocity \( \Omega_i \) and the outer cylinder is stationary. Let \( z_R \) be the liquid height at the outer cylinder.

b) when a single vertical cylindrical rod of radius \( R_i \) is rotating at a constant angular velocity \( \Omega_i \) in a large body of quiescent liquid. Let \( z_{R_o} \) be the liquid height far away from the rotating rod. 

*Hint:* Use the result from a).

c) when the liquid is in the annular space between two vertical coaxial cylinders, whose inner and outer wetted surfaces have radii of \( R \) and \( \kappa R \), respectively. Here, the outer cylinder is rotating at a constant angular velocity \( \Omega_o \) and the inner cylinder is stationary. Let \( z_R \) be the liquid height at the outer cylinder.

d) when the liquid is in a vertical cylindrical vessel of radius \( R \), which is rotating about its own axis at a constant angular velocity \( \Omega_o \). Let \( z_R \) be the liquid height at the vessel wall.

*Hint:* Use the result from c).

Solution

**Simplification of the continuity equation**

In steady laminar flow, the liquid is expected to travel in a circular motion with only the tangential component of velocity. The radial and axial components of velocity are zero; so, \( u_r = 0 \) and \( u_z = 0 \).

For an incompressible fluid, the continuity equation gives \( \nabla \cdot \vec{u} = 0 \). In cylindrical coordinates,

\[
\frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial u_\theta}{\partial \theta} = 0
\]  

(5.1)

So, \( u_\theta = u_\theta(r, z) \). If end effects are neglected, then \( u_\theta \) does not depend on \( z \). Thus, \( u_\theta = u_\theta(r) \).
Simplification of the equation of motion

There is no pressure gradient in the \( \theta \)-direction. The components of the equation of motion simplify to

- \( r \)-component: \(-\rho u_\theta^2 r = -\frac{\partial p}{\partial r}\)
- \( \theta \)-component: \(0 = \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} (ru_\theta) \right]\)
- \( z \)-component: \(0 = -\frac{\partial p}{\partial z} - \rho g \) (\( z \) points upwards)

where in the equation for the \( \theta \)-component the partial derivatives have been replaced with ordinary ones, since \( u_\theta \) is a function only of \( r \) as was shown in the previous step.

As discussed later, the \( r \)-component provides the radial pressure distribution due to centrifugal forces and the \( z \)-component gives the axial pressure distribution due to gravitational forces (the hydrostatic head effect).

Solution of the differential equation and the velocity profile

The \( \theta \)-component of the equation of motion is integrated to get the velocity profile:

\[
\frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} (ru_\theta) \right] = 0 \Rightarrow \frac{1}{r} \frac{d}{dr} (ru_\theta) = C_1 \Rightarrow ru_\theta = C_1 \frac{r^2}{2} + C_2
\]

and therefore, the velocity profile is described by the equation

\[
u_\theta(r) = C_1 \frac{r^2}{2} + C_2
\]

The no-slip boundary conditions at the two cylindrical surfaces are

\[
u_\theta(r = \kappa R) = \Omega_i \kappa R \quad \Leftrightarrow \quad \Omega_i \kappa R = C_1 \frac{\kappa R}{\kappa R} + C_2
\]

\[
u_\theta(r = R) = \Omega_o R \quad \Leftrightarrow \quad \Omega_o R = C_1 \frac{R}{R} + C_2
\]

Solving equations (5.4) and (5.5) for the integration constants we find that they are given by

\[ (\Omega_i - \Omega_o) \kappa R = C_2 \left( \frac{1}{\kappa R} - \frac{\kappa}{R} \right), \]

\[ (\Omega_i \kappa R - \Omega_o \frac{R}{\kappa}) = C_1 \frac{\kappa R - \frac{R}{\kappa}}{2} \]

Substituting the above values for \( C_1 \) and \( C_2 \) in equation (5.3), the velocity profile is obtained as

\[
u_\theta(r) = \frac{\Omega_o - \Omega_i \kappa^2}{1 - \kappa^2} r + \frac{\Omega_i - \Omega_o \kappa^2 R^2}{1 - \kappa^2} \left[ (\Omega_o - \Omega_i \kappa^2) \frac{r}{\kappa R} + (\Omega_i - \Omega_o) \frac{\kappa R}{r} \right]
\]

The velocity distribution given by the above expression can be written in the following alternative form:

\[
u_\theta(r) = \frac{\Omega_o \kappa R}{1 - \kappa^2} \left( \frac{r}{\kappa R} - \frac{\kappa R}{r} \right) + \frac{\Omega_i \kappa^2 R}{1 - \kappa^2} \left( \frac{R}{r} - \frac{r}{R} \right)
\]

In the above form, the first term on the right-hand-side corresponds to the velocity profile when the inner cylinder is held stationary and the outer cylinder is rotating with an angular velocity \( \Omega_o \). The second term on the right-hand-side of the above equation corresponds to the velocity profile when the outer cylinder is fixed and the inner cylinder is rotating with an angular velocity \( \Omega_i \).
5.a Free surface shape for rotating inner cylinder and fixed outer cylinder.

Now, consider the case where the inner cylinder is rotating at a constant angular velocity $\Omega_i$ and the outer cylinder is stationary. With $\Omega_o = 0$, the velocity profile in (5.7) is substituted into the $r$-component of the equation of motion to get

$$\frac{\partial p}{\partial r} = \rho \frac{\partial}{\partial r} \left( \frac{u_r^2}{r} \right) = \rho \frac{\Omega_i^2 \kappa^4 R_i^2}{r (1 - \kappa^2)^2} \left( \frac{R_i^2}{r^2} - 2 + \frac{r^2}{R_i^2} \right)$$

(5.a.1)

Integrating the above equation we find the pressure profile as

$$p(r, z) = \frac{\rho \Omega_i^2 \kappa^4 R_i^2}{(1 - \kappa^2)^2} \left( -\frac{R_i^2}{2r^2} - 2 \ln r + \frac{r^2}{2R_i^2} \right) + f_1(z)$$

(5.a.2)

It is clear that the pressure profile is independent of $\theta$ due to the axial symmetry present. The above expression for the pressure $p$ is now substituted into the $z$-component of the equation of motion to obtain

$$\frac{\partial p}{\partial z} = -\rho g \Rightarrow \frac{df_1}{dz} = -\rho g \Rightarrow f_1(z) = -\rho gz + C_3$$

(5.a.3)

On substituting the expression for $f_1$ in (5.a.3) we find that

$$p(r, z) = \frac{\rho \Omega_i^2 \kappa^4 R_i^2}{(1 - \kappa^2)^2} \left( -\frac{R_i^2}{2r^2} - 2 \ln r + \frac{r^2}{2R_i^2} \right) - \rho gz + C_3$$

The constant $C_3$ is determined from the requirement that at the outer cylinder the liquid height be $z_R$. At that point, the pressure is equal to the atmospheric pressure, $p_{atm}$, therefore

$$p(r = R, z = z_R) = p_{atm} \Rightarrow p_{atm} = \frac{\rho \Omega_i^2 \kappa^4 R_i^2}{(1 - \kappa^2)^2} (-2 \ln R) - \rho gz_R + C_3$$

Subtraction of the above two equations allows for elimination of $C_3$ and the determination of the pressure distribution in the liquid as

$$p(r, z) - p_{atm} = \frac{\rho \Omega_i^2 \kappa^4 R_i^2}{(1 - \kappa^2)^2} \left( -\frac{R_i^2}{2r^2} - 2 \ln r + \frac{r^2}{2R_i^2} \right) + \rho g(z_R - z)$$

(5.a.4)

Since $p = p_{atm}$ at every point across the liquid-air interface (boundary), the shape of the free liquid surface is obtained as

$$z_R - z = \frac{1}{2g} \left( \frac{\Omega_i R_i^2}{1 - \kappa^2} \right)^2 \left[ \left( \frac{R_i}{r} \right)^2 + 4 \ln \left( \frac{r}{R_i} \right) - \left( \frac{r}{R_i} \right)^2 \right].$$

(5.a.5)

5.b Free surface shape for rotating cylinder in an infinite expanse of liquid.

The result in a) may be used to derive the free surface shape when a single vertical cylindrical rod of radius $R_i$ is rotating at a constant angular velocity $\Omega_i$ in a large body of quiescent liquid.

On substituting $R = R_o$ and $\kappa = R_i/R_o$, equations (5.7) (with $\Omega_o = 0$) and (5.a.5) yield

$$u_\theta = \frac{\Omega_i R_i^2}{1 - R_i^2/R_o^2} \left( \frac{1}{r} - \frac{r}{R_o^2} \right) \quad \text{and} \quad z_{R_o} - z = \frac{1}{2g} \left( \frac{\Omega_i R_i^2}{1 - R_i^2/R_o^2} \right)^2 \left[ \left( \frac{1}{r} \right)^2 + 4 \ln \left( \frac{r}{R_o} \right) - \left( \frac{r}{R_o} \right)^2 \right]$$

(5.b.1)

The above equations give the velocity profile and the shape of the free liquid surface for the case where the inner cylinder of radius $R_i$ is rotating at a constant angular velocity $\Omega_i$ and the outer cylinder of radius $R_o$ is stationary. Here, $z_{R_o}$ is the liquid height far away from the rotating rod.

In the limit as $R_o$ tends to infinity, the velocity profile and free surface shape for a single vertical rod rotating in a large body of quiescent liquid are obtained as

$$u_\theta = \frac{\Omega_i R_i^2}{r} \quad \text{and} \quad z_{R_o} - z = \frac{\Omega_i^2 R_i^4}{2gr^2}.$$

(5.b.2)
5. c  Free surface shape for rotating outer cylinder and fixed inner cylinder.

Next, consider the case where the outer cylinder is rotating at a constant angular velocity $\Omega_o$ and the inner cylinder is stationary. With $\Omega_i = 0$, the velocity profile in equation (5.7) is substituted into the $r$-component of the equation of motion to get

$$\frac{\partial p}{\partial r} = \frac{\rho u^2_o}{r} = \rho \frac{\Omega^2_o \kappa^2 R^2}{r (1-\kappa^2)} \left( \frac{\kappa^2 R^2}{r^2} - 2 + \frac{r^2}{\kappa^2 R^2} \right)$$

(5.c.1)

Integration gives

$$p(r, z) = \frac{\rho \Omega^2_o \kappa^2 R^2}{(1-\kappa^2)} \left( \frac{\kappa^2 R^2}{2r^2} - 2 \ln r + \frac{r^2}{2\kappa^2 R^2} \right) + f_2(z)$$

(5.c.2)

The above expression for the pressure $p$ is now substituted into the $z$-component of the equation of motion to obtain

$$\frac{\partial p}{\partial z} = -\rho g \Rightarrow \frac{df_2}{dz} = -\rho g \Rightarrow f_2(z) = -\rho gz + C_4$$

(5.c.3)

On substituting the expression for $f_2$ in equation (5.c.2), we get

$$p(r, z) = \frac{\rho \Omega^2_o \kappa^2 R^2}{(1-\kappa^2)} \left( \frac{\kappa^2 R^2}{2r^2} - 2 \ln r + \frac{r^2}{2\kappa^2 R^2} \right) - \rho gz + C_4$$

Again as in a), the constant $C_4$ is determined from the requirement that at the outer cylinder the liquid height be $z_R$. At that point, the pressure is equal to the atmospheric pressure, $p_{atm}$, therefore

$$p(r = R, z = z_R) = p_{atm} \Rightarrow p_{atm} = \frac{\rho \Omega^2_o \kappa^2 R^2}{(1-\kappa^2)} \left( \frac{\kappa^2}{2} - 2 \ln \frac{r}{R} + \frac{1}{2\kappa^2} \left( 1 - \frac{r^2}{R^2} \right) \right) - \rho gz_R + C_4$$

Subtraction of the above two equations allows for elimination of $C_4$ and the determination of the pressure distribution in the liquid as

$$p(r, z) - p_{atm} = \frac{\rho \Omega^2_o \kappa^2 R^2}{(1-\kappa^2)} \left[ -\frac{\kappa^2}{2} \left( \frac{r^2}{R^2} - 1 \right) - 2 \ln \left( \frac{r}{R} \right) - \frac{1}{2\kappa^2} \left( 1 - \frac{r^2}{R^2} \right) \right] + \rho g(z_R - z)$$

(5.c.4)

Since $p = p_{atm}$ at every point across the liquid-air interface (boundary), the shape of the free liquid surface is obtained as

$$z_R - z = \frac{1}{2g} \left( \frac{\Omega_o R}{1 - \kappa^2} \right)^2 \left[ \kappa^4 \left( \frac{R^2}{r^2} - 1 \right) + 4\kappa^2 \ln \left( \frac{r}{R} \right) + \left( 1 - \frac{r^2}{R^2} \right) \right].$$

(5.c.5)

An alternative approach is to obtain the above equation from the result of a) by substituting $r = \kappa R$ in equation (5.a.5) to get

$$z_R - z = \frac{1}{2g} \left( \frac{\Omega_o \kappa^2 R^2}{1 - \kappa^2} \right)^2 \left( \frac{1}{\kappa^2} + 4\ln \kappa - \kappa^2 \right)$$

(5.c.6)

The above expression gives the difference in elevations of the liquid surface at the outer and inner cylinders. Subtracting equation (5.c.6) from equation (5.a.5) yields

$$z_{\kappa R} - z = \frac{1}{2g} \left( \frac{\Omega_i \kappa^2 R^2}{1 - \kappa^2} \right)^2 \left[ \left( \frac{R^2}{r^2} - 1 \right) - 4\ln \left( \frac{r}{R} \right) - \left( 1 - \frac{r^2}{R^2} \right) \right]$$

(5.c.7)

The cylinders must now be swapped, i.e., the rotating cylinder should be made the outer cylinder rather than the inner one. So, let $\kappa R = R_o$ and $R = \beta R_o$. On substituting $\kappa = 1/\beta$ and $R = \beta R_o$ in the above equation, we get

$$z_{R_o} - z = \frac{1}{2g} \left( \frac{\Omega_i \kappa^2 R^2}{\beta^2 - 1} \right)^2 \left[ \left( \frac{\beta^2 R_o^2}{r^2} - 1 \right) - 4\ln \left( \frac{r}{R_o} \right) - \left( 1 - \frac{r^2}{R_o^2} \right) \right]$$

(5.c.8)

On recognizing that $\beta$ (which is a dummy parameter) can be simply replaced by $\kappa$, $R_o$ may be replaced by $R$, and $\Omega_i$ by $\Omega_o$, the above equation is identical to equation (5.c.5).
5.d  Free surface shape for cylindrical vessel rotating about its own axis.

