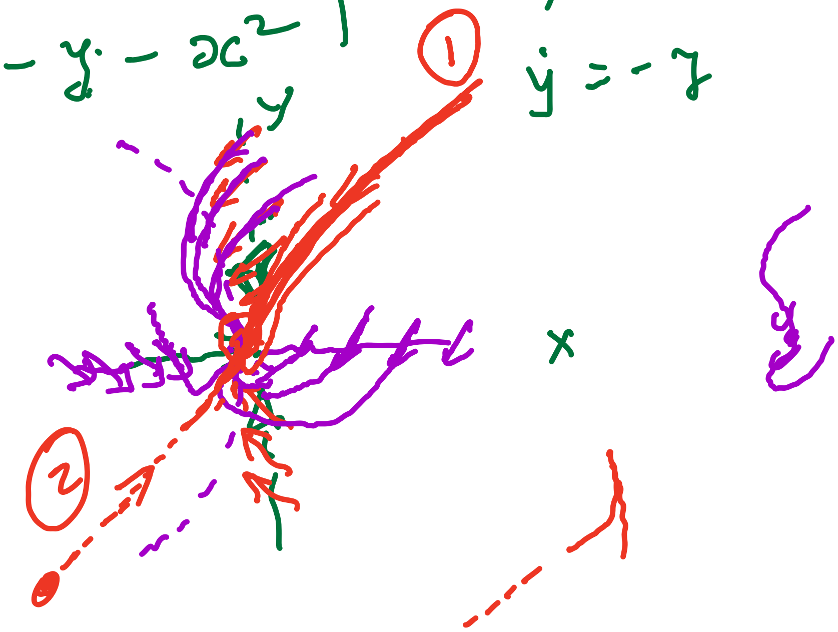
Παρασκευή 3 Ιουνίου

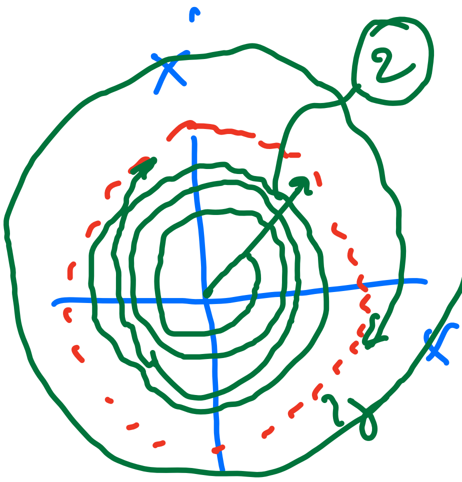
$$\dot x = -2x - y^2$$
$$\dot y = -y - x^2$$

$(0,0)$

$$\dot x = -2x$$
$$\dot y = -y$$



$$\ddot x + \varepsilon \dot x (x^2 - 1) + x = 0$$



$\varepsilon \ll 1$

$$\boxed{x \simeq 2\cos t}$$

$$\oint_\gamma \frac{d}{dt} E \, dt = 0$$

$$E = \frac{\dot x^2}{2} + \frac{x^2}{2}$$

$x \simeq 2\cos t$ $\quad$ $\cancel{\varepsilon t}$ $\quad t \sim \frac{1}{\varepsilon}$

$x(t, \varepsilon t)$

$$\boxed{\ddot{x} + 2\varepsilon\dot{x} + x = 0}$$

$x(0) = 1$

$\dot{x}(0) = 0$

$$x = e^{-\varepsilon t}\cos\left(\sqrt{1-\varepsilon^2}\, t\right)$$

$$\sim e^{-\varepsilon t}\cos t$$

$$\boxed{e^{-\varepsilon t} = \boxed{1 - \varepsilon t} + \cdots\cdots}$$

$\boxed{t \gg 1/\varepsilon}$

συγκλίνει

$\sum f_n^{(\varepsilon)} \to f(\varepsilon)$

σταθερό

ε , ~~για~~ ~~κάποιο~~ $n \to \infty$

$|f - \sum_n(\varepsilon)| \to 0$

Poincaré

$\boxed{\text{Ασυμπτωτικές σειρές}}$

$$\sum^{n} f_n(\varepsilon) \to f(\varepsilon)$$

αλλά $\varepsilon \to 0$

Θέλουμε

Ασυμπτωτική σύγκλιση   μ δεδομένο

$$\sum^{M} f_n(\varepsilon) \to f(\varepsilon)$$

όταν
$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon) - \sum^{M} f_n(\varepsilon)}{f_M(\varepsilon)} = 0$$

$$\boxed{\ddot{x} + x = \cos\omega t}$$

$$x = \alpha \cos t + \beta \sin t + \frac{\cos \omega t}{1-\omega^2}$$

$|\omega| = 1$

$$\ddot{x} - x = e^{-t}$$

$$x = \alpha \cos t + \beta \sin t - \frac{\cos\omega t - \cos t}{(\omega-1)(\omega+1)}$$

$$= \alpha \cos t + \beta \sin t - \frac{t}{(\omega+1)} \left(\frac{\cos\omega t - \cos t}{\omega t - t} \right)$$

$$\lim_{\omega \to 1} x(t;\omega) = \alpha \cos t + \beta \sin t + \frac{t \sin t}{2}$$

