The nonlinear Schrödinger (NLS) equation

Modulational instability

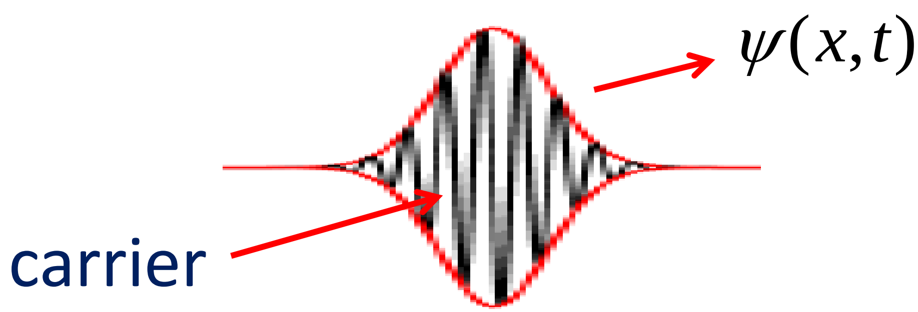
and envelope solitons

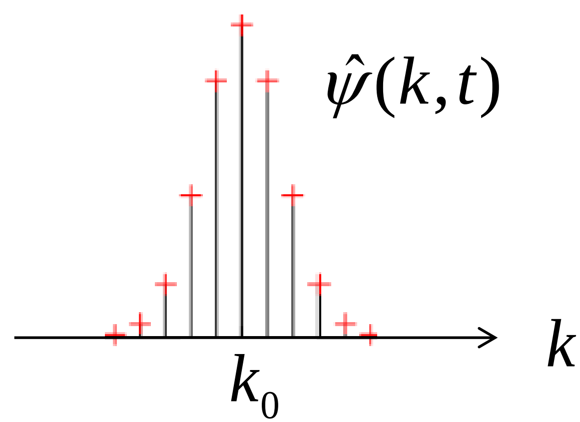
Nonlinear wave envelopes

Consider a **wavepacket**, composed by a carrier wave, namely a plane wave of the form $\exp[i(k_0 x - \omega_0 t)]$, which is modulated by a generally complex field envelope $u(x, t)$, resulting in the real field:

$$\psi(x,t) = \mathrm{Re}\{u(x,t)\exp[i(k_0 x - \omega_0 t)]\}$$

The spectrum of the **wavepacket** is **located around k_0 (and ω_0)**





Assume that the wave obeys the **nonlinear dispersion relation:**

$$\omega = \omega(k, I), \quad I = |u|^2$$

The nonlinear Schrödinger (NLS) equation

Since the spectrum is located around k_0, we may **Taylor expand the dispersion relation**, i.e., $\omega(k)$ **around k_0:**

$$\omega=\omega(k,I)\approx\omega(k_0)+\left.\frac{\partial\omega}{\partial k}\right|_{k=k_0}(k-k_0)+\frac{1}{2}\left.\frac{\partial^2\omega}{\partial k^2}\right|_{k=k_0}(k-k_0)^2+\ldots+\left.\frac{\partial\omega}{\partial I}\right|_{I=0}I+\ldots$$

$$\Rightarrow \omega-\omega_0\approx\omega_0'(k-k_0)+\frac{1}{2}\omega_0''(k-k_0)^2-gI,\quad g=-\left.\frac{\partial\omega}{\partial I}\right|_{I=0}$$

Using: $\omega-\omega_0\mapsto i\partial_t,\quad k-k_0\mapsto -i\partial_x$ we obtain the **operator**:

$$i\partial_t=-i\upsilon_g\partial_x-\frac{1}{2}\omega_0''\partial_x^2-gI,\qquad \upsilon_g=\omega_0'$$

and operating on $u(x,t)$:

$$i(u_t+\upsilon_g u_x)+\frac{1}{2}\omega_0''u_{xx}+g|u|^2u=0$$ **NLS equation**

Galilei transformation and normalization

For the NLS equation: $i(u_t + \upsilon_g u_x) + \frac{1}{2}\omega_0'' u_{xx} + g|u|^2 u = 0$

we introduce the **Galilei transformation:** $x \mapsto x - \upsilon_g t, \quad t \mapsto t$

and cast the NLS into the form: $iu_t + \frac{1}{2}\omega_0'' u_{xx} + g|u|^2 u = 0$