The result in c) may be used to derive the free surface shape when a liquid is in a vertical cylindrical vessel of radius \( R \), which is rotating about its own axis at a constant angular velocity \( \Omega_0 \).

In the limit as \( \kappa \) tends to zero, equation (5.7) (with \( \Omega_i = 0 \)) and equation (5.c.5) yield the velocity profile and free surface shape as

\[
\begin{align*}
u_\theta &= \Omega_0 R \\
z_R - z &= \frac{\Omega_0^2 R^2}{2g} \left[ 1 - \left( \frac{r}{R} \right)^2 \right]
\end{align*}
\]  

(5.d.1)

Note that as \( \kappa \) tends to zero, the first and second terms in square brackets in equation (5.c.5) also tend to zero. The velocity profile suggests that each element of the liquid in a rotating cylindrical vessel moves like an element of a rigid body. The free surface of the rotating liquid in a cylindrical vessel is a paraboloid of revolution [since equation (5.d.1) corresponds to a parabola].
Problem 6

A plastic resin is in a vertical cylindrical vessel of radius $R$, which is rotating about its own axis at a constant angular velocity $\Omega$.

![Parabolic mirror from free surface shape of rotating liquid.](image)

**a)** Determine the shape of the free surface $z(r)$ at steady state (neglecting end effects, if any). Let $z_R$ be the liquid height at the vessel wall.

**b)** Show that the radius of curvature (of the free surface shape) is given by

$$r_c = \left[1 + \left(\frac{dz}{dr}\right)^2\right]^{3/2} \left|\frac{d^2z}{dr^2}\right|^{-1}$$

**c)** The cylinder is rotated until the resin hardens in order to fabricate the backing for a parabolic mirror. Determine the angular velocity (in rpm) necessary for a mirror of focal length 175 cm.

*Hint:* The focal length $f$ is one-half the radius of curvature $r_c$ at the axis.

Solution

**Simplification of the continuity equation**

In steady laminar flow, the liquid is expected to travel in a circular motion with only the tangential component of velocity. The radial and axial components of velocity are zero; so, $u_r = 0$ and $u_z = 0$. For an incompressible fluid, the continuity equation gives $\nabla \cdot \vec{u} = 0$. In cylindrical coordinates this is written as follows:

$$\frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial u_\theta}{\partial \theta} = 0$$

So, $u_\theta = u_\theta(r, z)$. If end effects are neglected, then $u_\theta$ cannot depend on $z$. Thus, $u_\theta = u_\theta(r)$.

**Simplification of the equation of motion**

There is no pressure gradient in the $\theta$-direction. The components of the equation of motion simplify to

- $r$-component: $-\rho \frac{u_\theta^2}{r} = -\frac{\partial p}{\partial r}$
- $\theta$-component: $0 = \frac{d}{dr} \left[ \frac{1}{r} \frac{d}{dr} (ru_\theta) \right]$
- $z$-component: $0 = -\frac{\partial p}{\partial z} - \rho g$ (z points upwards)

where in the equation for the $\theta$-component the partial derivatives have been replaced with ordinary ones, since $u_\theta$ is a function only of $r$ as was shown in the previous step. The $r$-component provides the radial pressure distribution due to centrifugal forces and the $z$-component gives the axial pressure distribution due to gravitational forces (the hydrostatic head effect).
Solution of the differential equation and the velocity profile

The θ-component of the equation of motion can be integrated to get the velocity profile:

\[
\frac{1}{r} \frac{d}{dr} (ru_\theta) = C_1 \quad \Rightarrow \quad ru - \theta = \frac{C_1 r^2}{2} + C_2 \quad \Rightarrow \quad u_\theta(r) = \frac{C_1}{2} r + \frac{C_2}{r}
\]  

(6.1)

Since the velocity \( u_\theta \) is finite at \( r = 0 \), the integration constant \( C_2 \) must be zero. Because \( u_\theta(r = R) = \Omega R \), \( C_1 = 2\Omega \). Thus,

\[
u_\theta(r) = \Omega r
\]

(6.2)

The above velocity profile suggests that each element of the liquid in a rotating cylindrical vessel moves like an element of a rigid body.

6.a Free surface shape for rotating liquid in cylinder.

Next, the velocity profile is substituted into the \( r \)-component of the equation of motion to get

\[
\frac{\partial p}{\partial r} = \frac{\rho u_\theta^2}{r} = \rho \Omega^2 r
\]

(6.a.1)

Equation (6.a.1) is easily integrated to obtain

\[
p(r, z) = \frac{\rho \Omega^2}{2} r^2 + f_1(z)
\]

(6.a.2)

The above expression for the pressure \( p \) is now substituted into the \( z \)-component of the equation of motion to obtain

\[
\frac{dp}{dz} = -\rho g \quad \Rightarrow \quad \frac{df_1}{dz} = -\rho g \quad \Rightarrow \quad f_1(z) = -\rho gz + C_3
\]

(6.a.3)

On substituting the expression for \( f_1 \) in equation (6.a.2),

\[
p(r, z) = \frac{\rho \Omega^2}{2} r^2 - \rho gz + C_3
\]

The constant \( C_3 \) can be found from the following boundary condition:

\[
p(r = R, z = z_R) = p_{atm} \quad \Rightarrow \quad p_{atm} = \frac{\rho \Omega^2}{2} R^2 - \rho gz_R + C_3
\]

where \( z_R \) is the liquid height at the vessel wall and \( p_{atm} \) denotes the atmospheric pressure.

Subtraction of the above two equations eliminates \( C_3 \) and gives the pressure distribution in the liquid as

\[
p(r, z) - p_{atm} = \frac{\rho \Omega^2}{2} (r^2 - R^2) + \rho g (z_R - z)
\]

(6.a.4)

Since \( p = p_{atm} \) at the liquid-air interface, the shape of the free liquid surface is finally obtained as

\[
z_R - z = \frac{\Omega^2}{2g} (R^2 - r^2)
\]

(6.a.5)

The free surface of the rotating liquid in a cylindrical vessel is a paraboloid of revolution [since equation (6.a.5) corresponds to a parabola].
6.b General expression for radius of curvature.

The circle that is tangent to a given curve at point \( P \), whose center lies on the concave side of the curve and which has the same curvature as the curve has at \( P \), is referred to as the circle of curvature. Its radius \((CP)\) defines the radius of curvature at \( P \). The circle of curvature has its first and second derivatives respectively equal to the first and second derivatives of the curve itself at \( P \). If the coordinates of the center \( C \) and point \( P \) are \((r_1, z_1)\) and \((r, z)\) respectively, then the radius of curvature \( CP \) is given by

\[
r_c = \sqrt{(r - r_1)^2 + (z - z_1)^2} \tag{6.b.1}
\]

Now, the radius \( CP \) is perpendicular to the tangent to the curve at \( P \). So, the slope of the radius \( CP \) is given by

\[
- \frac{dr}{dz} = \frac{z_1 - z}{r_1 - r}
\]

Thus, the first and second derivatives of the curve at point \( P \) are

\[
\frac{dz}{dr} = \frac{r - r_1}{z_1 - z} \quad \text{and} \quad \frac{d^2z}{dr^2} = \frac{1}{z_1 - z} + \frac{r - r_1}{(z_1 - z)^2} \frac{dz}{dr} = 1 + \left(\frac{dz}{dr}\right)^2
\]

On substituting the above derivatives in equation (6.b.1), the general expression for the radius of curvature is obtained as

\[
r_c = \left[1 + \left(\frac{dz}{dr}\right)^2\right]^{3/2} \left|\frac{d^2z}{dr^2}\right|^{-1} \tag{6.b.2}
\]

6.c Angular velocity necessary for parabolic mirror fabrication.

Let \( z_0 \) be the height of the liquid surface at the centerline of the cylindrical vessel. Substituting \( z = z_0 \) at \( r = 0 \) in equation (6.a.5) gives

\[
z_R z_0 = \frac{\Omega^2 R^2}{2g}.
\]

Thus, an alternative expression (in terms of \( z_0 \)) for the shape of the free surface is

\[
z = z_0 + \frac{\Omega^2}{2g} r^2 \tag{6.c.1}
\]

Differentiating the above expression twice gives

\[
\frac{dz}{dr} = \frac{\Omega^2}{g} r \quad \text{and} \quad \frac{d^2z}{dr^2} = \frac{\Omega^2}{g}
\]

We can use the above derivatives to calculate the radius of curvature of the parabola. By substituting the derivatives into equation (6.b.2), the radius of curvature of the parabola is given by

\[
r_c = \left(1 + \frac{\Omega^4 r^2}{g^2}\right)^{3/2} \frac{g}{\Omega^2} \tag{6.c.2}
\]

Since the focal length \( f \) of the mirror is one–half the radius of curvature \( r_c \) at the axis,

\[
f = \frac{1}{2} r_c (r = 0) = \frac{g}{2\Omega^2}
\]

and therefore, the rotational speed necessary to manufacture a parabolic mirror of focal length \( f \) is

\[
\Omega = \sqrt{\frac{g}{2f}} \tag{6.c.3}
\]

On substituting the focal length \( f = 1.75 \) m and the gravitational acceleration \( g = 9.81 \) m/s\(^2\) in the above equation, the rotational speed may be calculated as \( \Omega = 1.68 \text{ rad/s} = 1.68 \times 60 / (2\pi) \text{ rpm} = 16.00 \text{ rpm} \).
Problem 7

Consider an incompressible isothermal fluid in laminar flow between two concentric spheres, whose inner and outer wetted surfaces have radii of $\kappa R$ and $R$, respectively. The inner and outer spheres are rotating at constant angular velocities $\Omega_i$ and $\Omega_o$, respectively. The spheres rotate slowly enough that the creeping flow assumption is valid.

![Figure 7.1: Flow between two slowly rotating spheres.](image)

a) Determine the steady-state velocity distribution in the fluid (for small values of $\Omega_i$ and $\Omega_o$).

b) Find the torques on the two spheres required to maintain the flow of a Newtonian fluid.

c) Simplify the velocity and torque expressions for the case of a single solid sphere of radius $R_i$ rotating slowly at a constant angular velocity $\Omega_i$ in a very large body of quiescent fluid.

Solution

7.a  The steady-state velocity distribution.

**Simplification of the continuity equation**

In steady laminar flow, the liquid is expected to travel in a circular motion (for low values of $\Omega_i$ and $\Omega_o$) with only the tangential component of velocity. The radial and polar components of velocity are zero; so, $u_r = 0$ and $u_\theta = 0$.

For an incompressible fluid, the continuity equation gives $\nabla \cdot \vec{u} = 0$. In spherical coordinates this is written as follows:

$$
\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r \eta \mu \theta} \frac{\partial}{\partial \theta} (u_\theta \eta \mu \theta) + \frac{1}{r \eta \mu \phi} \frac{\partial}{\partial \phi} (u_\phi) = 0
$$

So, $u_\phi = u_\phi(r, \theta)$.

