Nonlinear dispersive wave equations

Basic physically significant models

Constructing linear dispersive PDEs

- **Recall:** Given a linear dispersive PDE, the substitution:

$$\partial_t \mapsto -i\omega,\ \ \partial_x \mapsto ik$$

leads to the dispersion relation, $\boldsymbol{D(\omega,k) = 0}$.

- **Reversely:** Using: $\omega \mapsto i\partial_t,\ \ k \mapsto -i\partial_x$

the dispersion relation will become a PDE:

$$Q(i\partial_t, -i\partial_x)u = 0$$

where $\boldsymbol{Q}$ is the operator corresponding to $\boldsymbol{D}$ and $\boldsymbol{u(x,t)}$ is a field accounting for the wave motion (e.g., the wave amplitude on the free surface of water, electric field envelope, etc)

  - **This way, we can construct, via simple dispersion relations, a wealth of linear dispersive equations.**

Unidirectional propagation

Consider, e.g., the **polynomial dispersion relation**:

$$\omega = \omega(k) = \alpha_0 + \alpha_1 k + \alpha_2 k^2 + \alpha_3 k^3 + \cdots ,$$

Using: $\omega \mapsto i\partial_t, \; k \mapsto -i\partial_x$ we obtain the operator:

$$i\partial_t = \alpha_0 - i\alpha_1 k\partial_x - \alpha_2\partial_x^2 - i\alpha_3\partial_x^3 + \cdots$$

Examples

1) Choose: $i\partial_t = \cancel{\alpha_0} - i\alpha_1 k\partial_x - \cancel{\alpha_2\partial_x^2} - \cancel{i\alpha_3\partial_x^3} + \cdots$, $a_1 = c$

and obtain: $u_t + cu_x = 0$ **Transport equation** (dispersionless)

2) Choose: $i\partial_t = \cancel{\alpha_0} - i\alpha_1 k\partial_x - \cancel{\alpha_2\partial_x^2} - i\alpha_3\partial_x^3 + \cdots$, $a_1 = c$, $a_3 = \gamma$

and obtain: $u_t + cu_x + \gamma u_{xxx} = 0$ **Linearized KdV** (dispersive)

3) Choosing $a_0 = -i\Gamma, a_2 = -iD \rightarrow$ $u_t + cu_x + \underbrace{\gamma u_{xxx}}_{\text{dispersion}} = \underbrace{\Gamma u}_{\text{linear loss}} + \underbrace{\mathrm{D}u_{\mathrm{xx}}}_{\text{diffusion}}$

Introducing nonlinearity

➢ **Nonlinearity** can be introduced in the linear models as follows. ***A fundamental property of <u>any</u> nonlinear wave, is the field amplitude dependence of the phase velocity.***

- In the simplest possible case, and using $\boldsymbol{v}_{\mathbf{p}} = \boldsymbol{c}$, this dependence assumes the following **polynomial** form:

$$c = c_0(1 + \beta_1 u + \beta_2 u^2 + \cdots)$$

Example

Transport equation:

$$\left.\begin{aligned} u_t + c u_x &= 0 \\ c &= c_0(1+\beta_1 u) \end{aligned}\right\} \to \boxed{u_t + c_0(1+\beta_1 u)u_x = 0}$$

Galilei transformation and **rescale of time:**

$$\left.\begin{aligned} &u_t + c_0 u_x + \beta_1 u u_x = 0 \\ &x \mapsto x - c_0 t,\ \ t \mapsto t \end{aligned}\right\} \to \left.\begin{aligned} &u_t + \beta_1 u u_x = 0 \\ &t \mapsto (1/\beta_1)t \end{aligned}\right\} \to \boxed{u_t + u u_x = 0}$$ **Hopf equation**

