Burgers equation

Regularized (viscous) shock waves

Cole-Hopf transformation

Introducing the Burgers equation

Burgers' equation is a fundamental PDE occurring in various areas of applied mathematics and physics, such as **fluid mechanics**, **nonlinear acoustics**, **gas dynamics**, **traffic flow**,… It has the form:

$$\boxed{\underbrace{u_t + uu_x} = \nu u_{xx}}$$

Hopf equation **diffusion term**

(nonlinear transport equation)

For instance, for an incompressible fluid, **the fluid velocity** satisfies the **Navier-Stokes (NS) equations**:

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = \nu \Delta \mathbf{v} - \frac{1}{\rho} \nabla p\,, \qquad \nabla \cdot \mathbf{v} = 0$$

kinematic **viscocity** **density** **pressure**

In (1+1)-dimensions, and in the absence of pressure, NS equations reduce to Burgers equation [Burgers (1939)]

The linear counterpart of the Burgers equation

The linear counterpart of the Burgers equation is of the form:

$$\boxed{u_t + cu_x = \nu u_{xx},}$$

Complex dispersion relation: $\omega = ck - i\nu k^2 \quad \Rightarrow$

plane waves $\propto \exp[i(kx - \omega t)]$, are of the form:

$$u = u_0 \exp[ik(x - ct)] \exp(-\nu k^2 t)$$

right-going traveling wave

decaying amplitude

short waves attenuate faster than long ones

For $c = 0$ (diffusion equation), plane waves are:

$$\propto \exp(ikx) \exp(-\nu k^2 t)$$

These are not traveling waves – they oscillate in x and decay in t

Burgers equation – the role of diffusion

$$\boxed{u_t + uu_x = \nu u_{xx}}$$

- One expects that **the nonlinear term uu_x**, will **tend to steepen the wave** **up to the formation of a shock** (and the eventual break up of the wave), while **the diffusion νu_{xx}** is expected to have a **smoothing out and broadening effect**.

- It is therefore reasonable to ask **whether the presence of diffusion can prevent the appearance of a discontinuous shock wave**.

- Indeed, as we will show below, there exist traveling waves, in the form of **viscous shocks** which, **for any finite ν**, **remain smooth and well-defined for all times**.

Traveling wave solutions

We seek **traveling wave solutions** of the Burgers equation

$$u_t + uu_x = \nu u_{xx}$$

of the form: $u(x,t) = u(\xi),$ $\xi = x - ct,$

where c is the **unknown velocity**. Taking into regard that:

$$u_t = -cu'(\xi),\ u_x = u'(\xi),\ \text{and}\ u_{xx} = u''(\xi)$$

we obtain the 2nd-order ODE: $\underline{-cu' + uu' - \nu u'' = 0.}$

Then, noting that: $uu' = (u^2/2)'$, we integrate wrt. ξ and obtain

$$-cu + \frac{1}{2}u^2 - \nu u' = K,$$

where K is a constant of integration

The associated dynamical system

We have thus derived the 1st-order autonomous nonlinear ODE:

$$-cu + \frac{1}{2}u^2 - \nu u' = K \qquad (1)$$

which admits non-constant solutions which **either tend to infinity or to one of the equilibrium points, as $t \to \pm\infty$.** Since we are interested in obtaining bounded solutions, we rewrite (1) as:

$$\boxed{2\nu u' = u^2 - 2cu - 2K = 0,}$$

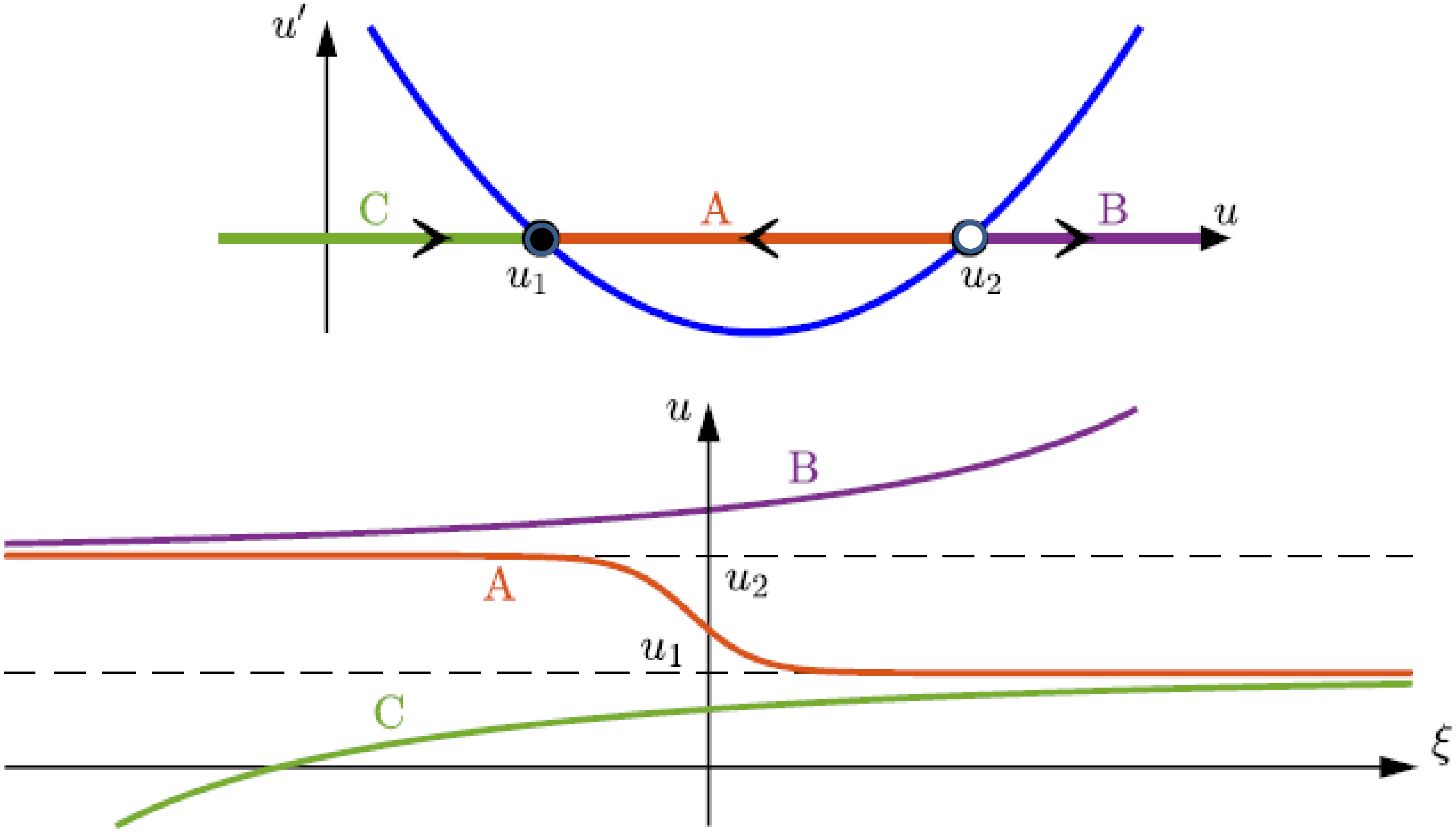
and require that the equilibrium points, i.e., **the roots $u_{1,2}$ of the quadratic polynomial** $u^2 - 2cu - 2K = 0$ **be real**. The roots are:

$$\boxed{u_1 = c - \sqrt{c^2 + 2K}, \quad u_2 = c + \sqrt{c^2 + 2K},}$$

We thus require that: $\underline{c^2 + 2K > 0}$, and thus: $\underline{u_1 < u_2.}$

Fixed points and phase plane

Dynamical system: $\boxed{\frac{du}{d\xi} = \frac{1}{2\nu}(u - u_1)(u - u_2)}$



Bounded solutions of the Burgers equation occur for: $\boxed{u_1 < u < u_2}$

The traveling shock wave solution

Integrating the ODE: $\frac{du}{d\xi} = \frac{1}{2\nu}(u - u_1)(u - u_2)$ we obtain:

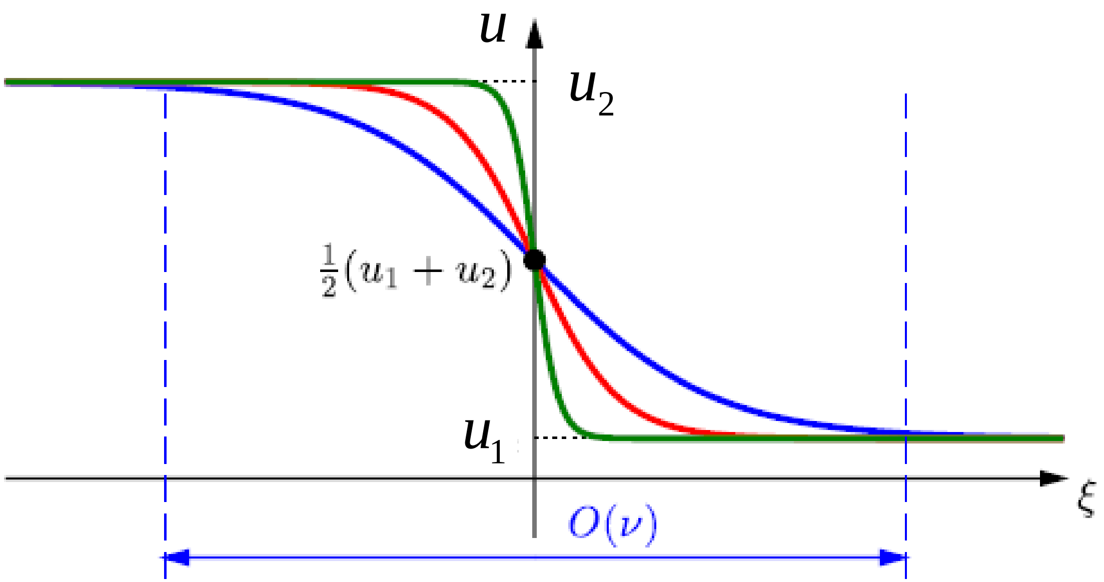
$$\int \frac{du}{(u - u_1)(u - u_2)} = \frac{1}{u_2 - u_1} \ln\left(\frac{u_2 - u}{u - u_1}\right) = \frac{1}{2\nu}(\xi - \delta)$$

where δ is a constant of integration. Then, solving the above equation for u, we obtain:

$$\boxed{\begin{aligned} u(\xi) &= \frac{u_2 + u_1 \exp[\alpha(\xi - \delta)]}{1 + \exp[\alpha(\xi - \delta)]} \\ &= u_1 + \frac{u_2 - u_1}{1 + \exp[\alpha(\xi - \delta)]} \\ &= \frac{1}{2}(u_2 + u_1) - \frac{1}{2}(u_2 - u_1) \tanh\left(\frac{u_2 - u_1}{4\nu}(\xi - \delta)\right) \end{aligned}}$$

where $\boxed{\alpha = \frac{1}{2\nu}(u_2 - u_1) > 0.}$

Structure of the traveling shock wave



- The shock wave (SW) asymptotes the equilibrium points:

$$\boxed{\lim_{\xi\to-\infty} u(\xi) = u_2, \qquad \lim_{\xi\to+\infty} u(\xi) = u_1}$$

- Using: $u_1 = c - \sqrt{c^2+2K}, \quad u_2 = c + \sqrt{c^2+2K},$ we obtain:

$\boxed{c = \frac{1}{2}(u_1+u_2)}$ velocity of the SW consistent with **RH condition!**

- "Shock thickness" $\nu/(u_2-u_1)$: **diffusion prevents breaks up!**

The Cole-Hopf transformation

Background

- It was devised independently by **Eberhard Hopf** **(German)** **(1950)** and **Julian Cole** **(American)** **(1951)** [and also, even earlier, by **V. Florin** **(Russian)** **(1948)**].

E. Hopf

J. D. Cole

- It is a ***remarkable*** **nonlinear transformation** that reduces the **Burgers equation** to the **linear diffusion equation.** This way, the nonlinear Burgers equation can be explicitly solved.

- The Cole-Hopf transformation is a **milestone in the field of nonlinear PDEs**, and has inspired —among others— important developments in the **theory of solitons**

Introducing the Cole-Hopf transformation (I)

- Express the Burgers equation in a **conservation law** form:

$$u_t = \left(\nu u_x - \frac{1}{2}u^2\right)_x \qquad (1)$$

- Introduce the **"potential" function *U(x, t)***, which is actually the **antiderivative of *u*(*x*, *t*)**. The potential function $U(x,t)$ satisfies:

$$U_x = u, \quad U_t = \nu u_x - \frac{1}{2}u^2 \qquad (2)$$

[To see this, use the compatibility condition $u_{xt} = u_{tx}$ and (1)]

- Combine Eqs. (2) to derive the **Hamilton-Jacobi equation**:

$$U_t + \frac{1}{2}U_x^2 = \nu U_{xx}.$$

- Introduce the **Cole-Hopf relation**:

$$U = -2\nu \ln \Phi$$

Introducing the Cole-Hopf transformation (II)

To this end, substitute the **Cole-Hopf relation**: $U = -2\nu \ln \Phi$

into the **Hamilton-Jacobi equation**: $U_t + \frac{1}{2}U_x^2 = \nu U_{xx}$.