**Simplification of the equation of motion**

When the flow is very slow, the term $\rho (\vec{u} \cdot \nabla) \vec{u}$ in the equation of motion can be neglected because it is quadratic in the velocity. This is referred to as the creeping flow assumption (in other words, we ignore convective acceleration). Thus, for steady creeping flow, the entire left-hand side of the equation of motion is zero and we get

$$
0 = -\nabla P + \mu \nabla^2 \vec{u} + \rho g = -\nabla P + \mu \nabla^2 \vec{u}
$$

where $P$ is the modified pressure given by $P = p + \rho gh$ and $h$ is the elevation in the gravitational field (i.e., the distance upward in the direction opposite to gravity from some chosen reference plane). In this problem, $h = z = r \sigma \nu \theta$. The above equation is referred to as the Stokes flow equation or the creeping flow equation of motion. The components of the creeping flow equation in spherical coordinates simplify to
\[ r \text{- component: } 0 = - \frac{\partial P}{\partial r} \]

\[ \theta \text{- component: } 0 = - \frac{1}{r} \frac{dP}{d\theta} \]

\[ \phi \text{- component: } 0 = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial u_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[ \frac{1}{\eta \mu} \frac{\partial (u_\phi \eta \mu)}{\partial \theta} \right] \]

There is no dependence on the angle \( \phi \) due to symmetry about the vertical axis. The above partial differential equation must be solved for the velocity distribution \( u_\phi(r, \theta) \).

**Solution of the differential equation and the velocity profile**

The no-slip boundary conditions at the two spherical surfaces are

\[ u_\phi(r = \kappa R, \theta) = \Omega_i \kappa R \eta \mu \theta \] \hspace{1cm} (7.a.1)

\[ u_\phi(r = R, \theta) = \Omega_o \eta \mu \theta \] \hspace{1cm} (7.a.2)

From the boundary conditions, the form \( u_\phi(r, \theta) = f(r) \eta \mu \theta \) appears to be an educated guess to solve the partial differential equation for the velocity profile. On substituting this form in the \( \phi \)-component of the equation of motion and simplifying, we get

\[ r^2 \frac{d^2 f}{dr^2} + 2r \frac{df}{dr} - 2f = 0 \] \hspace{1cm} (7.a.3)

The above ordinary differential equation may be solved by substituting \( f(r) = r^n \) (i.e. using the power series method). The resulting characteristic equation is \( n^2 + n - 2 = 0 \), whose solution is \( n = 1, -2 \). Thus,

\[ f(r) = C_1 r + \frac{C_2}{r^2} \] and \( u_\phi(r, \theta) = \left( C_1 r + \frac{C_2}{r^2} \right) \eta \mu \theta \] \hspace{1cm} (7.a.4)

On imposing the boundary conditions (7.a.1) and (7.a.2), we find that

\[ \Omega_i \kappa R = C_1 \kappa R + \frac{C_2}{\kappa^2 R^2} \] and \( \Omega_o R = C_1 R + \frac{C_2}{R^2} \)

The integration constants are thus given by

\[ C_1 = \frac{\Omega_o - \Omega_i \kappa^3}{1 - \kappa^3} \] and \( C_2 = (\Omega_i - \Omega_o) \frac{\kappa^3 R^3}{1 - \kappa^3} \)

On substituting for \( C_1 \) and \( C_2 \) in (7.a.4), the velocity profile is obtained as

\[ u_\phi(r, \theta) = \left[ \Omega_o - \Omega_i \kappa^3 \right] r + \left( \Omega_i - \Omega_o \right) \frac{\kappa^3 R^3}{1 - \kappa^3} \frac{1}{r^2} \eta \mu \theta = \frac{\kappa R}{1 - \kappa^3} \left[ \left( \Omega_o - \Omega_i \kappa^3 \right) \frac{r}{\kappa R} + \left( \Omega_i - \Omega_o \right) \left( \frac{\kappa R}{r} \right) \right] \eta \mu \theta \] \hspace{1cm} (7.a.5)

The velocity distribution given by the above expression can be written in the following alternative form:

\[ u_\phi(r, \theta) = \left[ \frac{\Omega_o \kappa R}{1 - \kappa^3} \left( \frac{r}{\kappa R} - \frac{\kappa^2 R^2}{r^2} \right) + \frac{\kappa^3 R}{1 - \kappa^3} \left( \frac{R^2}{r^2} - \frac{r}{R} \right) \right] \eta \mu \theta \] \hspace{1cm} (7.a.6)

In the above form, the first term on the right-hand-side corresponds to the velocity profile when the inner sphere is held stationary and the outer sphere is rotating with an angular velocity \( \Omega_o \). The second term on the right-hand-side of the above equation corresponds to the velocity profile when the outer sphere is fixed and the inner sphere is rotating with an angular velocity \( \Omega_i \).
7.b  Determination of momentum flux and torque.

From the velocity profile, the momentum flux (shear stress) is determined as

\[
\tau_{r\phi} = -\mu \frac{\partial}{\partial r} \left( \frac{u_{\phi}}{r} \right) = 3\mu \frac{\Omega_i - \Omega_o}{1 - \kappa^3} \frac{\kappa^3 R^3}{r^3} \eta \mu \theta
\]  \hspace{1cm} (7.b.1)

The torque required to rotate the inner sphere is obtained as the integral over the sphere surface of the product of the force exerted on the fluid by the solid surface element and the lever arm \(\kappa R \eta \mu \theta\). The force required on the inner sphere is the product of the outward momentum flux and the wetted surface area of the inner sphere. Thus, the differential torque on the inner sphere surface element is

\[
dT_z = (\tau_{r\phi})|_{r=\kappa R} (2\pi \kappa^2 R^2 \eta \mu \theta d\theta)(\kappa R \eta \mu \theta) = 3\mu \frac{\Omega_i - \Omega_o}{1 - \kappa^3} \eta \mu \theta (2\pi \kappa^2 R^2 \eta \mu \theta d\theta)(\kappa R \eta \mu \theta)
\]  \hspace{1cm} (7.b.2)

On integrating from 0 to \(\pi\), the torque needed on the inner sphere is

\[
T_z = 6\pi \mu (\Omega_i - \Omega_o) \frac{\kappa^3 R^3}{1 - \kappa^3} \int_0^\pi \eta \mu^3 \theta d\theta = 8\pi \mu (\Omega_i - \Omega_o) \frac{\kappa^3 R^3}{1 - \kappa^3}
\]  \hspace{1cm} (7.b.3)

Note that the integral above evaluates to \(4/3\).

Similarly, the differential torque needed on the outer sphere surface element is obtained as the product of the inward momentum flux, the wetted surface area of the outer sphere, and the lever arm. Thus,

\[
dT_z = (\tau_{r\phi})|_{r=R} (2\pi R^2 \eta \mu \theta d\theta)(R \eta \mu \theta) = 3\mu \frac{\Omega_i - \Omega_o}{1 - \kappa^3} \eta \mu \theta (2\pi R^2 \eta \mu \theta d\theta)(R \eta \mu \theta)
\]  \hspace{1cm} (7.b.4)

On integrating from 0 to \(\pi\), the torque needed on the inner sphere is

\[
T_z = 6\pi \mu (\Omega_o - \Omega_i) \frac{\kappa^3 R^3}{1 - \kappa^3} \int_0^\pi \eta \mu^3 \theta d\theta = 8\pi \mu (\Omega_o - \Omega_i) \frac{\kappa^3 R^3}{1 - \kappa^3}
\]  \hspace{1cm} (7.b.5)

7.c  Limiting case of single rotating sphere.

For the case of a single solid sphere of radius \(R_i\) rotating slowly at a constant angular velocity \(\Omega_i\) in a very large body of quiescent fluid, let \(R = R_o\) and \(\kappa = R_i/R_o\). Then, on letting \(\Omega_o = 0\), equations (7.a.6) and (7.b.3) give

\[
u_{\phi}(r, \theta) = \Omega_i \frac{R_i^3}{r^2} \eta \mu \theta \quad \text{and} \quad T_z = 8\pi \mu \Omega_i \frac{R_i^3}{(1/R_i)^3} \frac{1}{(1/R_i)^3}
\]  \hspace{1cm} (7.c.1)

The above equations hold when the outer sphere of radius \(R_o\) is stationary and the inner sphere of radius \(R_i\) is rotating at an angular velocity \(\Omega_i\). In the limit as \(R_o\) tends to infinity, the results for a single rotating sphere in an infinite body of quiescent fluid are obtained as

\[
u_{\phi}(r, \theta) = \Omega_i \frac{R_i^3}{r^2} \eta \mu \theta \quad \text{and} \quad T_z = 8\pi \mu \Omega_i R_i^3
\]  \hspace{1cm} (7.c.2)
Problem 8

An isothermal, incompressible fluid of density $\rho$ flows radially outward owing to a pressure difference between two fixed porous, concentric spherical shells of radii $\kappa R$ and $R$. Note that the velocity is not zero at the solid surfaces. Assume negligible end effects and steady laminar flow in the region $\kappa R \leq r \leq R$.

![Figure 8.1: Radial flow between two porous concentric spheres.](image)

a) Simplify the equation of continuity to show that $r^2 u_r = \text{constant}$.

b) Simplify the equation of motion for a Newtonian fluid of viscosity $\mu$.

c) Obtain the pressure profile $P(r)$ in terms of $P_R$ and $u_R$, the pressure and velocity at the sphere of radius $R$.

d) Determine the nonzero components of the viscous stress tensor for the Newtonian case.

Solution

8.a Simplification of continuity equation.

Since the steady laminar flow is directed radially outward, only the radial velocity component $v_r$ exists. The other two components of velocity are zero; so, $u_\theta = 0$ and $u_\phi = 0$. For an incompressible flow, the continuity equation gives $\nabla \cdot \vec{u} = 0$. In spherical coordinates, this is written in fully expanded form as follows:

$$
\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 u_r) + \frac{1}{r \eta \mu \theta} \frac{\partial}{\partial \theta} (u_\theta \eta \mu \theta) + \frac{1}{r \eta \mu \theta} \frac{\partial}{\partial \phi} (u_\phi) = 0 \Rightarrow \frac{\partial}{\partial r} (r^2 u_r) = 0
$$

After integrating the above simplified continuity equation, we find that $r^2 u_r = f(\theta, \phi)$. However, due to the symmetry present in the problem, there is no dependence expected on the angles $\theta$ and $\phi$. In other words, $r^2 u_r$ must be equal to a constant, $C$. This is simply explained from the fact that mass (or volume, if density $\rho$ is constant) is conserved; so,

$$
\rho (4\pi r^2 u_r) = w \quad \text{is constant, and} \quad C = \frac{w}{4\pi \rho}.
$$

8.b Simplification of the equation of motion for a Newtonian fluid.

The equation of motion is

$$
\rho \frac{D\vec{u}}{Dt} = -\nabla p - \nabla \cdot \tau_{ab} + \rho \vec{g}
$$

The above equation is simply Newton’s second law of motion for a fluid element. It states that, on a per unit volume basis, the mass multiplied by the acceleration is because of three forces, namely the pressure force, the viscous force, and the gravity force. For an incompressible, Newtonian fluid, the term $\nabla \cdot \tau_{ab} = \mu \nabla^2 \vec{u}$ and the equation of motion yields the Navier - Stokes equation. On noting that $r^2 u_r = \text{constant}$ from the continuity equation, the components of the equation of motion for steady flow in spherical coordinates may be simplified as given below.
Here, \( P = p + \rho g h \), where \( h \) is the elevation or the height above some arbitrary datum plane, to avoid calculating the components of the gravitational acceleration vector \( \vec{g} \) in spherical coordinates. Here,

\[
P = p + \rho g r \sigma \nu \theta,
\]
giving

\[
\frac{\partial P}{\partial r} = \frac{\partial p}{\partial r} + \rho g \sigma \nu \theta \quad \text{and} \quad \frac{\partial P}{\partial \theta} = \frac{\partial p}{\partial \theta} - \rho g r \eta \mu \theta.
\]

Then, the components of the equation of motion in terms of the modified pressure are

\[
\begin{align*}
\text{r\textendash component:} & \quad 0 = \rho \left( u_r \frac{\partial u_r}{\partial r} \right) = -\frac{\partial P}{\partial r} \\
\text{\theta\textendash component:} & \quad 0 = \frac{1}{r} \frac{dp}{d\theta} + \rho g \eta \mu \theta \\
\text{\phi\textendash component:} & \quad 0 = \frac{\partial P}{\partial \phi}
\end{align*}
\]

Note that equations (8.b.2) – (8.b.4) could have been directly obtained from the following form of the equation of motion:

\[
\rho \frac{D\vec{u}}{Dt} = -\vec{\nabla} P - \vec{\nabla} \cdot \tau_{ab}
\]

in which the modified pressure \( P \) includes both the pressure and gravitational terms.