Furthermore, using the rescaling: $t \mapsto \omega_0'' t, \quad u \mapsto \sqrt{|g/\omega_0''|}\, u$

the NLS is expressed as:

$$iu_t + \frac{1}{2}u_{xx} + s|u|^2 u = 0, \qquad s = \begin{cases} +1 & \textbf{Focusing NLS} \\ -1 & \textbf{Defocusing NLS} \end{cases}$$

An example from nonlinear optics

Wave equation – resulting from **Maxwell's equations** –

describing **optical beam propagation in a nonlinear medium**:

$$\Delta \mathbf{E} - \frac{1}{c^2}\mathbf{E}_{tt} = \frac{1}{\varepsilon_0 c^2}\mathbf{P}_{tt}$$

$$\mathbf{P} = \mathbf{P}_L + \mathbf{P}_{NL} = \varepsilon_0 \chi_L \mathbf{E} + \varepsilon_0 \chi_{NL} \mathbf{EEE}$$

Electric field: $\mathbf{E}(\mathbf{r},t) = \frac{1}{2}\{u(x,y,z)\exp[i(k_0 z - \omega_0 t)] + \text{c.c.}\}$

Propagation along z

Wave equation:

$$u_{zz} + 2ik_0 u_z + \Delta_\perp u + \left[-k_0^2 + \frac{\omega_0^2}{c^2}\underbrace{(1+\chi_L)}_{n^2}\right]u + \frac{\omega_0^2}{c^2}\chi_{NL}|u|^2 u = 0$$