Then, observing that:

$$U_t = -2\nu\frac{\Phi_t}{\Phi}, \quad U_x = -2\nu\frac{\Phi_x}{\Phi}, \quad U_x^2 = 4\nu^2\frac{\Phi_x^2}{\Phi^2}, \quad U_{xx} = -2\nu\frac{\Phi\Phi_{xx} - \Phi_x^2}{\Phi^2}$$

the **Hamilton-Jacobi** equation transforms into the **linear diffusion equation for Φ**:

$$\Phi_t = \nu\Phi_{xx}$$

Thus, If $\Phi(x, t)$ is any nonzero solution of the **diffusion equation**

$$\Phi_t = \nu\Phi_{xx}$$

then $u(x,t) = \frac{\partial}{\partial x}\left[-2\nu \ln \Phi(x,t)\right] = -2\nu\frac{\Phi_x}{\Phi}$

satisfies the **Burgers equation:** $u_t + uu_x = \nu u_{xx}$

Burgers equation vs. diffusion equation

Consider the following Cauchy problem for the Burgers equation:

$$u_t + uu_x = \nu u_{xx},$$
$$u(x,0) = u_0(x), \quad -\infty < x < +\infty$$

Then, employing the Cole-Hopf relation, we can determine $\Phi_0(x)$ from $u_0(x)$, i.e., the **initial condition for the diffusion equation** from the one for the above Burgers equation:

$$u_0(x) = -2\nu \frac{\partial}{\partial x}\left[\ln \Phi_0(x)\right] \Rightarrow \Phi_0(x) = \exp\left[-\frac{1}{2\nu}\int_0^x u_0(x')dx'\right]$$

Thus, thanks to the **Cole-Hopf transformation**, instead of solving **the nonlinear problem**, we only need to solve the **linear problem**:

$$\Phi_t = \nu \Phi_{xx},$$
$$\Phi(x,0) = \Phi_0(x) = \exp\left[-\frac{1}{2\nu}\int_0^x u_0(x')dx'\right], \quad -\infty < x < +\infty$$

Solution of the Burgers equation (I)

The general solution of the Cauchy problem for the diffusion equation can be expressed in terms of the **convolution integral**:

$$\Phi(x,t) = \int_{-\infty}^{+\infty} G(x-y)\Phi_0(y)dy$$

where: $G(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \exp\left(-\frac{x^2}{4\nu t}\right)$ is the **fundamental solution** (or **Green's function**) of the diffusion equation.

Then, we use the general solution in the form:

$$\Phi(x,t) = \frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(x-y)^2}{4\nu t}\right] \Phi_0(y)dy$$

and find its derivative with respect to x:

$$\Phi_x(x,t) = -\frac{1}{\sqrt{4\pi\nu t}} \int_{-\infty}^{+\infty} \exp\left[-\frac{(x-y)^2}{4\nu t}\right] \left(\frac{x-y}{2\nu t}\right) \Phi_0(y)dy$$

Solution of the Burgers equation (II)

Finally, we can **construct the solution $u(x, t)$ of the Burgers equation** by means of the **Cole-Hopf transformation**:

$$u(x,t) = -2\nu \frac{\Phi_x}{\Phi} = \frac{\int_{-\infty}^{+\infty} \frac{x-y}{t} \exp\left[-\frac{(x-y)^2}{4\nu t}\right] \Phi_0(y) dy}{\int_{-\infty}^{+\infty} \exp\left[-\frac{(x-y)^2}{4\nu t}\right] \Phi_0(y) dy}$$

We can also use the equation: $\Phi_0(x) = \exp\left[-\frac{1}{2\nu}\int_0^x u_0(x')dx'\right]$

and rewrite the solution of the Burgers equation as follows:

$$u(x,t) = \frac{\int_{-\infty}^{+\infty} \frac{(x-y)}{t} \exp\left[-\frac{1}{2\nu} F(x,y,t)\right] dy}{\int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2\nu} F(x,y,t)\right] dy}$$