From equations (8.b.3) and (8.b.4), \( P \)'s partial derivatives with respect to \( \theta \) and \( \phi \) vanish, therefore \( P \) is a function of only \( r \). Substituting \( P = P(r) \) and \( u_r = u_r(r) \) in equation (8.b.2) then gives

\[
\frac{dP}{dr} = -\rho u_r \frac{du_r}{dr}.
\]

8.c Pressure profile.

To calculate the pressure profile, we start by substituting \( u_r = C/r^2 \) into equation (8.b.6), which then gives

\[
\frac{dP}{dr} = 2\rho \frac{C^2}{r^5}
\]

Integration then produces

\[
P(r) = -\rho C^2 \cdot \frac{2}{2r^4} + C_1
\]

To determine the integration constants, \( C \) and \( C_1 \), we make use of the boundary conditions. Since the pressure is \( P(r = R) = P_R \) and the velocity is \( u_r(r = R) = v_R \), the integration constants may be evaluated as

\[
C_1 = P_R + \rho \frac{C^2}{2R^4} \quad \text{and} \quad C = R^2 v_R
\]

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Then, substitution in equation (8.c.1) yields the pressure profile as

\[ P(r) = P_R + \frac{\rho C^2}{2R^4} \left[ 1 - \left( \frac{R}{r} \right)^4 \right] = P_R + \frac{1}{2} \rho u_R^2 \left[ 1 - \left( \frac{R}{r} \right)^4 \right] \] (8.c.2)

8.d Determination of nonzero components of the viscous stress tensor.

For an incompressible, Newtonian fluid, the viscous stress tensor is given by \( \tau_{ab} = -\mu (\partial_b u_a + \partial_a u_b) \). Since the only velocity component that exists is \( u_r(r) = C/r^2 \), the nonzero components of the viscous stress tensor are

\[ \tau_{rr} = -2\mu \frac{du_r}{dr} = \frac{4\mu C}{r^3} \quad \text{and} \quad \tau_{\theta\theta} = \tau_{\phi\phi} = -2\mu \frac{v_r}{r} = -\frac{2\mu C}{r^3} \] (8.d.1)

Note that the shear stresses are all zero. The normal stresses in equation (8.b.6) are partly compressive and partly tensile. Not only is the \( \phi \)-component of \( \vec{\nabla} \cdot \tau \) directly zero, but the \( r \)-component and the \( \theta \)-component are zero too as evaluated below.

\[ \vec{\nabla} \cdot \tau \mid_r = \frac{1}{r^2} \frac{d}{dr} (r^2 \tau_{rr}) - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r} = -\frac{4\mu C}{r^4} + \frac{4\mu C}{r^4} = 0 \] (8.d.2)

\[ \vec{\nabla} \cdot \tau \mid_\theta = \frac{1}{r \eta \mu \theta} \frac{\partial}{\partial \theta} (\tau_{\theta\theta} \eta \mu \theta) - \frac{\tau_{\phi\phi} \sigma \phi \theta}{r} = -\frac{2\mu C \sigma \phi \theta}{r} + \frac{2\mu C \sigma \phi \theta}{r} = 0 \] (8.d.3)

Since \( \vec{\nabla} \cdot \tau = 0 \), viscous forces can be neglected in this flow and the Euler equation for inviscid fluids may be directly used.
**Problem 9**

A parallel – disk viscometer consists of two circular disks of radius $R$ separated by a small gap $B$ (with $R \gg B$). A fluid of constant density $\rho$, whose viscosity $\mu$ is to be measured, is placed in the gap between the disks. The lower disk at $z = 0$ is fixed. The torque $T_z$ necessary to rotate the upper disk (at $z = B$) with a constant angular velocity $\Omega$ is measured. The task here is to deduce a working equation for the viscosity when the angular velocity $\Omega$ is small (creeping flow).

![Figure 9.1: Parallel – disk viscometer.](image)

Figure 9.1: Parallel – disk viscometer.

a) Simplify the equations of continuity and motion to describe the flow in the parallel – disk viscometer.
b) Obtain the tangential velocity profile after writing down appropriate boundary conditions.
c) Derive the formula for determining the viscosity $\mu$ of a Newtonian fluid from measurements of the torque $T_z$ and angular velocity $\Omega$ in a parallel – disk viscometer. Neglect the pressure term.

**Solution**

9.a  **Simplification of the continuity equation.**

In steady laminar flow, the fluid is expected to travel in a circular motion (for small values of $\Omega$). Only the tangential component of velocity exists. The radial and axial components of velocity are zero; so, $u_r = 0$ and $u_z = 0$.

For an incompressible fluid, the continuity equation gives $\nabla \cdot \vec{u} = 0$.

In cylindrical coordinates,

$$
\frac{1}{r} \frac{\partial}{\partial r} (ru_r) + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial u_\theta}{\partial \theta} = 0
$$

So, $u_\theta = u_\theta(r, z)$.

Next, we simplify the equation of motion. This is none other than the Navier–Stokes equation. For a Newtonian fluid, the components of the Navier – Stokes equation may be simplified as given below.

$$
\begin{align*}
\text{r – component:} & \quad 0 = -\rho u_\theta^2/r = -\frac{\partial p}{\partial r} \quad (9.a.1) \\
\text{\theta – component:} & \quad 0 = \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (ru_\theta) \right) \quad (9.a.2) \\
\text{z – component:} & \quad 0 = -\frac{\partial p}{\partial z} - \rho g \quad (9.a.3)
\end{align*}
$$

The symmetry inherent in the problem suggests that there is no pressure gradient in the $\theta$-direction; so, $p = p(r, z)$.

9.b  **Velocity profile.**

The no-slip boundary conditions at the two disk surfaces are

$$
\begin{align*}
\text{at } r = R & \quad u_\theta(r, z = 0) = 0 \quad (9.b.1) \\
\text{at } r = B & \quad u_\theta(r, z = B) = \Omega r \quad (9.b.2)
\end{align*}
$$
It is possible to make an educated guess for the form of the tangential velocity taking a clue from equation (9.b.2). Thus, it is reasonable to postulate \( u_{\theta}(r, z) = rf(z) \). Substituting this form in equation (9.a.2) then gives
\[
\frac{d^2 f(z)}{dz^2} = 0 \Rightarrow \frac{df(z)}{dz} = C_1 \Rightarrow f(z) = C_1 z + C_2
\] (9.b.3)

Equation (9.b.1) gives \( f(0) = 0 \), therefore \( C_2 = 0 \). On the other hand, equation (9.b.2) gives \( f(B) = \Omega \), and therefore \( C_1 = \Omega / B \). On substituting \( f(z) = \Omega z / B \), the tangential velocity profile is ultimately obtained as
\[
u_{\theta}(r, z) = \Omega B r z
\] (9.b.4)

The above result is expected and could have been guessed by considering a fluid contained between two parallel plates (with the lower plate at \( z = 0 \) held stationary and the upper plate at \( z = B \) moving at a constant velocity \( V \)). Then, the linear velocity distribution is given by \( u = V z / B \), which is the velocity profile at any \( r \) in the parallel–disk viscometer. On noting that \( V = \Omega r \) at the upper plate, equation (9.b.4) is obtained.

### 9.c Determination of torque.

From the velocity profile in equation (9.b.4), the momentum flux (shear stress) is determined as
\[
\tau_{z\theta} = -\mu \left( \frac{\partial u_{\theta}}{\partial z} \right) = -\frac{\mu \Omega}{B} r \quad \text{and} \quad \tau_{r\theta} = -\mu r \frac{\partial}{\partial r} \left( \frac{u_{\theta}}{r} \right) = 0
\] (9.c.1)

The differential torque \( d\vec{T} \) is rigorously obtained as the cross product of the lever arm \( r \) and the differential force \( d\vec{F} \) exerted on the fluid by a solid surface element. Thus, \( d\vec{T} = \vec{r} \times d\vec{F} \), where \( d\vec{F} = \hat{n} \cdot (\vec{p} + \tau) dS \), \( \vec{r} \) is the vector drawn from the axis of rotation to the element of surface area \( dS \), \( \hat{n} \) is the unit normal to the surface, \( p \) is the pressure, \( \delta \) is the identity (unit) tensor, and \( \tau \) is the viscous stress tensor. For the parallel–disk viscometer,
\[
d\vec{T} = \vec{r} \times (\hat{n} \cdot \tau) dS = r\delta \times (-\delta_z) \cdot \tau_{z\theta} \delta \cdot \delta \cdot (2\pi r dr)
\] (9.c.2)

The pressure term is neglected in equation (9.c.2). Since \( \delta_z \cdot \delta_z = 1 \) and \( \delta_r \times \delta_\theta = \delta_z \), we obtain
\[
d\vec{T} = r(-\tau_{z\theta})(2\pi r dr)\delta_z
\]

Thus, the \( z \)-component of the differential torque is \( dT_z = r(-\tau_{z\theta})(2\pi r dr) \), which is simply the product of the lever arm, the momentum flux and the differential surface area. Integrating over the area of the upper disk after substituting for \( \tau_{z\theta} \) from equation (9.c.1) yields
\[
T_z(z = B) = -\int_0^R \tau_{z\theta}(z = B) 2\pi r^2 dr = \frac{2\pi \mu \Omega}{B} \int_0^R r^3 dr = \frac{\pi \mu \Omega R^4}{2B}
\] (9.c.3)

Equation (9.c.3) provides the formula for determining the viscosity \( \mu \) of a Newtonian fluid from measurements of the torque \( T_z \) and angular velocity \( \Omega \) in a parallel–disk viscometer as \( \mu = 2BT_z/(\pi \Omega R^4) \). Note that the shear stress is not uniform [as given by equation (9.c.1)] in a parallel–disk viscometer, whereas it is uniform throughout the gap in a cone–and–plate viscometer.
Problem 10

Consider a fluid (of density \( \rho \)) in incompressible, laminar flow in a plane narrow slit of length \( L \) and width \( W \) formed by two flat parallel walls that are a distance \( 2B \) apart. End effects may be neglected because \( B \ll W \ll L \). The fluid flows under the influence of a pressure difference \( \Delta p \), gravity or both.

\[
\text{Figure 10.1: Fluid flow in plane narrow slit.}
\]

a) Determine the steady-state velocity distribution for a non-Newtonian fluid that is described by the Bingham model.

b) Obtain the mass flow rate for a Bingham fluid in slit flow.

Solution

10.a Shear stress distribution.

For axial flow in rectangular Cartesian coordinates, the differential equation for the momentum flux is

\[
\frac{d\tau_{xz}}{dx} = \frac{\Delta P}{L} \tag{10.a.1}
\]

where \( P \) is a modified pressure, which is the sum of both the pressure and gravity terms, i.e., \( \Delta P = \Delta p + \rho g L \sigma \nu \beta \). Here, \( \beta \) is the angle of inclination of the \( z \)-axis with the vertical.

On integration, this gives the expression for the shear stress \( \tau_{xz} \) for laminar flow in a plane narrow slit as

\[
\tau_{xz} = \frac{\Delta P}{L} x \tag{10.a.2}
\]

It must be noted that the above momentum flux expression holds for both Newtonian and non-Newtonian fluids (and does not depend on the type of fluid). Further, \( \tau_{xz}(x = 0) = 0 \) in slit flow based on symmetry arguments, i.e., the velocity profile is symmetric about the midplane \( x = 0 \).

The Bingham model

For viscoplastic materials (e.g., thick suspensions and pastes), there is no flow until a critical stress (called the yield stress \( \tau_0 \)) is reached. To describe such a material that exhibits a yield stress, the simplest model is the Bingham model given below.

\[
\eta \to \infty \quad \text{or} \quad \frac{d\nu_z}{dx} = 0, \quad \text{if} \quad |\tau_{xz}| \leq \tau_0 \tag{10.a.3}
\]

\[
\eta = \mu_0 + \frac{\tau_0}{\pm d\nu_z/dx} \quad \text{or} \quad \tau_{xz} = \mp \mu_0 \frac{d\nu_z}{dx} \pm \tau_0, \quad \text{if} \quad |\tau_{xz}| \geq \tau_0 \tag{10.a.4}
\]
Here, \( \eta \) is the non-Newtonian viscosity and \( \mu_0 \) is a Bingham model parameter with units of viscosity. In equation (10.a.4), the positive sign is used with \( \tau_0 \) and the negative sign with \( du_z/dx \) when \( \tau_{xz} \) is positive. On the other hand, the negative sign is used with \( \tau_0 \) and the positive sign with \( du_z/dx \) when \( \tau_{xz} \) is negative.