$\boxed{\text{secular growth}}$

κοσμική αύξηση

διαιώνιο

$$\ddot{x} + \varepsilon \underline{\dot{x}(x^2-1)} + x = 0 \qquad x(0) = r_0$$
$$\dot{x}(0) = 0$$

$$\underline{x = x_0 + \varepsilon x_1 + \dots}$$

לא בהכרח נכון!

$$\forall \varepsilon \qquad \lim_{\varepsilon \to 0} \frac{x - x_0}{x_0} = 0$$

$$x_0(0) + \varepsilon x_1(0) + \dots = r_0$$

... רגולרי

$$\varepsilon = 0 \qquad x_0(0) = r_0$$

$$\cancel{r_0} + \varepsilon x_1(0) + \dots = \cancel{r_0}$$

$$\varepsilon \dot{x}_1(0) + \dots = 0$$

גוזרים $\quad x_1(0) + \varepsilon x_2(0) + \dots = 0$

$$\underline{\varepsilon = 0} \qquad x_1(0) = 0 \qquad \dots$$

$$\ddot{x}_0 + \varepsilon \underline{\ddot{x}_1} + \dots + \varepsilon(\dot{x}_0 + \varepsilon \dot{x}_1 + \dots)((x_0 + \varepsilon x_1 + \dots)^2 - 1) +$$

$$+ x_0 + \varepsilon \underline{x_1} + \dots = 0$$

$$(\ddot{x}_0 + x_0) + \varepsilon(\ddot{x}_1 + x_1 + \dot{x}_0(x_0^2 - 1)) +$$
$$\varepsilon^2(\dots) + \dots = 0$$

$$\forall \varepsilon$$

$$\ddot{x}_0 + x_0 = 0 \qquad x_0(0) = r_0 \quad \dot{x}_0(0) = 0$$

$$\ddot{x}_1 + x_1 = -\dot{x}_0 (x_0^2 - 1) \qquad x_1(0) = 0, \dot{x}_1(0) = 0$$

$$\cdot \quad \cdot \quad \cdot \cdot \cdot \quad \cdots$$

$$x_0 = r_0 \cos t$$

$$\ddot{x}_1 + x_1 = + r_0 \sin t \, (r_0^2 \overbrace{\cos^2 t}^{1 - \sin^2 t} - 1)$$

$$= r_0 \sin t \left((r_0^2 - 1) - r_0^2 \sin^2 t \right)$$

$$x_0 = r_0 \cos t$$

$$\underline{\ddot{x}_1 + x_1} = \underline{r_0 (r_0^2 - 1) \sin t} - r_0^3 \sin^3 t$$

$$(\sin t)^3 = \frac{(e^{it} - e^{-it})^3}{2^3 i^3} =$$

$$= \frac{e^{3it} - 3 e^{2it} e^{-it} + 3 e^{it} e^{-2it} - e^{-3it}}{-2^3 i}$$

$$= -\frac{e^{3it} - e^{-3it}}{2i} \frac{1}{4} + \frac{3}{4} \frac{e^{it} - e^{-it}}{2i}$$

$$\sin^3 t = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t$$

$$\ddot{x}_1 + x_1 = \sin t\,\overbrace{\left[r_0(r_0^2-1) - \frac{3r_0^3}{4}\right]}^{B} + \frac{r_0^3}{4}\sin 3t$$

$$x_1 = A\sin 3t + Bt\cos t$$

$$\frac{x-x_0}{x_0} = \frac{\varepsilon A\sin 3t + \cancel{\varepsilon t B\cos t}}{\cos t} \to 0$$