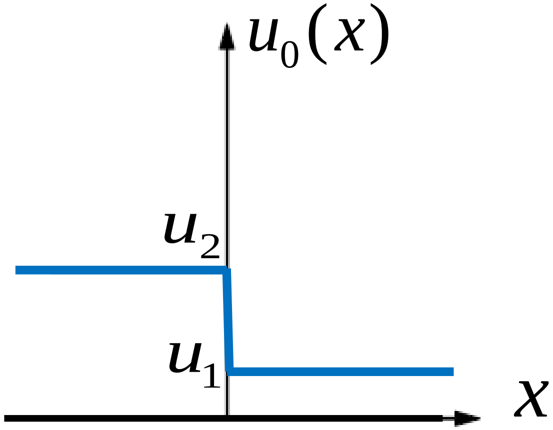
where $F(x,y,t)$ is given by: $F(x,y,t) = \frac{(x-y)^2}{2t} + \int_0^y u_0(y')dy'$

Riemann problem* for the Burgers equation

Consider the following IVP for the Burgers equation:

$$u_t + uu_x = \nu u_{xx},$$
$$u(x,0) = u_0(x),$$

where $u_0(x)$ reads: $u_0(x) = \begin{cases} u_2 & \text{for } x < 0 \\ u_1 & \text{for } x > 0 \end{cases}$



This initial condition has the form of a step-like shock, which will evolve to a genuine shock wave **in the inviscid limit of $\nu = 0$**

We wish to employ the results of the analysis above, and find the solution of this Riemann problem for the Burgers equation

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* Recall that a **Riemann problem** is an initial value problem for a hyperbolic PDE (or a system thereof) in which the initial data is piecewise constant with a discontinuity.

Solution via the Cole-Hopf transformation (I)

Recall that:
$$\boxed{u(x,t) = \frac{\int_{-\infty}^{+\infty} \frac{(x-y)}{t} \exp\left[-\frac{1}{2\nu} F(x,y,t)\right] dy}{\int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2\nu} F(x,y,t)\right] dy}}$$

We can find:
$$u(x,t) = \frac{\int_{-\infty}^{+\infty} \frac{x-y}{t} R(y) \exp\left[\frac{-(x-y)^2}{4\nu t}\right] dy}{\int_{-\infty}^{+\infty} R(y) \exp\left[\frac{-(x-y)^2}{4\nu t}\right] dy} \qquad (1)$$

where R is given by:
$$R(x) = \begin{cases} \exp\left(-\frac{u_2 x}{2\nu}\right) & \text{for } x < 0, \\ \exp\left(-\frac{u_1 x}{2\nu}\right) & \text{for } x > 0. \end{cases}$$

Then, upon manipulating the integrals in Eq. (1), we can find that the solution can be rewritten in the following form:

Solution via the Cole-Hopf transformation (II)

$$u(x,t) = u_1 + \frac{u_2 - u_1}{1 + \frac{\mathrm{erfc}\left(-\frac{x-u_1 t}{2\sqrt{\nu t}}\right)}{\mathrm{erfc}\left(\frac{x-u_2 t}{2\sqrt{\nu t}}\right)} \exp\left[\frac{u_2-u_1}{2\nu}(x-ct)\right]}$$

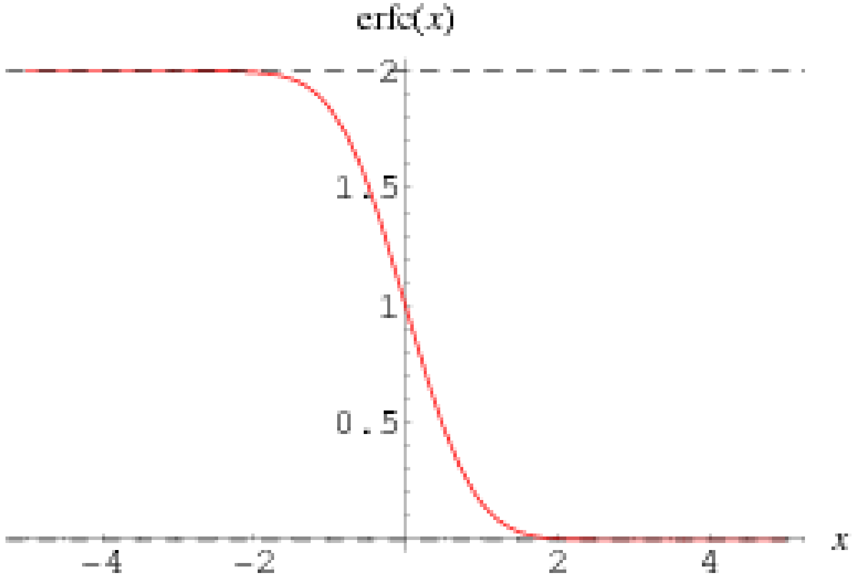
where: $\underline{c = (1/2)(u_1 + u_2)}$ as per the **Rankine-Hugoniot condition!**