**Inner plug-flow region**

Let \( \tau_{xz} = +\tau_0 \) at \( x = +x_0 \). Then, from equation (10.a.2), \( x_0 \) is given by \( \tau_0 = (\Delta P/L)x_0 \). Let the inner region \((-x_0 \leq x \leq x_0)\) where the shear stress is less than the yield stress \((-\tau_0 \leq \tau_{xz} \leq \tau_0)\) be denoted by subscript \( i \). Since \( \tau_{xz} \) is finite and \( \tau_{xz} = -\eta(du_z/dx) \), the Bingham model [as per equation (10.a.3)] gives \( (du_z/dx) = 0 \). Integration gives \( u_{zi} = C_1 \) (constant velocity), which implies the fluid is in plug flow in the inner region.

**Outer region**

Let the outer region be denoted by subscript \( o \). For the region \( x_0 \leq x \leq B \), the velocity decreases with increasing \( x \) and therefore \( du_z/dx \leq 0 \). On the other hand, for the region \(-B \leq x \leq -x_0 \), the velocity decreases with decreasing \( x \) and therefore \( du_z/dx \geq 0 \). Thus, the Bingham model according to equation (10.a.4) is given by

\[
\tau_{xz} = -\mu_0 \frac{du_{zo}}{dx} + \tau_0, \quad \text{or} \quad \eta = \mu_0 + \frac{\tau_0}{-du_{zo}/dx}, \quad \text{for} \quad x_0 \leq x \leq B \tag{10.a.5}
\]

\[
\tau_{xz} = -\mu_0 \frac{du_{zo}}{dx} - \tau_0, \quad \text{or} \quad \eta = \mu_0 + \frac{\tau_0}{+du_{zo}/dx}, \quad \text{for} \quad -B \leq x \leq -x_0 \tag{10.a.6}
\]

To obtain the velocity profile for \( x_0 \leq x \leq B \), equations (10.a.2) and (10.a.5) may be combined to eliminate \( \tau_{xz} \) and get

\[
\frac{du_{zo}}{dx} = -\frac{\Delta P}{\mu_0 L} x + \frac{\tau_0}{\mu_0}, \quad \text{for} \quad x_0 \leq x \leq B \tag{10.a.7}
\]

Integration gives

\[
u_{zo} = -\frac{\Delta P}{2\mu_0 L} x^2 + \frac{\tau_0}{\mu_0} x + C_2
\]

Imposing the boundary condition that \( u_{zo}(x = B) = 0 \) then yields

\[
C_2 = \frac{\Delta PB^2}{2\mu_0 L} - \frac{\tau_0 B}{\mu_0}
\]

Thus,

\[
u_{zo}(x) = \frac{\Delta PB^2}{2\mu_0 L} \left[ 1 - \left( \frac{x}{B} \right)^2 \right] - \frac{\tau_0 B}{\mu_0} \left( 1 - \frac{x}{B} \right), \tag{10.a.8}
\]

**Velocity distribution**

For the inner plug-flow region, it was earlier found that \( u_{zi} = C_1 \). To determine \( C_1 \), it may be noted that \( u_{zi}(x = x_0) = u_{zo} \). Then, equation (10.a.8) along with \( \tau_0 = (\Delta P/L)x_0 \) may be simplified to give

\[
C_1 = \frac{\Delta PB^2}{2\mu_0 L} \left( 1 - \frac{x_0}{B} \right)^2
\]

Thus, the final results for the velocity profile are

\[
u_{zi}(x) = \frac{\Delta PB^2}{2\mu_0 L} \left( 1 - \frac{x_0}{B} \right)^2, \quad \text{for} \quad |x| \leq x_0 \tag{10.a.9}
\]

\[
u_{zo}(x) = \frac{\Delta PB^2}{2\mu_0 L} \left[ 1 - \left( \frac{x}{B} \right)^2 \right] - \frac{\tau_0 B}{\mu_0} \left( 1 - \frac{|x|}{B} \right), \quad \text{for} \quad x_0 \leq |x| \leq B \tag{10.a.10}
\]

The velocity profile is flat in the inner region as given by equation (10.a.9) and is parabolic in the outer region as given by equation (10.a.10).
10.b General expression for mass flow rate.

Since the velocity profile is symmetric about the midplane $x = 0$, the mass flow rate may be obtained by integrating the velocity profile over half the cross section of the slit, as shown below.

\[ w = \int_{-B}^{B} \rho u_z W dx = 2 \int_{0}^{B} \rho u_z W dx \quad (10.b.1) \]

Rather than insert two separate expressions from equations (10.a.9) and (10.a.10) for $u_z$ and integrate in two regions, it is easier to integrate by parts.

\[ w = 2 W \rho \left[ u_z x \right]_{0}^{B} - \int_{0}^{B} x \frac{du_z}{dx} dx \]

\[ = 2 W \rho \int_{0}^{B} x \left( -\frac{du_z}{dx} \right) dx \quad (10.b.2) \]

The first term in the square brackets above is zero at both limits on using the no-slip boundary condition $(u_z(x = B) = 0)$ at the upper limit. From equation (10.a.2), $x/B = \tau_xz/\tau_B$ where $\tau_B = \Delta PB/L$ is the wall shear stress. Thus, a general expression for the mass rate of flow in a plane narrow slit is

\[ w = \frac{2WB^2}{\tau_B^2} \int_{0}^{\tau_B} \tau_xz \left( -\frac{du_z}{dx} \right) d\tau_xz. \]

\[ (10.b.3) \]

**Mass flow rate for Bingham fluid** For a material with a yield stress, the lower limit of integration is reset to $\tau_0$ as per equation (10.a.3). Then, substituting equation (10.a.5) and integrating yields

\[ w = \frac{2WB^2}{\mu_0\tau_B^2} \int_{0}^{\tau_B} \tau_xz (\tau_xz - \tau_0) d\tau_xz = \frac{2WB^2}{\mu_0\tau_B^2} \left[ \frac{1}{3} (\tau_B^3 - \tau_0^3) - \frac{1}{2} (\tau_0^2 - \tau_0^3) \right] \]

\[ (10.b.4) \]

The final expression for the mass flow rate is

\[ w = \frac{2\Delta PB^3}{3\mu_0 L} \left[ 1 - \frac{3\tau_0}{2\tau_B} + \frac{1}{2} \left( \frac{\tau_0}{\tau_B} \right)^3 \right] \]

\[ (10.b.5) \]

Note that $\tau_B = \Delta PB/L$ is the wall shear stress and $\tau_0$ is the yield stress. Since no flow occurs below the yield stress (that is, when $\tau_B \leq \tau_0$), the above expression is valid only for $\tau_B > \tau_0$. For $\tau_0 = 0$ and $\mu_0 = \mu$, the Bingham model simplifies to the Newtonian model and equation (10.b.5) reduces to the Newtonian result, i.e.,

\[ w = 2\Delta PB^3 W/(3\mu L). \]

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Problem 11

Consider a fluid (of density \( \rho \)) in incompressible, laminar flow in a long circular tube of radius \( R \) and length \( L \). End effects may be neglected because the tube length \( L \) is relatively large compared to the tube radius \( R \). The fluid flows under the influence of a pressure difference \( \Delta p \), gravity or both.

Figure 11.1: Fluid flow in circular tube.

a) Determine the steady-state velocity distribution for a non-Newtonian fluid that is described by the Bingham model.

b) Obtain the mass rate of flow for a Bingham fluid in a circular tube.

Solution

11.a Shear stress distribution.

For axial flow in cylindrical coordinates, the differential equation for the momentum flux is

\[
\frac{d}{dr}(r\tau_{rz}) = \frac{\Delta P}{L} r
\]

where \( P \) is a modified pressure, which is the sum of both the pressure and gravity terms, i.e., \( \Delta P = \Delta p + \rho g L \sigma \nu \beta \). Here, \( \beta \) is the angle of inclination of the tube axis with the vertical.

On integration (with the condition that \( \tau_{rz} \) must be finite at \( r = 0 \)), this gives the expression for the shear stress \( \tau_{rz} \) for steady laminar flow in a circular tube as

\[
\tau_{rz} = \frac{\Delta P}{2L} r
\]

It must be noted that the above momentum flux expression holds for both Newtonian and non-Newtonian fluids (and does not depend on the type of fluid).

The Bingham model

For viscoplastic materials (e.g., thick suspensions and pastes), there is no flow until a critical stress (called the yield stress \( \tau_0 \)) is reached. To describe such a material that exhibits a yield stress, the simplest model is the Bingham model given below.

\[
\eta \rightarrow \infty \quad \text{or} \quad \frac{du_z}{dr} = 0, \quad \text{if} \quad |\tau_{rz}| \leq \tau_0
\]

\[
\eta = \mu_0 + \frac{\tau_0}{\pm du_z/dr}, \quad \tau_{rz} = -\mu_0 \frac{du_z}{dr} \pm \tau_0, \quad \text{if} \quad |\tau_{rz}| \geq \tau_0
\]

Here, \( \eta \) is the non-Newtonian viscosity and \( \mu_0 \) is a Bingham model parameter with units of viscosity. In equation (11.a.4), the positive sign is used with \( \tau_0 \) and the negative sign with \( du_z/dx \) when \( \tau_{rz} \) is positive. On the other hand, the negative sign is used with \( \tau_0 \) and the positive sign with \( du_z/dx \) when \( \tau_{rz} \) is negative.
**Inner plug-flow region**

Let $\tau_{rz} = +\tau_0$ at $r = r_0$. Then, from equation (11.a.2), $r_0$ is given by $\tau_0 = (\Delta P/2L)r_0$. Let the inner region ($r \leq r_0$) where the shear stress is less than the yield stress ($\tau_{rz} \leq \tau_0$) be denoted by subscript $i$. Since $\tau_{rz}$ is finite and $\tau_{rz} = -\eta(du_z/dr)$, the Bingham model [as per equation (11.a.3)] gives $(du_z/dr) = 0$. Integration gives $u_z = C_1$ (constant velocity), which implies the fluid is in plug flow in the inner region.

**Outer region**

Let the outer region be denoted by subscript $o$. For the region $r_0 \leq r \leq R$, the velocity decreases with increasing $r$ and therefore $du_z/dr \leq 0$ and $\tau_{rz} \geq \tau_0$. Thus, the Bingham model according to equation (11.a.4) is given by

$$\tau_{rz} = -\mu_0 \frac{du_{zo}}{dr} + \tau_0, \quad \text{or} \quad \eta = \mu_0 + \frac{\tau_0}{-\frac{du_{zo}}{dr}}, \quad \text{for} \quad r_0 \leq r \leq R \quad (11.a.5)$$

Integration gives

$$u_{zo} = -\frac{\Delta P}{2\mu_0 L} r + \frac{\tau_0}{\mu_0} r + C_2$$

Imposing the boundary condition that $u_{zo}(r = R) = 0$ then yields

$$C_2 = \frac{\Delta PR^2}{4\mu_0 L} - \frac{\tau_0 R}{\mu_0}$$

Thus the velocity distribution in the outer region is

$$u_{zo}(r) = \frac{\Delta PR^2}{4\mu_0 L} \left[ 1 - \left( \frac{r}{R} \right)^2 \right] - \frac{\tau_0 R}{\mu_0} \left( 1 - \frac{r}{R} \right) \quad (11.a.7)$$

For the inner plug-flow region, it was earlier found that $u_z = C_1$. To determine $C_1$, it may be noted that $u_z(r = r_0) = u_{zo}$. Then, equation (11.a.7) along with $\tau_0 = (\Delta P/2L)r_0$ may be simplified to give

$$C_1 = \frac{\Delta PR^2}{4\mu_0 L} \left( 1 - \frac{r_0}{R} \right)^2$$

Thus, the velocity of the plug flow in the inner region is

$$u_z(r) = \frac{\Delta PR^2}{4\mu_0 L} \left( 1 - \frac{r_0}{R} \right)^2, \quad \text{for} \quad |r| \leq r_0 \quad (11.a.8)$$

Here, $r_0 = 2L\tau_0/\Delta P$ is the radius of the plug-flow region in the central part of the tube. The velocity profile is parabolic in the outer region as given by equation (11.a.7) and is flat in the inner region as given by equation (11.a.8).