A physical origin of the Hopf equation

- **Motion of a fluid / Gas dynamics** is governed by the system:
  - **Continuity equation:** $\rho_t + \nabla(\rho \mathbf{v}) = 0,$
  - **Euler equation:** $\mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v} = -(1/\rho)\,\nabla p$

  together with a "equation of state": $p = p(\rho)$

- Kinetic theory of gases – ideal gas: $pV = NkT \longrightarrow p = (\rho/m)kT$

- In the limiting case: $m \to \infty$ the pressure vanishes: $p = 0$

Thus, for "heavy particles" the gas dynamics equations decouple:

$$\mathbf{v}_t + (\mathbf{v}\nabla)\mathbf{v} = 0, \quad \rho_t + \nabla(\rho \mathbf{v}) = 0$$

- In the case of 1D flow, the velocity field is governed by:

$v_t + vv_x = 0$ **Hopf (Riemann / inviscid Burgers) equation**

The KdV equation (and its cousins)

a) Choose: $i\partial_t = \cancel{\alpha_0} - i\alpha_1 k\partial_x - \cancel{\alpha_2\partial_x^2} - i\alpha_3\partial_x^3 + \cdots$, $a_1=c$, $a_3=\gamma$

and obtain: $u_t + cu_x + \gamma u_{xxx} = 0$ Linearized KdV

b) Introduce nonlinearity through c:

$$\left.\begin{array}{l} u_t + cu_x + \gamma u_{xxx} = 0 \\ c = c_0(1+\beta u) \end{array}\right\} \to u_t + c_0(1+\beta u)u_x + \gamma u_{xxx} = 0$$

c) Galilei transformation:

$$\left.\begin{array}{l} u_t + c_0 u_x + \beta u u_x + \gamma u_{xxx} = 0 \\ x \mapsto x - c_0 t,\ \ t \mapsto t \end{array}\right\} \to \boxed{u_t + \beta u u_x + \gamma u_{xxx} = 0}$$ **KdV equation**

Members of the KdV family

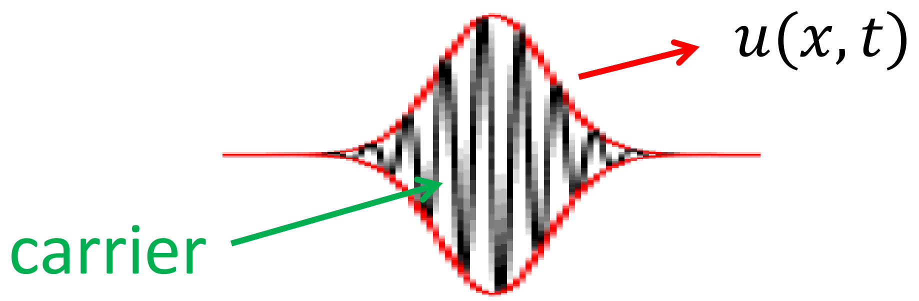
$u_t + c_0 u_x + \beta u u_x + \gamma u_{xxx} + \nu u_{xxxxx} = 0$	**5th-order KdV**
$u_t + c_0 u_x + \beta u u_x + \delta u^2 u_x + \gamma u_{xxx} = 0$	**Gardner equation**
$u_t + \beta u u_x + \gamma u_{xxx} = \mu u_{xx}$	**KdV-Burgers**

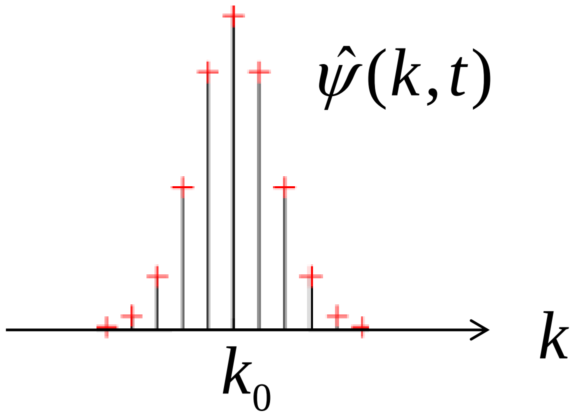
Nonlinear wave envelopes

Consider a **wavepacket**, composed by a carrier wave, namely a plane wave of the form $\exp[i(k_0 x - \omega_0 t)]$, which is modulated by a generally complex field envelope $u(x, t)$, resulting in the real field:

$$\psi(x,t) = \mathrm{Re}\{u(x,t)\exp[i(k_0 x - \omega_0 t)]\}$$

The spectrum of the wavepacket is located around k_0 (and ω_0)