while erfc(x) is the **complementary error function** defined as:

$$\mathrm{erfc}(x) \equiv 1 - \mathrm{erf}(x)$$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_0^x \exp(-x'^2)dx'$$

$$= \frac{2}{\sqrt{\pi}} \int_x^{+\infty} \exp(-x'^2)dx'$$



Asymptotic form of the solution

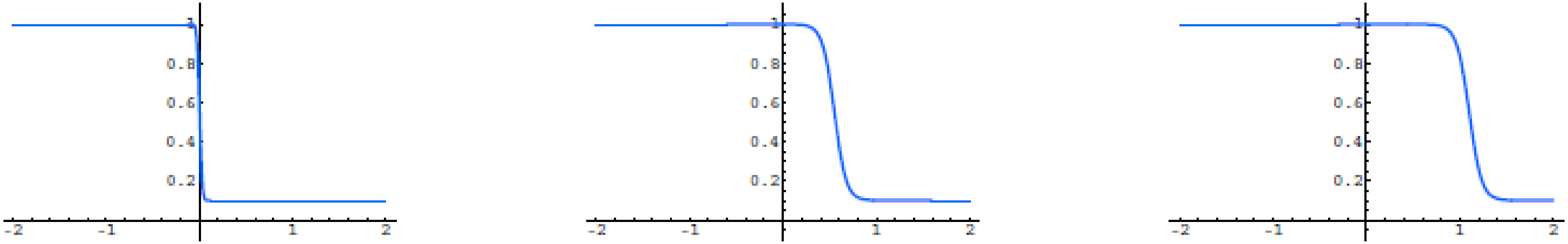
For fixed $x - ct$, the asymptotic behavior of the solution as $t \longrightarrow +\infty$ is:

$$\boxed{\lim_{t \to +\infty} u(x,t) = u_1 + \frac{u_2 - u_1}{1 + \exp\left[\frac{u_2 - u_1}{2\nu}(x - ct)\right]}}$$

which is identical to the equilibrium solution of the Burgers equation!

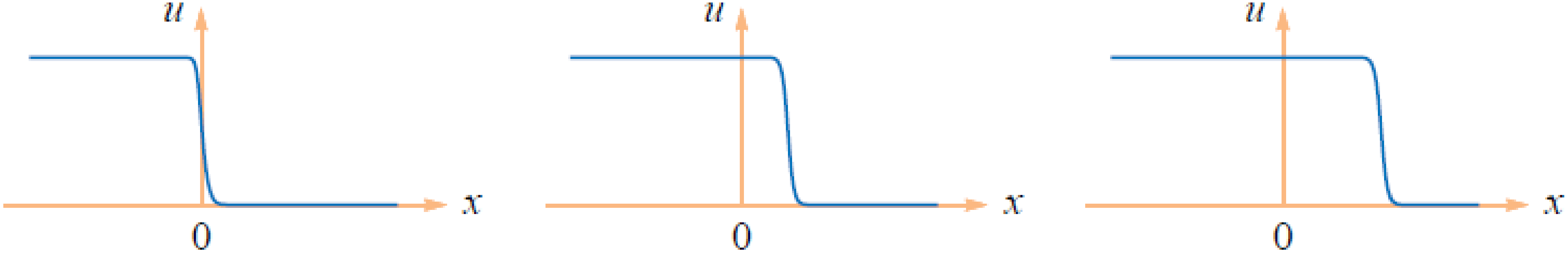
The results of the above analysis can be interpreted as follows.

The **initial discontinuity** is **eventually smoothed**, with the solution developing a continuously varying transition layer between the two asymptotic values u_2 and u_1.

