Linear dispersive wave equations

The effect of dispersion

Linear PDEs and useful notions

- Consider a differentiable scalar function $u(x,t)$, a partial differential operator L, and the PDE: $L[u] = 0$

  Let u_1 and u_2 two different solutions of the PDE; the latter is said to be linear iff: $L[u_1+u_2] = L[u_1] + L[u_2] = 0$

- **Dispersive wave equations**: existence of **plane waves**:

$$u(x,\mathrm{t}) = u_0 \exp(\mathrm{i}\theta), \quad \theta = kx\text{-}\omega t, \quad k, \omega \in \mathbb{R}$$

➡ Temporal period: $T=2\pi/\omega$.

➡ Spatial period (wavelength): $\lambda=2\pi/k$

- Long waves: correspond to **small *k***
- Short waves: correspond to **large *k***

Dispersion relation

Substituting $u = u_0 \exp[i(kx - \omega t)]$ into the PDE we can find that u_0 factors out (because the equation is linear), while k and ω should be related by an equation of the form:

$D(\omega,k) = 0$ or $\omega = \omega(k)$ **Dispersion relation**

so that the plane wave satisfies $L[u] = 0$

Examples

Transport equation	$u_t + cu_x = 0$	$\mapsto D(\omega,k) = \omega - ck = 0$
2nd-order wave equation	$u_{tt} - c^2 u_{xx} = 0$	$\mapsto D(\omega,k) = \omega^2 - c^2 k^2 = 0$
Schrödinger equation	$iu_t + u_{xx} = 0$	$\mapsto D(\omega,k) = \omega - k^2 = 0$
Linearized KdV equation	$u_t + u_{xxx} = 0$	$\mapsto D(\omega,k) = \omega + k^3 = 0$

Phase and group velocities

Given the dispersion relation $\omega = \omega(k)$ we can find:

- Phase velocity: $v_p = \omega/k$

[Plane wave: $u(x,t) \sim \exp(i\theta)$, $\theta = kx - \omega t = k[x-(\omega/k)t] = k(x - v_p t)$]

- Group velocity: $v_g = \partial\omega/\partial k \equiv \omega'(k)$

Let a pulse-shaped wave be represented as a sum of Fourier harmonics, with the dispersion relation $\omega = \omega(k)$. In the case of two harmonics with close wavenumbers and frequencies:

$$u(x,t) = a\cos\left[\left(k_0 - \frac{1}{2}\Delta k\right)x - \left(\omega_0 - \frac{1}{2}\Delta\omega\right)t\right] + a\cos\left[\left(k_0 + \frac{1}{2}\Delta k\right)x - \left(\omega_0 + \frac{1}{2}\Delta\omega\right)t\right] \Rightarrow$$

$$u(x,t) = 2a\cos[(\Delta k)x - (\Delta\omega)t]\cos(k_0 x - \omega_0 t)$$

$$\omega_0 = \omega(k_0),$$

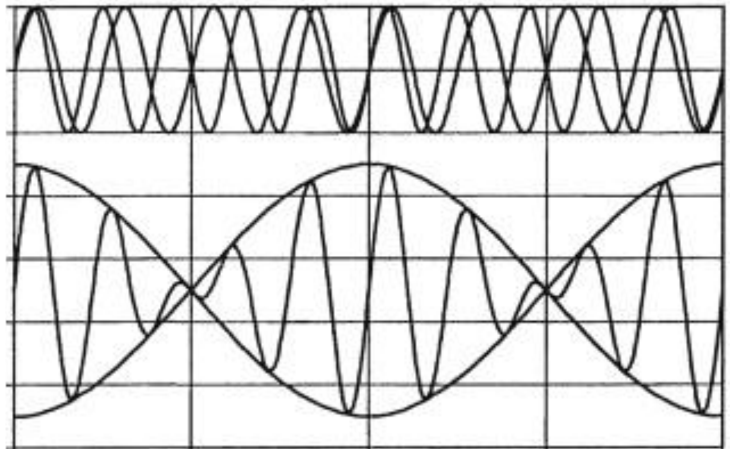
$$\Delta\omega = \left.\frac{d\omega}{dk}\right|_{k=k_0}\Delta k,$$

$$\Delta k << k_0$$

Wave's envelope

The sum of the two harmonics has the form of a **modulated wave:**

$$u(x,t) = \boxed{2a\cos[(\Delta k)x - (\Delta\omega)t]}\cos(k_0 x - \omega_0 t)$$



The **envelope function** $A(x,t) = 2a\cos[(\Delta k)x - (\Delta\omega)t]$
propagates with the group velocity: $v_g = \Delta\omega/\Delta k$