$$\underline{\varepsilon \to 0}$$

$$t > 1/\varepsilon$$

$$r_0^3 - r_0 - \frac{3r_0^3}{4} = 0$$

$$\frac{r_0^3}{4} = r_0$$

$$r_0 = 2\ !$$

$\ddot{x} + x + \underline{\varepsilon x^3} = 0$ Duffing

$x(0) = A$

$\dot{x}(0) = 0$

$x = x_0 + \varepsilon x_1 + \cdots\cdots$

$x_0(0) = A,\ \dot{x}_0(0) = 0$

$x_1(0) = 0 = \cdots$

$\dot{x}_1(0) = 0 \cdots$

$(\ddot{x}_0 + x_0) + \varepsilon(\ddot{x}_1 + x_1 + x_0^3) + \cdots\cdots = 0$

$x_0 = A\cos t$

$\ddot{x}_1 + x_1 = -A^3\cos^3 t$

$\cos^3 t = \left(\dfrac{e^{it} + e^{-it}}{2}\right)^3 = \dfrac{\cos 3t}{4} + \dfrac{3}{4}\cos t$

$= -A^3\dfrac{\cos 3t}{4} - \underbrace{\dfrac{3}{4}A\cos t}$

$\sim\ x_1 \sim \underline{t\sin t}$

$x(t) \cancel{x_0 =} = \underline{A\cos t} + \varepsilon[B\cos 3t$

$+ \underline{\Gamma\ t\sin t}]$

$$t \sim 1/\varepsilon$$

$$e^{-\varepsilon t} \approx \underline{1 - \varepsilon t}$$

<u>2 - χρόνοι</u>

$\boxed{t}$

$T = \varepsilon t \quad T$

$x(t, T)$

$$\frac{dx}{dt} = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial T}\frac{\partial T}{\partial t}$$

$$= \frac{\partial x}{\partial t} + \varepsilon \frac{\partial x}{\partial T}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T} \qquad e^{-\varepsilon t} \sin t$$

$$\frac{d}{dt}\Big(\underbrace{e^{-\overset{T}{\varepsilon t}} \sin t}_{x}\Big) = \underbrace{-\varepsilon e^{-\overset{T}{\varepsilon t}} \sin t}_{\left.\frac{\partial x}{\partial T}\right|_t} + \underbrace{e^{-\varepsilon t} \cos t}_{\left.\frac{\partial x}{\partial t}\right|_T}$$

$$x = x_0(t, T) + \varepsilon x_1(t, T) + \dots$$

$$\dot{x} = \frac{\partial x}{\partial t} + \varepsilon \frac{\partial x}{\partial T} = 0 \qquad x(0) = A, \quad \dot{x}(0) = 0$$

$X_0(0,0) = A, \quad X_n(0,0) = 0 \quad \forall\, n \geq 1$

$\dot{X}_0(0,0) = 0 \ \ldots\ldots \quad \forall\, n \geq 1$

$\ddot{X}$

$$\left(\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial T}\right)^2 (X_0 + \varepsilon X_1 + \cdots) +$$

$$+ X_0 + \varepsilon X_1 + \cdots + \varepsilon\left(\frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T}\right)(X_0 + \varepsilon X_1 + \cdots)\cdot$$

$$\left((X_0 + \varepsilon X_1 + \cdots)^2 - 1\right) = 0$$

$$\frac{\partial^2 X_0}{\partial t^2} + X_0 = 0 \qquad X_0(0,0) = \alpha, \quad \dot{X}(0,0) = 0$$

$$\frac{\partial^2 X_1}{\partial t^2} + X_1 = -2\frac{\partial^2}{\partial t \partial T} X_0 - \frac{\partial X_0}{\partial t}(X_0^2 - 1)$$

$X_1(0,0) = 0$

$\dot{X}_1(0,0) = 0$

. . . .

$$\to X_0 = A(T)\cos t + B(T)\sin t \quad \text{με } B(0) = 0$$

ή ~~κρατάμε~~ ως το λάβουμε σε μορφή

$$X_0 = A(T)e^{it} + A^*(T)e^{-it}$$

Θα συμβάλει αφού και το $\sin t$

$$\frac{\partial A}{\partial T} = \dot{A}$$

$$\frac{\partial^2 x_1}{\partial t^2} + x_1 = -2i\dot{A}e^{it} + c.c.$$

$$-\left(iA'e^{it} - iA^{*}e^{-it}\right)\left(A^2e^{2it} + 2|A|^2 + A^{*2}e^{-2it} - 1\right)$$

$$= e^{it}\left(\underline{-2i\dot{A} + iA - 2iA|A|^2 + i|A|^2A}\right)$$

$t \sim 1/\varepsilon$

$+ \; c.c.$

$+ \; e^{3it} + c.c.$

$$2\dot{A} = A - A|A|^2$$

A Airodi

$$\dot{A} = \frac{A}{2}\left(1 - |A|^2\right) \qquad \dot{A} = \frac{dA}{dT}$$

$$A = re^{i\theta} \qquad \dot{A} = \dot{r}e^{i\theta} + i\dot{\theta}re^{i\theta}$$