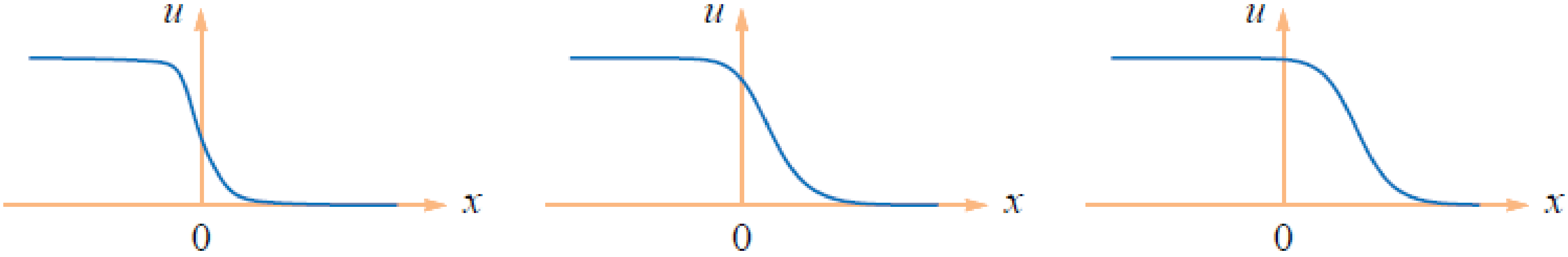


Spatial profiles of the solution, at $t = 0.01, 1$ and 2, for $u_2 = 1$, $u_1 = 0.1$ and $\nu = 0.03$

Profile of the solution



Shock wave solution of Burgers equation for low viscosity ν



Shock wave solution of Burgers equation for high viscosity ν

The smoothing effect is more pronounced for larger viscosity ν

The case of an initial single hump

Consider the following IVP for the Burgers equation:

$$u_t + uu_x = \nu u_{xx},$$
$$u(x,0) = u_0(x),$$

where $u_0(x)$ reads: $\boxed{u(x,0) = F(x) = A\delta(x)}$

This initial condition has the form of a delta function, which will evolve as the **fundamental solution of the diffusion equation in the limit of $\nu \rightarrow 0$**

As before, we wish to employ the results of the analysis above, and find the solution of this initial value problem for the full case, i.e., for the Burgers equation

Solution via the Cole-Hopf transformation (I)

Recall that:
$$u(x,t) = \frac{\int_{-\infty}^{+\infty} \frac{(x-y)}{t} \exp\left[-\frac{1}{2\nu}F(x,y,t)\right]dy}{\int_{-\infty}^{+\infty} \exp\left[-\frac{1}{2\nu}F(x,y,t)\right]dy} \qquad (1)$$

where $F(x,y,t)$ is given by:
$$F(x,y,t) = \frac{(x-y)^2}{2t} + \int_0^y u_0(y')dy'$$

For $u_0(x) = A\delta(x)$, we find:
$$F(x,y,t) = \begin{cases} \frac{(x-y)^2}{2t}, & \text{for } y < 0 \\ \frac{(x-y)^2}{2t} + A, & \text{for } y > 0 \end{cases}$$