**11.b General expression for mass flow rate.**

The mass flow rate may be obtained by integrating the velocity profile over the cross section of the circular tube as shown below.

$$w = 2\pi \rho \int_0^R u_z r dr \quad (11.b.1)$$

Rather than insert two separate expressions from equations (11.a.7) and (11.a.8) for $u_z$ and integrate in two regions, it is easier to integrate by parts.
\[ w = 2\pi \rho \left[ u_z \frac{r^2}{2} \left|_0^R \right. - \frac{1}{2} \int_0^R r^2 \frac{du_z}{dr} \right] = \pi \rho \int_0^R r^2 \left( -\frac{du_z}{dr} \right) dr \]  

(11.b.2)

The first term in the square brackets above is zero at both limits on using the no-slip boundary condition \( u_z(r = R) = 0 \) at the upper limit. From equation (11.a.2), \( r/R = \tau_{rz}/\tau_R \) where \( \tau_R = \Delta PR/2L \) is the wall shear stress. Thus, a general expression for the mass rate of flow in a circular tube is

\[ w = \frac{\pi R^3 \rho}{\tau_R^3} \int_0^{\tau_R} \tau_{rz}^2 \left( -\frac{du_z}{dr} \right) d\tau_{rz}. \]  

(11.b.3)

**Buckingham - Reiner equation for mass flow rate of Bingham fluid** For a material with a yield stress, the lower limit of integration is reset to \( \tau_0 \) as per equation (11.a.3). Then, substituting equation (11.a.4) and integrating yields

\[ w = \frac{\pi R^3 \rho}{\mu_0 \tau_R^3} \int_0^{\tau_R} \tau_{rz}^2 (\tau_{rz} - \tau_0) d\tau_{rz} = \frac{\pi R^3 \rho}{\mu_0 \tau_R^3} \left[ \frac{1}{4} (\tau_R^4 - \tau_0^4) \right] \]  

(11.b.4)

The final expression for the mass flow rate is

\[ w = \frac{\pi \Delta PR^4 \rho}{8\mu_0 L} \left[ 1 - \frac{4\tau_0}{3\tau_B} + \frac{1}{3} \left( \frac{\tau_0}{\tau_R} \right)^4 \right] \]  

(11.b.5)

The above expression is known as the Buckingham - Reiner equation. Note that \( \tau_R = \Delta PR/(2L) \) is the wall shear stress and \( \tau_0 \) is the yield stress. Since no flow occurs below the yield stress (that is, when \( \tau_R \leq \tau_0 \)), the above expression is valid only for \( \tau_R > \tau_0 \). For \( \tau_0 = 0 \) and \( \mu_0 = \mu \), the Bingham model simplifies to the Newtonian model and equation (11.b.5) reduces to the Hagen - Poiseuille equation, i.e., \( w = \pi \Delta PR^4 \rho/(8\mu_0 L) \).
Problem 12

Consider a fluid (of density $\rho$) in incompressible, laminar flow in a plane narrow slit of length $L$ and width $W$ formed by two flat parallel walls that are a distance $2B$ apart. End effects may be neglected because $B \ll W \ll L$. The fluid flows under the influence of a pressure difference $\Delta p$, gravity or both.

![Fluid flow in plane narrow slit.](image)

**Solution**

12.a  Steady-state velocity distribution.

For axial flow in rectangular Cartesian coordinates, the differential equation for the momentum flux is

$$\frac{d\tau_{xz}}{dx} = \frac{\Delta P}{L} \quad (12.a.1)$$

where $P$ is a modified pressure, which is the sum of both the pressure and gravity terms, i.e., $\Delta P = \Delta p + \rho g L \sigma v u \beta$. Here, $\beta$ is the angle of inclination of the $z$-axis with the vertical.

On integration, this gives the expression for the shear stress $\tau_{xz}$ for laminar flow in a plane narrow slit as

$$\tau_{xz} = \frac{\Delta P}{L} x \quad (12.a.2)$$

It must be noted that the above momentum flux expression holds for both Newtonian and non-Newtonian fluids (and does not depend on the type of fluid). Further, $\tau_{xz} = 0$ at $x = 0$ in slit flow based on symmetry arguments, i.e., the velocity profile is symmetric about the midplane $x = 0$.

For the region $0 \leq x \leq B$, the velocity decreases with increasing $x$ and therefore $\frac{du_z}{dx} \leq 0$. On the other hand, for the region $B \leq x \leq 0$, the velocity decreases with decreasing $x$ and therefore $\frac{du_z}{dx} \geq 0$. Thus, the power law model is given as

$$\tau_{xz} = \begin{cases} m \left( -\frac{du_z}{dx} \right)^n & \text{for } 0 \leq x \leq B, \\ -m \left( \frac{du_z}{dx} \right)^n & \text{for } -B \leq x \leq 0 \end{cases} \quad (12.a.3)$$

**Velocity distribution**

To obtain the velocity distribution for $0 \leq x \leq B$, equations (12.a.2) and the first of (12.a.3) are combined to eliminate $\tau_{xz}$ and get
\[-\frac{du_z}{dx} = \left(\frac{\Delta P x}{mL}\right)^{1/n} \quad \text{(for } 0 \leq x \leq B) \quad (12.a.4)\]

On integrating and using the no-slip boundary condition \((u_z = 0 \text{ at } x = B)\), we get

\[u_z = \left(\frac{\Delta PB}{mL}\right)^{1/n} \frac{B}{1 + 1/n} \left[ 1 - \left(\frac{x}{B}\right)^{1+1/n} \right] \quad \text{(for } 0 \leq x \leq B) \quad (12.a.5)\]

Similarly, for \(B \leq x \leq 0\), equations (12.a.2) and the second of (12.a.3) may be combined to eliminate \(\tau_{xz}\) and obtain

\[-\frac{du_z}{dx} = \left(\frac{\Delta P x}{mL}\right)^{1/n} \quad \text{(for } -B \leq x \leq 0) \quad (12.a.6)\]

Again, integrating and using the no-slip boundary condition \((u_z = 0 \text{ at } x = B)\) gives

\[u_z = \left(\frac{\Delta PB}{mL}\right)^{1/n} \frac{B}{1 + 1/n} \left[ 1 - \left(\frac{|x|}{B}\right)^{1+1/n} \right] \quad \text{(for } -B \leq x \leq 0) \quad (12.a.7)\]

Equations (12.a.5) and (12.a.7) may be combined and represented as a single equation for the velocity profile as follows:

\[u_z = \left(\frac{\Delta PB}{mL}\right)^{1/n} \frac{B}{1 + 1/n} \left[ 1 - \left(\frac{|x|}{B}\right)^{1+1/n} \right] \quad \text{(for } |x| \leq 0) \quad (12.a.8)\]

Note that the maximum velocity in the slit occurs at the midplane \(x = 0\).

### 12.b General expression for mass flow rate.

Since the velocity profile is symmetric about the midplane \(x = 0\), the mass flow rate may be obtained by integrating the velocity profile over half the cross section of the slit as shown below.

\[w = 2 \int_0^B \rho u_z W dx = 2 \left(\frac{\Delta PB}{mL}\right)^{1/n} \frac{B^2 W \rho}{1 + 1/n} \int_0^1 \left[ 1 - \left(\frac{x}{B}\right)^{1+1/n} \right] d\left(\frac{x}{B}\right) \quad (12.b.1)\]

which results in

\[w = \frac{2B^2 W \rho}{2 + 1/n} \left(\frac{\Delta PB}{mL}\right)^{1/n} \quad (12.b.2)\]

When the mass flow rate \(w\) in equation (12.b.2) is divided by the density \(\rho\) and the cross-sectional area \((2BW)\), an expression for the average velocity is obtained.

For \(n = 1\) and \(m = \mu\), equation (12.b.2) reduces to the Newtonian result, i.e., \(w = 2\Delta PB^3 W \rho/(3\mu L)\).
Problem 13

Consider a Newtonian liquid (of viscosity $\mu$ and density $\rho$) in laminar flow down an inclined flat plate of length $L$ and width $W$. The liquid flows as a falling film with negligible rippling under the influence of gravity. End effects may be neglected because $L$ and $W$ are large compared to the film thickness $\delta$.

Figure 13.1: Newtonian liquid flow in a falling film.

a) Determine the steady-state velocity distribution.
b) Obtain the mass rate of flow and average velocity in the falling film.
c) What is the force exerted by the liquid on the plate in the flow direction?
d) Derive the velocity distribution and average velocity for the case where $x$ is replaced by a coordinate $x'$ measured away from the plate (i.e., $x' = 0$ at the plate and $x' = \delta$ at the liquid-gas interface).

Solution

13.a  Shear stress distribution.

For axial flow in rectangular Cartesian coordinates, the differential equation for the momentum flux is

$$\frac{d\tau_{xz}}{dx} = \frac{\Delta P}{L}$$

where $P$ is a modified pressure, which is the sum of both the pressure and gravity terms, that is, $\Delta P = \Delta p + \rho g L \sigma v u \beta$. Here, $\beta$ is the angle of inclination of the $z$-axis with the vertical. Since the flow is solely under the influence of gravity, $\Delta p = 0$ and therefore $\Delta P/L = \rho g \sigma v u \beta$. On integration,

$$\tau_{xz} = \rho g x \sigma v u \beta + C_1$$

On using the boundary condition at the liquid-gas interface ($\tau_{xz} = 0$ at $x = 0$), the constant of integration $C_1$ is found to be zero. This gives the expression for the shear stress $\tau_{xz}$ for laminar flow in a falling film as

$$\tau_{xz}(x) = \rho g x \sigma v u \beta$$

It must be noted that the above momentum flux expression holds for both Newtonian and non-Newtonian fluids (and does not depend on the type of fluid).

The shear stress for a Newtonian fluid (as per Newton’s law of viscosity) is given by

$$\tau_{xz} = -\mu \frac{du_z}{dx}$$

Velocity distribution

To obtain the velocity distribution, there are two possible approaches.

In the first approach, equations (13.a.3) and (13.a.4) are combined to eliminate $\tau_{xz}$ and get a first-order differential equation for the velocity as given below.

$$\frac{du_z}{dx} = -\frac{\rho g \sigma v u \beta}{\mu} x$$

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On integrating, \( u_z = \rho g x^2 \sigma \nu \beta / 2\mu + C_2 \). On using the no-slip boundary condition at the solid surface \( (u_z = 0 \text{ at } x = \delta) \), \( C_2 = \rho g \delta^2 \sigma \nu \beta \). Thus, the final expression for the velocity distribution is parabolic as given below.

\[
    u_z = \frac{\rho g \delta^2 \sigma \nu \beta}{2\mu} \left[ 1 - \left( \frac{x}{\delta} \right)^2 \right] \quad (13.a.6)
\]

In the second approach, equations (13.a.1) and (13.a.4) are directly combined to eliminate \( \tau_{xz} \) and get a second-order differential equation for the velocity as given below.

\[-\mu \frac{d^2 u_z}{dx^2} = \rho g \sigma \nu \beta \quad (13.a.7)\]

Integration gives

\[
    \frac{du_z}{dx} = -\frac{\rho g \sigma \nu \beta}{\mu} x + C_1 \quad (13.a.8)
\]

On integrating again,

\[
    u_z = -\frac{\rho g x^2 \sigma \nu \beta}{2\mu} + C_1 x + C_2 \quad (13.a.9)
\]

On using the boundary condition at the liquid-gas interface \( (x = 0, \tau_{xz} = 0 \Rightarrow du_z/dx = 0) \), the constant of integration \( C_1 \) is found to be zero from equation (13.a.8). On using the other boundary condition at the solid surface \( (x = \delta, u_z = 0) \), equation (13.a.9) gives \( C_2 = \rho g \delta^2 \sigma \nu \beta / 2\mu \). On substituting \( C_1 \) and \( C_2 \) in equation (13.a.9), the parabolic velocity profile in equation (13.a.6) is again obtained.

13.b Mass flow rate.

The mass flow rate may be obtained by integrating the velocity profile given by equation (13.a.6) over the film thickness as shown below.

\[
    w = \int_0^\delta \rho u_z W dx = \frac{\rho^2 g W \delta^3 \sigma \nu \beta}{2\mu} \int_0^1 \left[ 1 - \left( \frac{x}{\delta} \right)^2 \right] d \left( \frac{x}{\delta} \right) \Rightarrow \quad (13.b.1)
\]

\[
    w = \frac{\rho^2 g W \delta^3 \sigma \nu \beta}{3\mu} \quad (13.b.2)
\]

When the mass flow rate \( w \) in equation (13.b.2) is divided by the density \( \rho \) and the film cross-sectional area \( (W \cdot \delta) \), an expression for the average velocity \( <u_z> \) is obtained. Also, the maximum velocity \( u_{z,max} \) occurs at the interface \( x = 0 \) and is readily obtained from equation (13.a.6). Thus, equations (13.a.6) and (13.b.2) give

\[
    <u_z> = \frac{\rho g \delta^2 \sigma \nu \beta}{3\mu} = \frac{2}{3} u_{z,max} \quad (13.b.3)
\]

13.c Force on the plate.

The \( z \)-component of the force of the fluid on the solid surface is given by the shear stress integrated over the wetted surface area. Using equation (13.a.3) gives

\[
    F_z = LW \tau_{xz}(x = \delta) = \rho g \delta LW \sigma \nu \beta \quad (13.c.1)
\]

As expected, the force exerted by the fluid on the plate is simply the \( z \)-component of the weight of the liquid film.
Change of coordinates.