Generalization: arbitrary wavepacket with narrow spectrum of k' s:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} A(k)\exp[i(kx - \omega(k)t)]dk$$

Wave's envelope and group velocity

Rewrite $u(x,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}A(k)\exp[i(kx-\omega(k)t)]dk$ as:

$$u(x,t)=\exp[i(k_0x-\omega_0t)\frac{1}{2\pi}\int_{-\infty}^{+\infty}A(k)\exp[i(k-k_0)x-(\omega(k)-\omega_0)t)]dk$$

and taking into account that the spectrum is narrow, expand the frequency in powers of (k-k_0): $\omega(k)-\omega_0\approx\omega'(k_0)(k-k_0)+...$

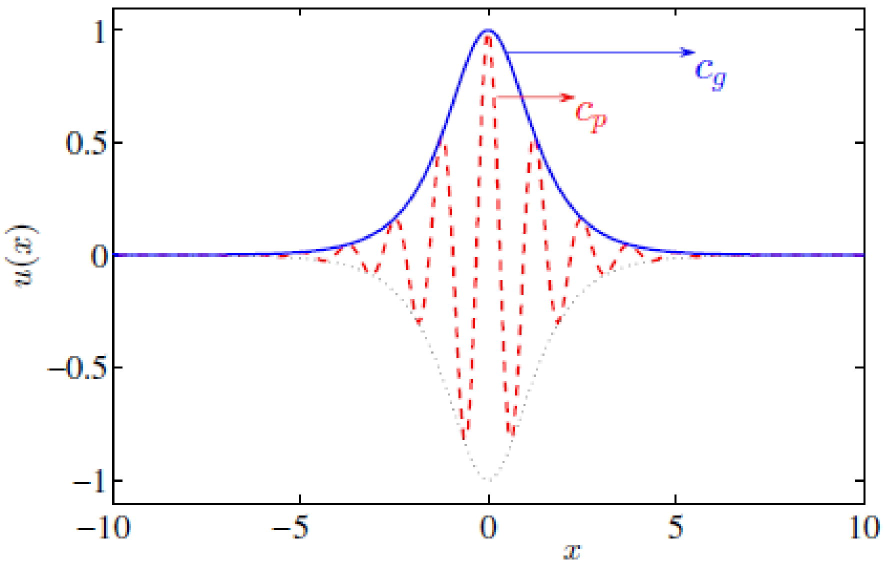
Then:

$$u(x,t)=\exp[i(k_0x-\omega_0t)\frac{1}{2\pi}\int_{-\infty}^{+\infty}A(k)\exp[i(k-k_0)(x-\omega'(k_0)t)]dk$$

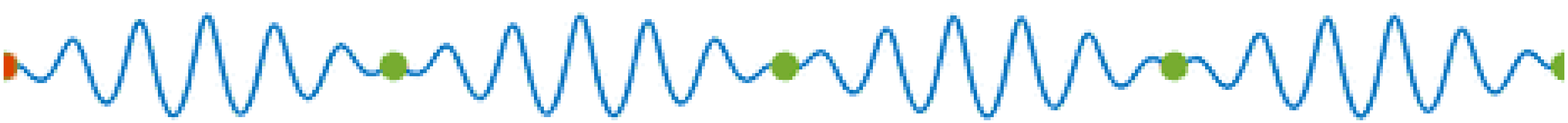
$$=\exp[i(k_0x-\omega_0t)A(x-\omega'(k_0)t)$$

which means that the envelope function $A(x,t)$ propagates with the group velocity $v_g=\left.\frac{d\omega}{dk}\right|_{k=k_0}=\left.\omega'(k)\right|_{k=k_0}$

Phase and group velocities – an example



An example of an envelope **(blue)** and a carrier wave **(red)**. The envelope moves with the **group velocity** c_g, while the carrier inside it moves with the **phase velocity** c_p.



The **red square** moves with the **phase velocity**, and the **green circles** propagate with the **group velocity**.