$|u_{zz}| << |k_0 u_z|$

Slowly-varying envelope approximation

$u_{xx} + u_{yy}$

Transverse Laplacian

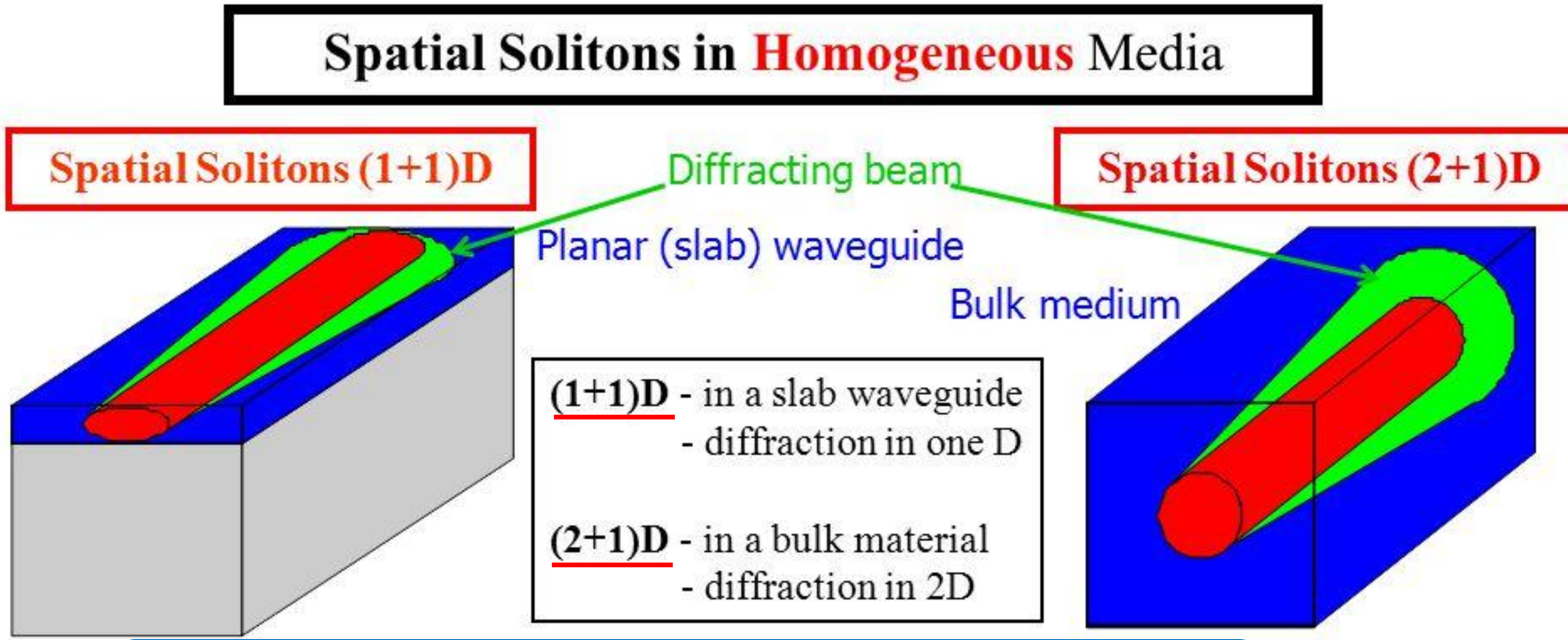
Linear dispersion relation

NLS – optical beams – spatial solitons

$$iu_z + \frac{1}{2k_0}\Delta_\perp u + \frac{\omega_0^2}{2k_0c^2}\chi_{NL}|u|^2 u = 0$$

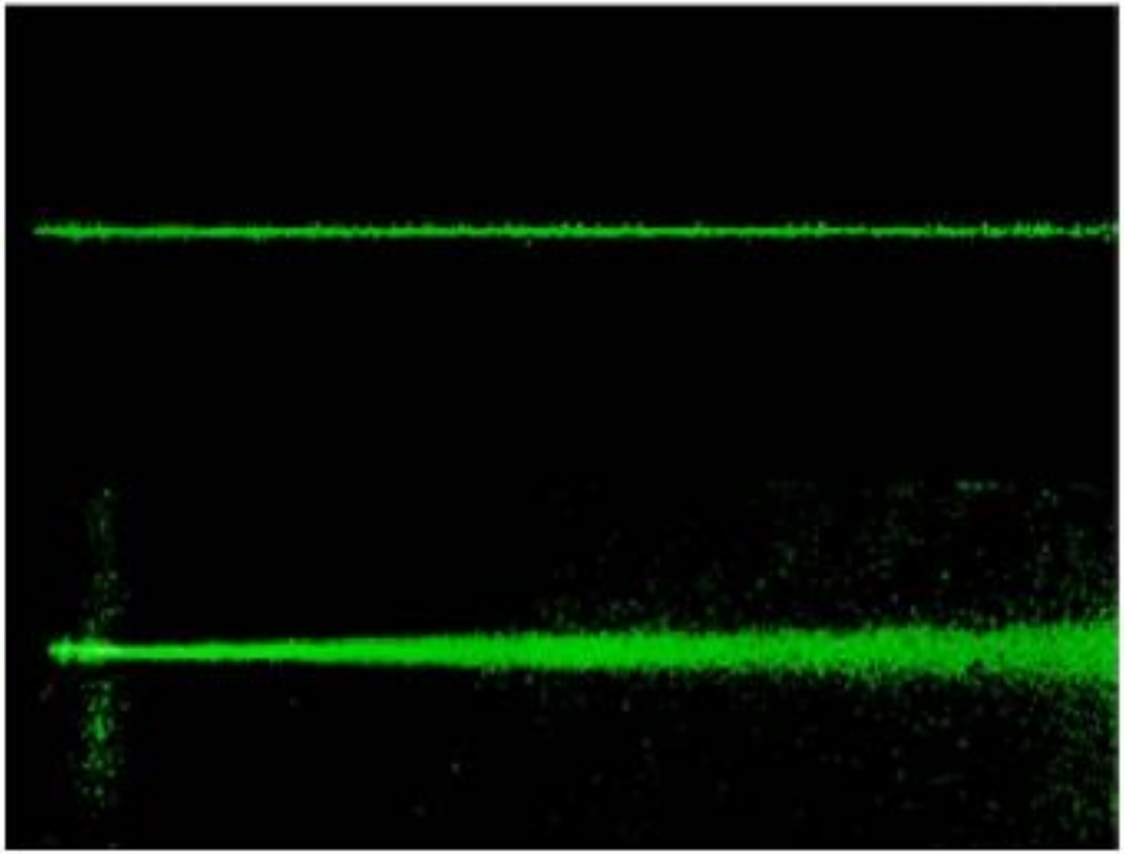
Diffraction

Nonlinearity



A *spatial soliton* is a shape invariant self guided beam of light or a *self-induced waveguide*