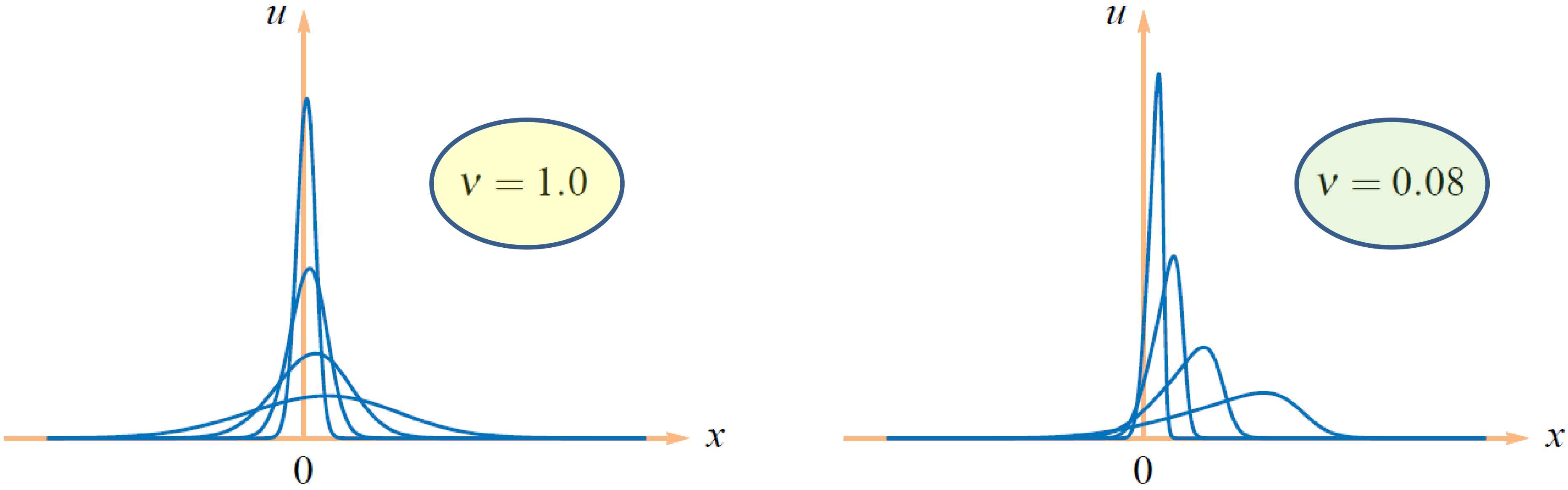
where the lower limit of integration was taken to be $-\infty$.

Then, upon manipulating the integrals in Eq. (1), we can find that the solution can be rewritten in the following form:

Solution via the Cole-Hopf transformation (II)

$$u(x,t) = \sqrt{\frac{\nu}{t}} \left[\frac{\left(e^{\frac{A}{2\nu}} - 1\right) e^{-\frac{x^2}{4\nu t}}}{\sqrt{\pi} + \left(e^{\frac{A}{2\nu}} - 1\right) \sqrt{\frac{\pi}{2}}\, \mathrm{erfc}\left(\frac{x}{\sqrt{4\nu t}}\right)} \right]$$

where erfc(x) is the **complementary error function.**



Evolution of the initial hump for **“large”** and **“small”** viscocity

The limiting case of $\nu \to \infty$

$$u(x,t) = \sqrt{\frac{\nu}{t}} \left[\frac{\left(e^{\frac{A}{2\nu}} - 1\right) e^{-\frac{x^2}{4\nu t}}}{\sqrt{\pi} + \left(e^{\frac{A}{2\nu}} - 1\right) \sqrt{\frac{\pi}{2}} \operatorname{erfc}\left(\frac{x}{\sqrt{4\nu t}}\right)} \right]$$

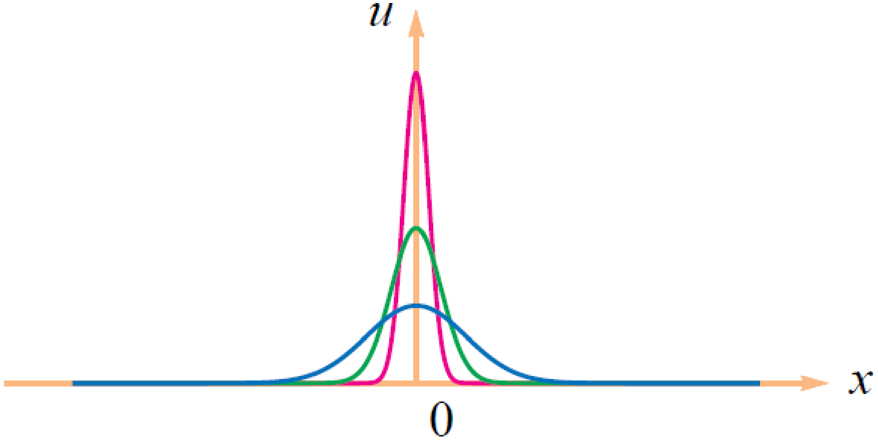
In the limit $\nu \to \infty$ we have:

$$\operatorname{erfc}\left(\frac{x}{\sqrt{4\nu t}}\right) \to 0$$

$$e^{\frac{A}{2\nu}} \to 1 + \frac{A}{2\nu}$$

and the solution becomes:

$$\boxed{u(x,t) = \frac{A}{\sqrt{4\pi\nu t}} e^{-\frac{x^2}{4\nu t}}}$$



which is the fundamental solution of the diffusion equation