Dispersive and non-dispersive PDEs

- **Recall:** Phase velocity: $v_p = \omega/k$ - Group velocity: $v_g = \partial\omega/\partial k \equiv \omega'(k)$
- If $\omega(k) \in \mathbb{R}$ and $\omega''(k) \neq 0$ or $v_p \neq v_g$: the PDE/wave: **Dispersive**

Examples

Transport equation $\rightarrow$ Non-dispersive equation

$$u_t + cu_x = 0 \quad \mapsto \quad \omega = ck \Rightarrow v_p = v_g = \omega/k,\ \omega''(k) = 0$$

2nd-order wave equation $\rightarrow$ Non-dispersive equation

$$u_{tt} - c^2 u_{xx} = 0 \ \mapsto \ \omega^2 = c^2k^2 \Rightarrow v_p = v_g = \pm\omega/k,\ \omega''(k) = 0$$

Schrödinger equation $\rightarrow$ Dispersive equation

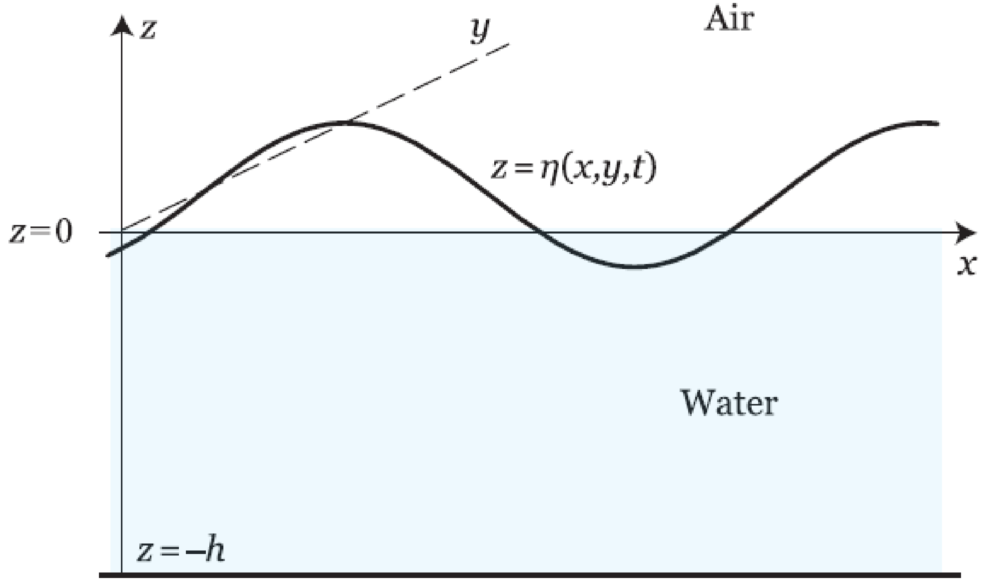
$$iu_t + u_{xx} = 0 \quad \mapsto \quad \omega = k^2 \Rightarrow v_p = k \neq v_g = 2k,\ \omega''(k) \neq 0$$

Linearized KdV equation $\rightarrow$ Dispersive equation

$$u_t + u_{xxx} = 0 \quad \mapsto \quad \omega = -k^3 \Rightarrow v_p = -k^2 \neq v_g = -3k^2,\ \omega''(k) \neq 0$$

A physical example: gravity water waves

As an example of a *dispersive wave system*, consider **surface water waves (WWs)**. If the surface of water is disturbed, then the gravity force will try to restore the equilibrium, which leads to the emergence of the surface water wave.



Dispersion relation: $\omega(k) = \sqrt{gk \tanh(kh)}$

Two physically relevant regimes:

- **Shallow WWs:** $\lambda >> h \Rightarrow kh << 1$
- **Deep WWs:** $\lambda << h \Rightarrow kh >> 1$

Shallow WWs – dispersionless case (I)

For shallow WWs, $kh \ll 1$, the dispersion relation reduces to:

$$\omega(k) = \sqrt{gk\tanh(kh)} \approx \sqrt{gh}\,k \quad \textbf{(Lagrange formula)}$$

Then, the phase of the plane waves is: $\theta = kx - \omega t = k\left(x - \sqrt{gh}\,t\right)$

and the water's free surface (1D) is: $\eta(x,t) = A\cos\left(k\left(x - \sqrt{gh}\,t\right)\right)$

Thus, **ALL** plane waves have the same phase velocity: $v_p = \sqrt{gh}$

NOTE: The shallow WWs regime is relevant to **tsunamis** that may result from earthquakes in the oceans. In this case, for ocean depth h = 4 km, and g = 10 m/s^2, one obtains:

$$v_p = \sqrt{gh} = 200\, m/s = 720\, km/h \quad (!)$$

Shallow WWs – dispersionless case (II)

■ Since **ALL** plane waves have the same phase velocity, a wavepacket composed by these harmonics also has the same velocity; hence, any initial disturbance **propagates undistorted.**

To show this , we will derive a PDE for $\eta(x,t)$ and solve the IVP.