Spatial solitons: an experimental result



Experimental demonstration of an optical spatial soliton propagating through a 5mm long nonlinear photorefractive crystal

Top: side-view of the soliton beam from scattered light

Bottom: normal diffraction of the same beam when the nonlinearity is ‘turned off’

NLS in hydrodynamic form

$$iu_t + \frac{1}{2}u_{xx} + s|u|^2 u = 0, \qquad s = \begin{cases} +1 & \textbf{Focusing NLS} \\ -1 & \textbf{Defocusing NLS} \end{cases}$$

Mandelung transformation: $u(x,t) = \sqrt{\rho(x,t)}\exp[i\varphi(x,t)]$

NLS:

$$\begin{cases} \rho_t + (\rho\varphi_x)_x = 0 \\ \varphi_t - s\rho + \frac{1}{2}\varphi_x^2 - \frac{1}{2}\rho^{-1/2}(\rho^{1/2})_{xx} = 0 \end{cases}$$

Compressible inviscid fluid:

$$\begin{cases} \rho_t + (\rho\varphi_x)_x = 0 \\ \upsilon_t + \upsilon\upsilon_x = -\frac{1}{\rho}P_x \end{cases}$$

$\upsilon = \varphi_x$ →

$$\upsilon_t + \upsilon\upsilon_x = s\rho_x + \frac{1}{2}\left[\rho^{-1/2}(\rho^{1/2})_{xx}\right]_x = 0$$

→ **Quantum-mechanical pressure**

$\rho =$ density, $\upsilon = \varphi_x =$ fluid velocity, $P = \frac{1}{2}\rho^2 =$ pressure $(s = -1)$

The simplest solution and its stability

NLS: $$\begin{cases} \rho_t + (\rho\varphi_x)_x = 0 \\ \varphi_t - s\rho + \frac{1}{2}\varphi_x^2 - \frac{1}{2}\rho^{-1/2}(\rho^{1/2})_{xx} = 0 \end{cases}$$

Solution:

$$\rho = \rho_0, \quad \varphi = s\rho_0 t$$

Spatially homogeneous

Stability: Consider the ansatz: $$\begin{cases} \rho(x,t) = \rho_0 + \varepsilon\rho_1(x,t) \\ \varphi = s\rho_0 t + \varepsilon\varphi_1(x,t) \end{cases}$$

small perurbations

Substituting, at O(ε), we obtain: $$\begin{cases} \rho_{1t} + \rho_0\varphi_{1xx} = 0 \\ \varphi_{1t} - s\rho_1 - \frac{1}{4\rho_0}\rho_{1xx} = 0 \end{cases}$$

Using: $\rho_1 = a\exp[i(Kx - \Omega t)] + \text{c.c.}$, $\varphi_1 = b\exp[i(Kx - \Omega t)] + \text{c.c.}$

we derive the linear system: $$\begin{pmatrix} -i\Omega & -\rho_0 K^2 \\ -s + \frac{1}{4\rho_0}K^2 & -i\Omega \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Modulational (Benjamin-Feir) instability

Linear **homogeneous** system:

$$\begin{pmatrix} -i\Omega & -\rho_0 K^2 \\ -s+\dfrac{1}{4\rho_0}K^2 & -i\Omega \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We obtain:

$$\det\begin{pmatrix} -i\Omega & -\rho_0 K^2 \\ -s+\dfrac{1}{4\rho_0}K^2 & -i\Omega \end{pmatrix}=0 \Rightarrow \Omega^2=\rho_0 K^2\left(-s+\frac{1}{4\rho_0}K^2\right)$$