Consider the coordinate \( x' \) starting from the plate (i.e., \( x' = 0 \) is the plate surface and \( x' = \delta \) is the liquid-gas interface). The form of the differential equation for the momentum flux given by equation (13.a.1) will remain unchanged. Thus, equation (13.a.2) for the new choice of coordinates will be

\[
\tau_{x'z} = \rho g x' \sigma vv \beta + C_1
\]  

(13.d.1)

On using the boundary condition at the liquid-gas interface \((\tau_{x'z} = 0 \text{ at } x' = \delta)\), the constant of integration is now found to be given by \( C_1 = g \delta \sigma vv \beta \). This gives the shear stress as

\[
\tau_{x'z} = \rho g (x' - \delta) \sigma vv \beta
\]  

(13.d.2)

Note that the momentum flux is in the negative \( x' \)-direction. On substituting Newton’s law of viscosity,

\[
\frac{du_z}{dx'} = \frac{\rho g \sigma vv \beta}{\mu}(\delta - x')
\]  

(13.d.3)

On integrating,

\[
u_z = \frac{\rho g}{\mu} \left( \delta x' \frac{1}{2} x'^2 \right) \sigma vv \beta + C_2
\]

On using the no-slip boundary condition at the solid surface \((u_z = 0 \text{ at } x' = 0)\), it is found that \( C_2 = 0 \). Thus, the final expression for the velocity distribution is

\[
u_z = \frac{\rho g \delta^2 \sigma vv \beta}{\mu} \left[ \frac{x'}{\delta} - \frac{1}{2} \left( \frac{x'}{\delta} \right)^2 \right]
\]  

(13.d.4)

The average velocity may be obtained by integrating the velocity profile given in equation (13.d.4) over the film thickness as shown below.

\[
\omega = \frac{1}{\delta} \int_0^\delta u_z dx' = \frac{\rho g \delta^2 \sigma vv \beta}{\mu} \int_0^1 \left[ x' - \frac{1}{2} \left( \frac{x'}{\delta} \right)^2 \right] d \left( \frac{x'}{\delta} \right)
\]  

(13.d.5)

Integration of equation (13.d.5) yields the same expression for \( <u_z> \) obtained earlier in equation (13.b.3). It must be noted that the two coordinates \( x \) and \( x' \) are related as follows: \( x + x' = \delta \). By substituting

\[
\left( \frac{x'}{\delta} \right)^2 = \left[ 1 - \left( \frac{x'}{\delta} \right) \right]^2 = 1 - 2 \left( \frac{x'}{\delta} \right) + \left( \frac{x'}{\delta} \right)^2
\]

equation (13.d.4) is directly obtained from equation (13.a.6).
**Problem 14**

Consider a liquid (of density $\rho$) in laminar flow down an inclined flat plate of length $L$ and width $W$. The fluid flows as a falling film with negligible rippling under the influence of gravity. End effects may be neglected because $L$ and $W$ are large compared to the film thickness $\delta$.

![Figure 14.1: Non-Newtonian liquid flow in a falling film.](image)

a) Determine the steady-state velocity distribution for a non-Newtonian fluid that obeys the power law model (e.g., a polymer liquid). Reduce the result to the Newtonian case.

b) Obtain the mass flow rate for a power law fluid. Simplify for a Newtonian fluid.

c) What is the force exerted by the fluid on the plate in the flow direction?

**Solution**

**14.a Shear stress distribution.**

For axial flow in rectangular Cartesian coordinates, the differential equation for the momentum flux is

$$\frac{d\tau_{xz}}{dx} = \frac{\Delta P}{L} \quad (14.a.1)$$

where $P$ is a modified pressure, which is the sum of both the pressure and gravity terms, that is, $\Delta P = \Delta p + \rho g L \sigma \nu \beta$. Here, $\beta$ is the angle of inclination of the $z$-axis with the vertical. Since the flow is solely under the influence of gravity, $\Delta p = 0$ and therefore $\Delta P/L = \rho g \sigma \nu \beta$. On integration,

$$\tau_{xz} = \rho g x \sigma \nu \beta + C_1 \quad (14.a.2)$$

On using the boundary condition at the gas-liquid interface ($\tau_{xz} = 0$ at $x = 0$), the constant of integration $C_1$ is found to be zero. This gives the expression for the shear stress tensor $\tau_{xz}$ for laminar flow in a falling film as

$$\tau_{xz} = \rho g x \sigma \nu \beta \quad (14.a.3)$$

It must be noted that the above momentum flux expression holds for both Newtonian and non-Newtonian fluids (and does not depend on the type of fluid).

In the falling film, the velocity decreases with increasing $x$ and therefore $du_z/dx \leq 0$. Thus, the power law model is given as

$$\tau_{xz} = m \left( -\frac{du_z}{dx} \right)^n \quad (14.a.4)$$

**Velocity distribution**

To obtain the velocity distribution, equations (14.a.3) and (14.a.4) are combined to eliminate $\tau_{xz}$ and get

$$-\frac{du_z}{dx} = \left( \frac{\rho g \sigma \nu \beta}{m} \right)^{1/n} x^{1/n} \quad (14.a.5)$$
On integrating,

\[ u_z = \left( \frac{\rho g \sigma \nu \beta}{m} \right)^{\frac{1}{n}} \frac{x^{\frac{1}{n}+1}}{\frac{1}{n}+1} + C_2 \]

On using the no-slip boundary condition at the solid surface \((u_z = 0 \text{ at } x = \delta)\),

\[ C_2 = \left( \frac{\rho g \sigma \nu \beta}{m} \right)^{\frac{1}{n}} \frac{\delta^{\frac{1}{n}+1}}{\frac{1}{n}+1} \]

Thus, the final expression for the velocity distribution is

\[ u_z = \left( \frac{\rho g \sigma \nu \beta}{m} \right)^{\frac{1}{n}} \frac{\delta^{\frac{1}{n}+1}}{\frac{1}{n}+1} \left[ 1 - \left( \frac{x}{\delta} \right)^{\frac{1}{n}+1} \right] \quad (14.a.6) \]

Equation (14.a.6) when simplified for a Newtonian fluid \((n = 1 \text{ and } m = \mu)\) gives the following parabolic velocity profile:

\[ u_z = \frac{\rho g \delta^{2} \sigma \nu \beta}{2\mu} \left[ 1 - \left( \frac{x}{\delta} \right)^{2} \right] \quad (14.a.7) \]

14.b Mass flow rate.

The mass flow rate may be obtained by integrating the velocity profile over the film thickness as shown below.

\[ w = \int_{0}^{\delta} \rho u_z W dx = \left( \frac{\rho g \delta \sigma \nu \beta}{m} \right)^{\frac{1}{n}} \frac{\delta^{\frac{1}{n}+1}}{\frac{1}{n}+1} \int_{0}^{1} \left[ 1 - \left( \frac{x}{\delta} \right)^{\frac{1}{n}+1} \right] d \left( \frac{x}{\delta} \right) \quad (14.b.1) \]

or,

\[ w = \frac{\delta^{\frac{1}{n}+2}}{\frac{1}{n}+2} \left( \frac{\rho g \delta \sigma \nu \beta}{m} \right)^{\frac{1}{n}} \quad (14.b.2) \]

Equation (14.b.2) for a Newtonian fluid \((n = 1 \text{ and } m = \mu)\) gives

\[ w = \frac{\rho g \delta^{3} \sigma \nu \beta}{3\mu} \quad (14.b.3) \]

When the mass flow rate \(w\) in either equation (14.b.2) or equation (14.b.3) is divided by the density \(\rho\) and the film cross-sectional area \(W \cdot \delta\), an expression for the average velocity \(\langle u_z \rangle\) is obtained. Also, the maximum velocity \(u_{z,\text{max}}\) occurs at the interface \(x = 0\) and is readily obtained from equation (14.a.6) or equation (14.a.7) respectively. Thus, equations (14.a.6) and (14.b.2) give

\[ \langle u_z \rangle = \frac{\delta}{\frac{1}{n}+2} \left( \frac{\rho g \delta \sigma \nu \beta}{m} \right)^{\frac{1}{n}} = \frac{\frac{1}{n}+1}{\frac{1}{n}+2} u_{z,\text{max}} \quad (14.b.4) \]

For \(n = 1 \text{ and } m = \mu\), equation (14.b.4) reduces to the Newtonian result given below:

\[ \langle u_z \rangle = \frac{\rho g \delta^{2} \sigma \nu \beta}{3\mu} = \frac{2}{3} u_{z,\text{max}} \quad (14.b.5) \]

14.c Force on the plate.

The \(z\)-component of the force of the fluid on the solid surface is given by the shear stress integrated over the wetted surface area. Using equation (14.a.3) gives

\[ F_z = LW \tau_{xz}(x = \delta) = \rho g \delta LW \sigma \nu \beta \quad (14.c.1) \]

As expected, the force exerted by the fluid on the plate is simply the \(z\)-component of the weight of the liquid film.
Problem 15

Consider a fluid (of constant density $\rho$) in incompressible, laminar flow in a tube of circular cross section, inclined at an angle $\beta$ to the vertical. End effects may be neglected because the tube length $L$ is relatively very large compared to the tube radius $R$. The fluid flows under the influence of both a pressure difference $\Delta p$ and gravity.

![Figure 15.1: Fluid flow in circular tube.](image)

a) Using a differential shell momentum balance, determine expressions for the steady-state shear stress distribution and the velocity profile for a Newtonian fluid (of constant viscosity $\mu$).

b) Obtain expressions for the maximum velocity, average velocity and the mass flow rate for pipe flow.

c) Find the force exerted by the fluid along the pipe wall.

Solution

Flow in pipes occurs in a large variety of situations in the real world and is studied in various engineering disciplines as well as in physics, chemistry, and biology.

15.a Shear stress distribution.

For a circular tube, the natural choice is cylindrical coordinates. Since the fluid flow is in the $z$-direction, $u_r = 0$, $\theta = 0$, and only $u_z$ exists. Further, $u_z$ is independent of $z$ and it is meaningful to postulate that the velocity and pressure are both functions of only $z$, that is $u_z = u_z(r)$, $p = p(z)$. The only nonvanishing components of the stress tensor are $\tau_{rz} = \tau_{zr}$, which depend only on $r$.

Consider now a thin cylindrical shell perpendicular to the radial direction and of length $L$. A ‘rate of $z$-momentum’ balance over this thin shell of thickness $\Delta r$ in the fluid is of the form:

\[
\text{Rate of } z\text{-momentum} = \text{In} + \text{Out} + \text{Generation} = \text{Accumulation}
\]

At steady-state, the accumulation term is zero. Momentum can go ‘in’ and ‘out’ of the shell by both the convective and molecular mechanisms. Since $u_z(r)$ is the same at both ends of the tube, the convective terms cancel out because $(\rho u_z^2 2\pi r \Delta r) |_{z=0} = (\rho u_z^2 2\pi r \Delta r) |_{z=L}$. Only the molecular term $(2\pi r L \tau_{rz})$ remains to be considered, whose ‘in’ and ‘out’ directions are taken in the positive direction of the $r$-axis. Generation of $z$-momentum occurs by the pressure force acting on the surface $(\rho u_z^2 2\pi r \Delta r)$ and gravity force acting on the volume $[(\rho g \sigma \nu \beta)^2 2\pi r \Delta r L]$. On substituting these contributions into the $z$-momentum balance, we get

\[
(2\pi r L \tau_{rz}) |_r - (2\pi r L \tau_{rz}) |_{r+\Delta r} + (p_0 - p_L)2\pi r \Delta r + (\rho g \sigma \nu \beta)2\pi r \Delta r L = 0
\]

(15.a.1)

Dividing the above equation by $2\pi L \Delta r$ yields

\[
\frac{(r \tau_{rz}) |_{r+\Delta r} - (r \tau_{rz}) |_r}{\Delta r} = \frac{p_0 - p_L + \rho g \sigma \nu \beta}{L}
\]

(15.a.2)

On taking the limit as $\Delta r \rightarrow 0$, the left-hand side of the above equation is the definition of the first derivative. The right-hand side may be written in a compact and convenient way by introducing the modified pressure $P$, which is the sum of the pressure and gravitational terms. The general definition of the modified pressure is
\[ P = p + \rho gh, \]  
where \( h \) is the height (in the direction opposed to gravity) above some arbitrary preselected datum plane. The advantages of using the modified pressure \( P \) are that (i) the components of the gravity vector \( g \) need not be calculated in cylindrical coordinates; (ii) the solution holds for any orientation of the tube axis; and (iii) the effects of both pressure and gravity are in general considered. Here, \( h \) is negative since the \( z \)-axis points downward, giving \( h = z_0 \nu \nu \beta \) and therefore \( P = p - \rho g z_0 \nu \nu \beta \). Thus, \( P_0 = p_0 \) at \( z = 0 \) and \( P_L = p_L \rho g L \nu \nu \beta \) at \( z = L \), giving \( p_0 - p_L + \rho g L \nu \nu \beta = P_0 - P_L \Delta P \). Thus, equation (15.a.2) gives

\[ \frac{d}{dr} (r \tau_r z) = \frac{\Delta P}{L} r. \]  

Equation (15.a.3) on integration leads to the following expression for the shear stress distribution:

\[ \tau_r z = \frac{\Delta P}{2L} r + \frac{C_1}{r} \]  

The constant of integration \( C_1 \) is determined later using boundary conditions.