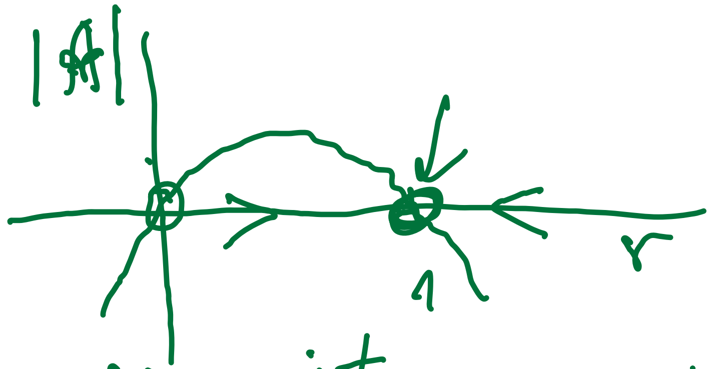
$$\dot{r}\cancel{e^{i\theta}} + ir\dot{\theta}\cancel{e^{i\theta}} = \frac{r\cancel{e^{i\theta}}}{2}\left(1 - |A|^2\right)$$

$$\boxed{\begin{gathered}\theta = 0 \\ \dot{r} = \frac{r}{2}\left(1 - r^2\right)\end{gathered}}$$

$$A = re^{i\theta}$$

$$A(0) = \frac{\alpha}{2}, \; \underline{\theta = 0}$$

14]



$x_0 = e^{it} + e^{-it} = 2\cos t$ ξi. !

$$\frac{2\,dr}{r(1-r^2)} = 2\left[\frac{1}{r} + \frac{r}{1-r^2}\right]dr = dT$$

$$2\log\frac{r}{r(0)} - \log\left(\frac{1-r^2}{1-r(0)^2}\right) = dT$$

$$\frac{\frac{r^2}{1-r^2}}{\frac{r_0^2}{1-r_0^2}} = e^T \qquad \Gamma = \frac{r_0^2}{1-r_0^2}e^T$$

$$\frac{r^2}{1-r^2} = \Gamma \qquad r^2(1+\Gamma) = \Gamma$$

$$r = \frac{r_0}{\sqrt{1-r_0^2}}\,\frac{e^{T/2}}{\sqrt{\frac{1-r_0^2+r_0^2e^T}{1-r_0^2}}}$$

$$r(T) = \frac{r_0 e^{T/2}}{\sqrt{1-r_0^2+r_0^2 e^{T}}} = \frac{r_0}{\sqrt{r_0^2+(1-r_0^2)e^{-T}}}$$

Αρχικά $\underline{x_0(0,0)=\alpha}$
η μηδενική αρχική ταχύτητα:

$$\left(\frac{\partial x_0}{\partial t}+\varepsilon\frac{\partial x_0}{\partial T}\right)+\varepsilon\left(\frac{\partial x_1}{\partial t}+\varepsilon\frac{\partial x_1}{\partial T}\right)+\dots=0$$

$$\frac{\partial x_0}{\partial t}+\varepsilon\left(\frac{\partial x_1}{\partial t}+\frac{\partial x_0}{\partial T}\right)+\varepsilon^2(\dots=0$$

για $\frac{\partial x_0}{\partial t}(0,0)$ και $\left.\frac{\partial x_1}{\partial t}\right|_0 = -\left.\frac{\partial x_0}{\partial T}\right|_0$

$$x_0 = r(T)e^{it}+r(T)e^{-it} = 2r(T)\cos t$$

συνεπώς

$\underline{για}$ $r(0)=\frac{\alpha}{2}$, $\left.\frac{\partial x_0}{\partial T}\right|_{T=0} = \left.\dot r(T)\right|_{T=0}$

και έτσι προσδιορίζεται
η σειρά των
αρχικών συνθηκών.

συνεπώς αν $x(0)=\alpha$
η πρώτη προσέγγιση με ισχύ τουλάχιστον
για $t \sim O(1/\varepsilon)$ θα είναι

$$x(t) = \frac{\alpha}{\sqrt{\frac{\alpha^2}{4} + \left(1-\frac{\alpha^2}{4}\right)e^{-\varepsilon t}}}\cos t$$

και όταν $t \to \infty$ $\quad x(t) \to 2\cos t$
που είναι ο οριακός κύκλος
που υπολογίσαμε με άλλα επιχειρήματα.