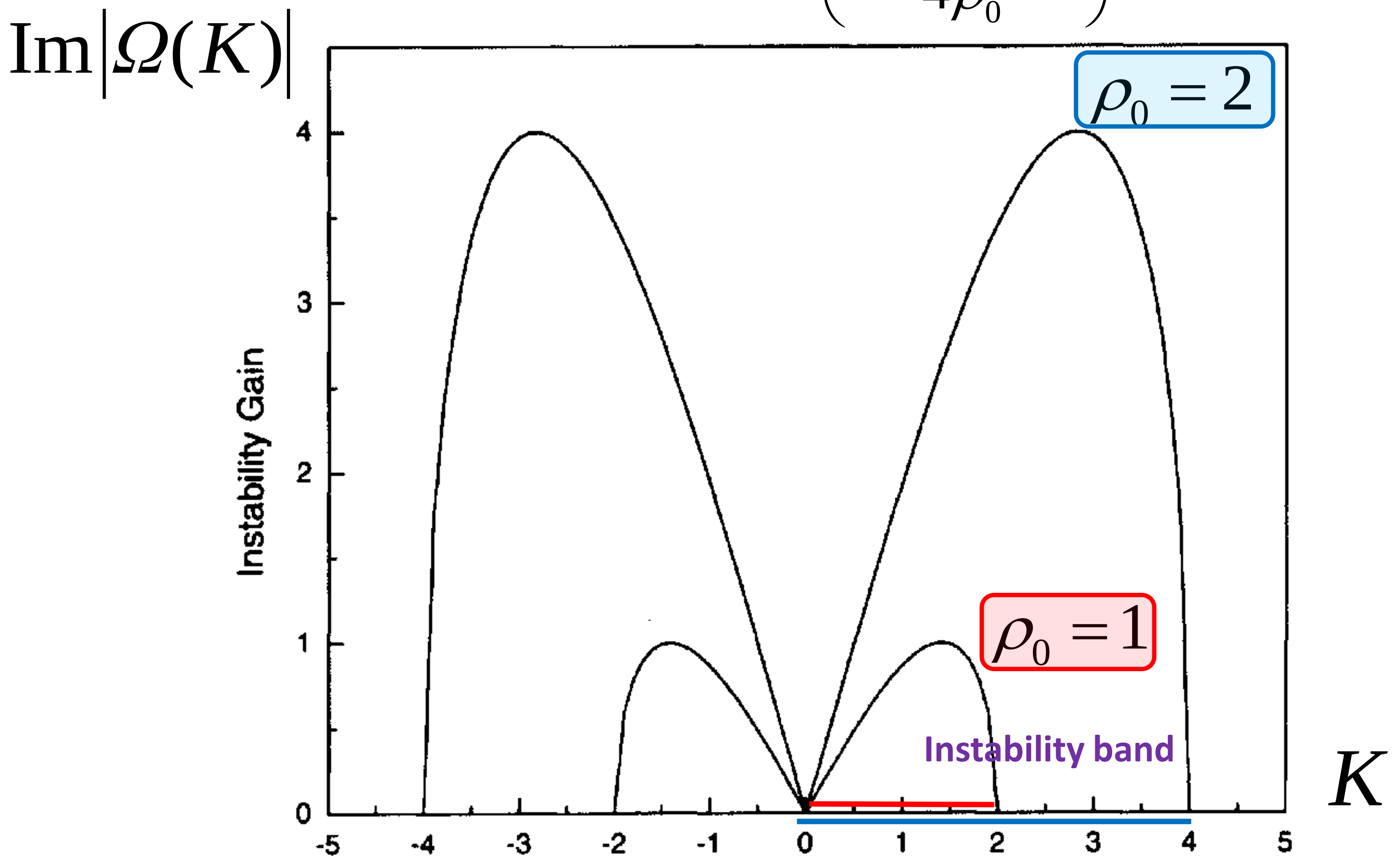
- The solution is stable if perturbations at **any** wave number K **do not grow** with evolution. This is the case as long as ***Ω* is real.**

In particular:

- If $s=-1 \Rightarrow \Omega \in \mathrm{R} \quad \forall K \rightarrow$ **NO instability**
- If $s=+1$ then for $|K| \le 2\sqrt{\rho_0} \Rightarrow \mathrm{Im}|\Omega(K)| = \sqrt{\rho_0}\,|K|\left(1-\dfrac{1}{4\rho_0}K^2\right)^{1/2} \neq 0$ — **instability**

Gain spectrum

$$\mathrm{Im}\left|\Omega(K)\right| = \sqrt{\rho_0}\left|K\right|\left(1-\frac{1}{4\rho_0}K^2\right)^{1/2}$$