Assume that the wave obeys the nonlinear dispersion relation:

$$\omega = \omega(k, I), \quad I = |u|^2$$

The nonlinear Schrödinger (NLS) equation

Since the spectrum is located around k_0, we may **Taylor expand the dispersion relation**, i.e., $\omega(k)$ **around k_0:**

$$\omega = \omega(k, I) \approx \omega(k_0) + \left.\frac{\partial \omega}{\partial k}\right|_{k=k_0} (k - k_0) + \frac{1}{2}\left.\frac{\partial^2 \omega}{\partial k^2}\right|_{k=k_0} (k - k_0)^2 + \ldots + \left.\frac{\partial \omega}{\partial I}\right|_{I=0} I + \ldots$$

$$\Rightarrow \boxed{\omega - \omega_0 \approx \omega_0'(k - k_0) + \frac{1}{2}\omega_0''(k - k_0)^2 - gI, \quad g = -\left.\frac{\partial \omega}{\partial I}\right|_{I=0}}$$

Using: $\omega - \omega_0 \mapsto i\partial_t, \quad k - k_0 \mapsto -i\partial_x$ we obtain the operator:

$$i\partial_t = -iv_g\partial_x - \frac{1}{2}\omega_0''\partial_x^2 - gI, \qquad v_g = \omega_0'$$

and operating on u(x,t):

$$\boxed{i(u_t + v_g u_x) + \frac{1}{2}\omega_0'' u_{xx} + g|u|^2 u = 0} \qquad \textbf{NLS equation}$$

A bidirectional model: Boussinesq

Consider, e.g., the **dispersion relation** of the **KdV equation**:

$$\omega = k(c - \gamma k^2)$$

and assume small dispersion and long waves, such that: $\gamma k^2 << c$

Then, approximate the **square** of the dispersion relation as:

$$\omega^2 \approx c^2 k^2 - 2c\gamma k^4$$

and use $\omega \mapsto i\partial_t,\; k \mapsto -i\partial_x$ to derive the linearized Boussinesq:

$$\boxed{u_{tt} - c^2 u_{xx} - 2\gamma c u_{xxxx} = 0}$$

Next, introduce nonlinearity (with small nonlinearity coefficient β):

$$c = c_0(1 + \beta u) \Rightarrow c^2 \approx c_0^2 + 2c_0 \beta u$$

and thus: $c^2 u_{xx} = \partial_x^2 (c^2 u) = \partial_x^2 [(c_0^2 + 2c_0\beta u)u = c_0^2 u_{xx} + 2c_0\beta (u^2)_{xx}$

leading to: $\boxed{u_{tt} - c_0^2 u_{xx} - 2\gamma c_0 u_{xxxx} + 2c_0 \beta (u^2)_{xx} = 0 \quad \textbf{Boussinesq equation}}$

A bidirectional model: Klein-Gordon

Consider again the **polynomial dispersion relation and assume**:

$$\omega = \omega(k) = \alpha_0 + \cancel{\alpha_1} k + \alpha_2 k^2 + \cancel{\alpha_3} k^3 + \cdots ,$$

Furthermore, consider long waves, such that: $k^2 \ll \alpha_0/\alpha_2$

Then, the square of the dispersion relation is approximated as:

$$\omega^2 \approx k^2 c_0^2 + m^2$$

where $c_0^2 = 2\alpha_0\alpha_2$ and $m^2 = \alpha_0^2$. This corresponds to:

$u_{tt} - c_0^2 u_{xx} + m^2 u = 0$ Klein Gordon equation

Nonlinear Klein Gordon

$$u_{tt} - c_0^2 u_{xx} = -V'(u) \qquad V(u) = \begin{cases} (1/2)m^2u^2 \longrightarrow \text{Linear KG} \\ (1/2)m^2u^2 - (1/4)\kappa^2 u^4 \longrightarrow \phi^4 \\ 1 - \cos(u) \longrightarrow \text{Sine-Gordon} \end{cases}$$