It is worth noting that equations (15.a.3) and (15.a.4) apply to both Newtonian and non-Newtonian fluids, and provide starting points for many fluid flow problems in cylindrical coordinates.

**Shear stress distribution and velocity profile**

Substituting Newton’s law of viscosity for \( \tau_r z \) in equation (15.a.4) gives

\[ -\mu \frac{du_z}{dr} = \frac{\Delta P}{2L} r + \frac{C_1}{r} \]  

The above differential equation is simply integrated to obtain the following velocity profile:

\[ u_z = \frac{\Delta P}{4\mu L} R^2 - \frac{C_1}{\mu} \ln r + C_2 \]  

The integration constants \( C_1 \) and \( C_2 \) are evaluated from the following boundary conditions:

**Condition 1**: At \( r = 0 \), \( \tau_r z \) and \( u_z \) must be finite.

**Condition 2**: At \( r = R \), \( u_z = 0 \).

From Condition 1 (which states that the momentum flux and velocity at the tube axis cannot be infinite), \( C_1 = 0 \). From Condition 2 (which is the no-slip condition at the fixed tube wall), \( C_2 = \frac{\Delta P R^2}{4\mu L} \). On substituting \( C_1 = 0 \) in equation (15.a.4), the final expression for the shear stress (or momentum flux) distribution is found to be linear as given by

\[ \tau_r z = \frac{\Delta P}{2L} r. \]  

Further, substitution of the integration constants into equation (15.a.6) gives the final expression for the velocity profile as

\[ u_z = \frac{\Delta P}{4\mu L} R^2 \left[ 1 - \left( \frac{r}{R} \right)^2 \right] \]  

It is observed that the velocity distribution for laminar, incompressible flow of a Newtonian fluid in a long circular tube is parabolic.

**15.b  Maximum velocity, average velocity, and mass flow rate**

From the velocity profile, various useful quantities may be derived.

- The maximum velocity occurs at \( r = 0 \) (where \( du_z/dr = 0 \)). Therefore,

\[ \max (u_z) = u_z(r = 0) = \frac{\Delta P}{4\mu L} R^2 \]  

(15.b.1)
• The average velocity is obtained by dividing the volumetric flow rate by the cross-sectional area as shown below.

\[ <u_z> = \frac{\int_0^R u_z 2\pi r dr}{\int_0^R 2\pi r dr} = \frac{2}{R^2} \int_0^R r u_z dr = \frac{\Delta P}{2\mu L} \int_0^R r \left[ 1 - \left( \frac{r}{R} \right)^2 \right] dr = \frac{\Delta P R^2}{8\mu L} = \frac{1}{2} \max(u_z) \quad (15.b.2) \]

Thus, the ratio of the average velocity to the maximum velocity for Newtonian fluid flow in a circular tube is \( \frac{1}{2} \).

• The mass rate of flow is obtained by integrating the velocity profile over the cross section of the circular tube as follows.

\[ w = \int_0^R \rho u_z 2\pi r dr = \pi R^2 \rho <u_z> \quad (15.b.3) \]

Thus, the mass flow rate is the product of the density \( \rho \), the cross-sectional area \( \pi R^2 \) and the average velocity \( <u_z> \). On substituting \( <u_z> \) from equation (15.b.2), the final expression for the mass rate of flow is

\[ w = \frac{\pi \Delta P \rho}{8\mu L} R^4 \quad (15.b.4) \]

The flow rate vs. pressure drop \( (w \text{ vs. } \Delta P) \) expression above is well-known as the Hagen-Poiseuille equation. It is a result worth noting because it provides the starting point for flow in many systems (e.g., flow in slightly tapered tubes).

15.c Force along the tube wall.

The \( z \)-component of the force, \( F_z \), exerted by the fluid on the tube wall is given by the shear stress integrated over the wetted surface area. Therefore, on using equation (15.a.7),

\[ F_z = 2\pi RL\tau_{rz}(r = R) = \pi R^2 \Delta P = \pi R^2 \Delta p + \pi R^2 \rho g L \sigma \nu \beta \quad (15.c.1) \]

where the pressure difference \( \Delta p = p_0 - p_L \). The above equation simply states that the viscous force is balanced by the net pressure force and the gravity force.
Problem 16

A fluid (of constant density $\rho$) is in incompressible, laminar flow through a tube of length $L$. The radius of the tube of circular cross section changes linearly from $R_0$ at the tube entrance ($z = 0$) to a slightly smaller value $R_L$ at the tube exit ($z = L$).

![Figure 16.1: Fluid flow in a slightly tapered tube.](image)

Using the lubrication approximation, determine the mass flow rate vs. pressure drop ($w$ vs. $\Delta P$) relationship for a Newtonian fluid (of constant viscosity $\mu$).

Solution

**Hagen-Poiseuille equation and lubrication approximation**

The mass flow rate vs. pressure drop ($w$ vs. $\Delta P$) relationship for a Newtonian fluid in a circular tube of constant radius $R$ is, as was derived in the previous problem

$$w = \frac{\pi \Delta P R^4}{8 \mu L}$$  \hspace{1cm} (16.1)

The above equation, which is the famous Hagen-Poiseuille equation, may be re-arranged as

$$\frac{\Delta P}{L} = \frac{8 \mu w}{\pi \rho R^4}$$  \hspace{1cm} (16.2)

For the tapered tube, note that the mass flow rate $w$ does not change with axial distance $z$. If the above equation is assumed to be approximately valid for a differential length $dz$ of the tube whose radius $R$ is slowly changing with axial distance $z$, then it may be re-written as

$$- \frac{dP}{dz} = \frac{8 \mu w}{\pi \rho R^4(z)}$$  \hspace{1cm} (16.3)

The approximation used above where a flow between non-parallel surfaces is treated locally as a flow between parallel surfaces is commonly called the lubrication approximation because it is often employed in the theory of lubrication. The lubrication approximation, simply speaking, is a local application of a one-dimensional solution and therefore may be referred to as a quasi-one-dimensional approach.

Equation (16.3) may be integrated to obtain the pressure drop across the tube on substituting the taper function $R(z)$, which is determined next.

**Taper function**

As the tube radius $R$ varies linearly from $R_0$ at the tube entrance ($z = 0$) to $R_L$ at the tube exit ($z = L$), the taper function may be expressed as

$$R(z) = R_0 + (R_L - R_0) \frac{z}{L}$$

On differentiating with respect to $z$, we get

$$\frac{dR}{dz} = \frac{R_L - R_0}{L}$$  \hspace{1cm} (16.4)

**Pressure $P$ as a function of radius $R$**
Equation (16.3) is readily integrated with respect to radius $R$ rather than axial distance $z$. Using equation (16.4) to eliminate $dz$ from equation (16.3) yields

$$-dP = \frac{8\mu w}{\pi \rho} \frac{L}{R_L - R_0} \frac{dR}{R^4} \quad (16.5)$$

Integrating the above equation between $z = 0$ and $z = L$, we get

$$\int_{P_0}^{P_L} -dP = \frac{8\mu w}{\pi \rho} \frac{L}{R_L - R_0} \int_{R_0}^{R_L} \frac{dR}{R^4} \quad (16.6)$$

and therefore,

$$\frac{P_0 - P_L}{L} = \frac{8\mu w}{\pi \rho} \frac{1}{3 (R_L - R_0)} \left( \frac{1}{R_0^3} - \frac{1}{R_L^3} \right) \quad (16.7)$$

Equation (16.7) may be re-arranged into the following standard form in terms of mass flow rate:

$$w = \frac{\pi \Delta P \rho}{8\mu L} R_0^4 \left[ \frac{3(\lambda - 1)}{1 - \lambda^{-3}} \right] = \frac{\pi \Delta P \rho}{8\mu L} R_0^4 \left[ \frac{3\lambda^3}{1 + \lambda + \lambda^2} \right] \quad (16.8)$$

where the taper ratio $\lambda = \frac{R_L}{R_0}$. The term in square brackets on the right-hand side of the above equation may be viewed as a taper correction to equation (16.1).
Problem 17

A fluid (of constant density \( \rho \)) is in incompressible, laminar flow through a tube of length \( L \). The radius of the tube of circular cross section changes linearly from \( R_0 \) at the tube entrance \( (z = 0) \) to a slightly smaller value \( R_L \) at the tube exit \( (z = L) \).

![Figure 17.1: Fluid flow in a slightly tapered tube.](image)

Using the lubrication approximation, determine the mass flow rate vs. pressure drop \( (w \text{ vs. } \Delta P) \) relationship for a fluid that can be described by the power law model (e.g., a polymeric liquid).

Solution

**Hagen-Poiseuille equation and lubrication approximation**

The mass flow rate vs. pressure drop \( (w \text{ vs. } \Delta P) \) relationship for a power law fluid in a circular tube of constant radius \( R \) is

\[
    w = \frac{\pi R^3 \rho}{\frac{1}{n} + 3} \left( \frac{\Delta P R}{2mL} \right)^{1/n} \tag{17.1}
\]

The above equation, which is the power law fluid analog of the Hagen-Poiseuille equation (originally for Newtonian fluids), may be re-arranged as

\[
    \frac{\Delta P}{L} = 2m \left[ \frac{3n + 1}{n} \frac{w}{\rho \pi} \right]^n \frac{1}{R^{3n+1}} \tag{17.2}
\]

For the tapered tube, note that the mass flow rate \( w \) does not change with axial distance \( z \). If the above equation is assumed to be approximately valid for a differential length \( dz \) of the tube whose radius \( R \) is slowly changing with axial distance \( z \), then it may be re-written as

\[
    -\frac{dP}{dz} = 2m \left[ \frac{3n + 1}{n} \frac{w}{\rho \pi} \right]^n \frac{1}{R^{3n+1}(z)} \tag{17.3}
\]

The approximation used above where a flow between non-parallel surfaces is treated locally as a flow between parallel surfaces is commonly called the lubrication approximation because it is often employed in the theory of lubrication. The lubrication approximation, simply speaking, is a local application of a one-dimensional solution and therefore may be referred to as a quasi-one-dimensional approach.

Equation (17.3) may be integrated to obtain the pressure drop across the tube on substituting the taper function \( R(z) \), which is determined next.

**Taper function**

As the tube radius \( R \) varies linearly from \( R_0 \) at the tube entrance \( (z = 0) \) to \( R_L \) at the tube exit \( (z = L) \), the taper function may be expressed as

\[
    R(z) = R_0 + (R_L - R_0) \frac{z}{L} \tag{17.4}
\]

On differentiating with respect to \( z \), we get

\[
    \frac{dR}{dz} = \frac{R_L - R_0}{L} \tag{17.4}
\]
Note that the taper function is not affected by the type of fluid (Newtonian or power-law); it is a purely geometrical characteristic of the system.

**Pressure $P$ as a function of radius $R$**

Equation (17.3) is readily integrated with respect to radius $R$ rather than axial distance $z$. Using equation (17.4) to eliminate $dz$ from equation (17.3) yields

$$-dP = 2m \left[ \frac{3n+1}{n} \frac{w}{\rho \pi} \right]^n \frac{L}{R_L - R_0} \frac{dR}{R^{3n+1}} \quad (17.5)$$

Integrating the above equation between $z = 0$ and $z = L$, we get

$$\int_{P_0}^{P_L} -dP = 2m \left[ \frac{3n+1}{n} \frac{w}{\rho \pi} \right]^n \frac{L}{R_L - R_0} \int_{R_0}^{R_L} \frac{dR}{R^{3n+1}} \quad (17.6)$$

and therefore,

$$\frac{P_0 - P_L}{L} = 2m \left[ \frac{3n+1}{n} \frac{w}{\rho \pi} \right]^n \frac{1}{3n(R_L - R_0)} \left( \frac{1}{R_0^{3n}} - \frac{1}{R_L^{3n}} \right) \quad (17.7)$$

Equation (17.7) may be re-arranged into the following standard form in terms of mass flow rate:

$$w = \frac{\pi \rho R_0^3}{\frac{1}{n} + 3} \left( \frac{\Delta P R_0}{2mL} \right)^{1/n} \left[ \frac{3n(\lambda - 1)}{1 - \lambda^{-3}} \right]^{1/n} \quad (17.8)$$

where the taper ratio $\lambda = \frac{R_L}{R_0}$. The term in square brackets on the right-hand side of the above equation may be viewed as a taper correction to equation (17.1).