An alternative path to the dispersion relation

We start from the linear system: $\begin{cases} \rho_{1t} + \rho_0 \varphi_{1xx} = 0 \\ \varphi_{1t} - s\rho_1 - \dfrac{1}{4\rho_0}\rho_{1xx} = 0 \end{cases}$

Using the compatibility condition: $\varphi_{1xxt} = \varphi_{1txx}$ we obtain the

linearized Boussinesq equation:

$$\rho_{1tt} + s\rho_0\, \rho_{1xx} + \frac{1}{4}\rho_{1xxxx} = 0$$

$$s = \begin{cases} +1 : \textit{elliptic} \\ -1 : \textit{hyperbolic} \end{cases}$$

+1 : elliptic **(focusing NLS)**

−1 : hyperbolic **(defocusing NLS)**

Assuming that: $\rho_1 \propto \exp[i(Kx - \Omega t)]$ we obtain the **dispersion relation**

for the **frequency** $\boldsymbol{\Omega}$ and **wavenumber** $\boldsymbol{K}$ of the perturbation:

$$\Omega^2 = \rho_0 K^2 \left(-s + \frac{1}{4\rho_0} K^2 \right)$$

- If $s = -1 \Rightarrow \Omega \in \mathrm{R} \;\; \forall K \rightarrow$ **NO instability**
- If $s = +1 \Rightarrow \exists K \in \mathrm{R} : \mathrm{Im}(\Omega) \neq 0 \rightarrow$ **instability**

Experimental evidence of Benjamin-Feir instability in deep water (Benjamin 1967)

→ NLS!

near the wavemaker

60 m downstream

frequency = 0.85 Hz, wavelength = 2.2 m,
water depth = 7.6 m

Wavetrains in deep water are unstable – they "disintegrate"

Behavior of the Fourier spectra (I)

We have: $u=\sqrt{\rho}\exp(i\varphi)=\sqrt{\rho_0+\rho_1}\exp(i\rho_0 t+i\varphi_1)$

with: $\rho_1=a\exp[i(Kx-\Omega t)]+\text{c.c.},\quad \varphi_1=b\exp[i(Kx-\Omega t)]+\text{c.c.}$

When the MI sets in: $\Omega=i\Omega_i$ and

$\rho_1=a\exp(\Omega_i t)\cos(K_i x+\theta)$, $\varphi_1=b\exp(\Omega_i t)\cos(K_i x+\theta)$, $K_i\in\left[0,\ 2\sqrt{\rho_0}\right]$

Thus, the **perturbed solution** reads:

$$u=\sqrt{\rho_0+a\,e^{\Omega_i t}\cos(K_i x+\theta)}\exp[i\rho_0 t+ib\,e^{\Omega_i t}\cos(K_i x+\theta)]$$

$$=\sqrt{\rho_0}\left[1+\frac{a}{\rho_0}e^{\Omega_i t}\cos(K_i x+\theta)\right]^{1/2}\exp\left(i\rho_0 t+iQ\sin\Phi\right)$$

where: $Q=b\exp(\Omega_i t),\quad \Phi=-K_i x-\theta-\pi/2$

Behavior of the Fourier spectra (II)

Solution: $u=\sqrt{\rho_0}\left[1+\frac{a}{\rho_0}e^{\Omega_i t}\cos(K_i x+\theta)\right]^{1/2}\exp\left(i\rho_0 t+iQ\sin\Phi\right)$

Using the identity: $\exp\left(iQ\sin\Phi\right)=\sum_{n=-\infty}^{+\infty}J_n(Q)\exp(in\Phi)$ we have:

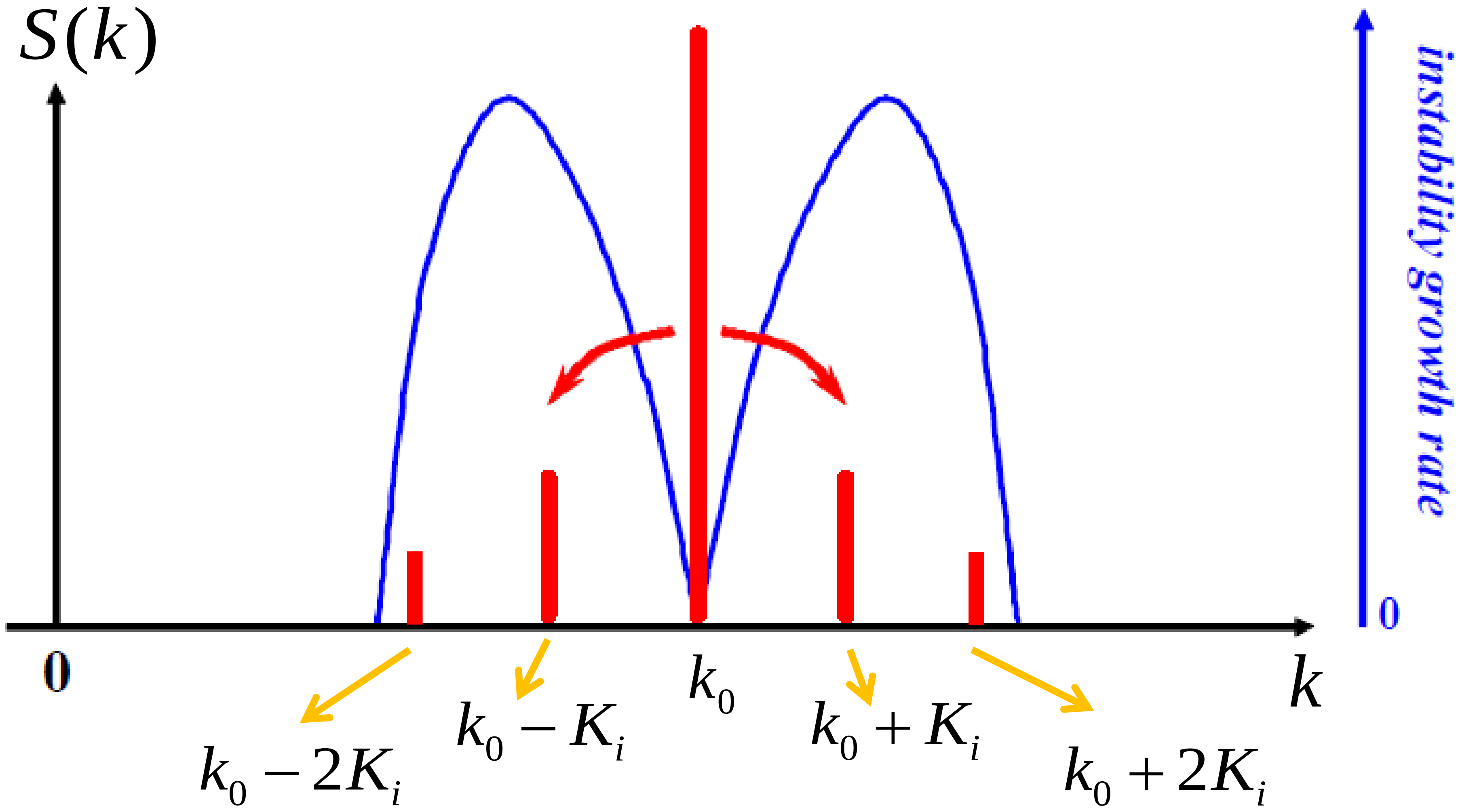
$$u=\sqrt{\rho_0}\left[1+\frac{a}{\rho_0}e^{\Omega_i t}\cos(K_i x+\theta)\right]^{1/2}\sum_{n=-\infty}^{+\infty}J_n(be^{\Omega_i t})\exp(in\Phi)\exp\left(i\rho_0 t\right)$$

Finally, recalling that: $\psi(x,t)=\mathrm{Re}\{u(x,t)\exp[i(k_0 x-\omega_0 t)]\}$ we find:

$$\psi=\sqrt{\rho_0}\left[1+\frac{a}{\rho_0}e^{\Omega_i t}\cos(K_i x+\theta)\right]^{1/2}\sum_{n=-\infty}^{+\infty}J_n(be^{\Omega_i t})$$
$$\times\exp\left[i(k_0-nK_i)x-i(\omega_0-\rho_0)t-in(\theta-\pi/2)\right]$$

Wavenumber generation!

Behavior of the Fourier spectra (III)



MI: a route to localization and envelope soliton formation