Recall that using : $\partial_t \mapsto -i\omega,\ \partial_x \mapsto ik \quad \Rightarrow$ PDE $\rightarrow$ ω = ω(k)
Reversely, $\omega \mapsto i\partial_t,\ k \mapsto -i\partial_x \Rightarrow \omega = \omega(k) \rightarrow$ PDE: $Q(i\partial_t, -i\partial_x)u = 0$

Thus: $\omega(k) = \sqrt{gh}k \mapsto \boxed{\eta_t + c\eta_x = 0,}\ c = \sqrt{gh}$

Let the initial disturbance $\eta(x,0) = \eta_0(x)$ presented as a sum of of its Fourier harmonics: $\eta(x,0) = \eta_0(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{\eta}_0(k)e^{ikx}dk$. Then:

$$\hat{\eta}_t + c\hat{\eta} = 0 \Rightarrow \hat{\eta}(k,t) = \hat{\eta}_0(k)e^{-ict} \Rightarrow \eta(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{\eta}_0(k)e^{i(kx-ct)}dk = \eta_0(x-ct)$$

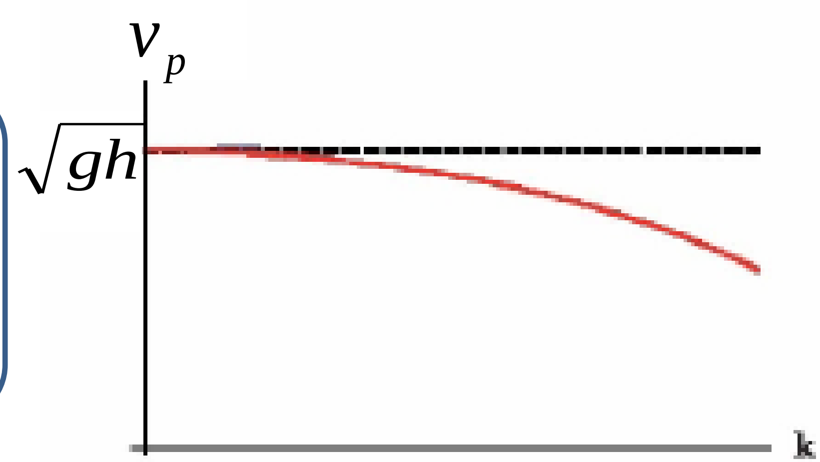
Shallow WWs – dispersive case (I)

- Take into regard **two terms** in the Taylor series expansion of the dispersion relation (for shallow WWs with *kh* << 1):

$$\omega(k) = \sqrt{gk\tanh(kh)} \approx \sqrt{gh}\,k\left(1 - \frac{1}{6}(hk)^2\right)$$

Thus, **the phase velocity now becomes *k*-dependent:**

$$v_p = \frac{\omega(k)}{k} = \sqrt{gh}\left(1 - \frac{1}{6}(hk)^2\right)$$



Harmonics with shorter wavelengths (longer waves) propagate faster → DISPERSION

Shallow WWs – dispersive case (II)

In the dispersive case, each harmonic:

$$\eta = A(k)\exp\left(i\left(kx-\omega(k)t\right)\right)$$

satisfy the dispersion relation: $\omega(k)=\sqrt{gh}\,k\left(1-\frac{1}{6}(hk)^2\right)$

and, hence, using again $\omega \mapsto i\partial_t,\ k \mapsto -i\partial_x$ they satisfy:

$$\eta_t+\sqrt{gh}\left(\eta_x+\frac{1}{6}h^2\eta_{xxx}\right)=0$$

Using: $x \mapsto (6^{1/3}/h)(x-\sqrt{gh}\,t),\ t \mapsto \sqrt{g/h}\,t$ we obtain:

$u_t+u_{xxx}=0 \mapsto \omega(k)=-k^3$ **Linearized KdV equation**

The Fourier Transform method (I)

Consider the **IVP**:

$$u_t = F(u, u_x, u_{xx}, \ldots) = \sum_{j=0}^{N} c_j \partial_x^j u_j,$$

$$u(x,0) = u_0(x) \quad -\infty < x < +\infty,\ t > 0,$$

1) Use the **Fourier Transform** $\hat{u}(k,t) = \int_{-\infty}^{+\infty} u(x,t)\exp(-ikx)dx$

and obtain: $\hat{u}(k,0) \ = \ \int_{-\infty}^{+\infty} u_0(x,0)\exp(-ikx)dx$

2) Find the time evolution of $\hat{u}(k,0)$, i.e., $\hat{u}(k,t)$ via the PDE:

$$\hat{u}_t(k,t) = -i\omega(k)\hat{u}(k,t), \quad \omega(k) = i\sum_{j=0}^{N} c_j (ik)^j$$

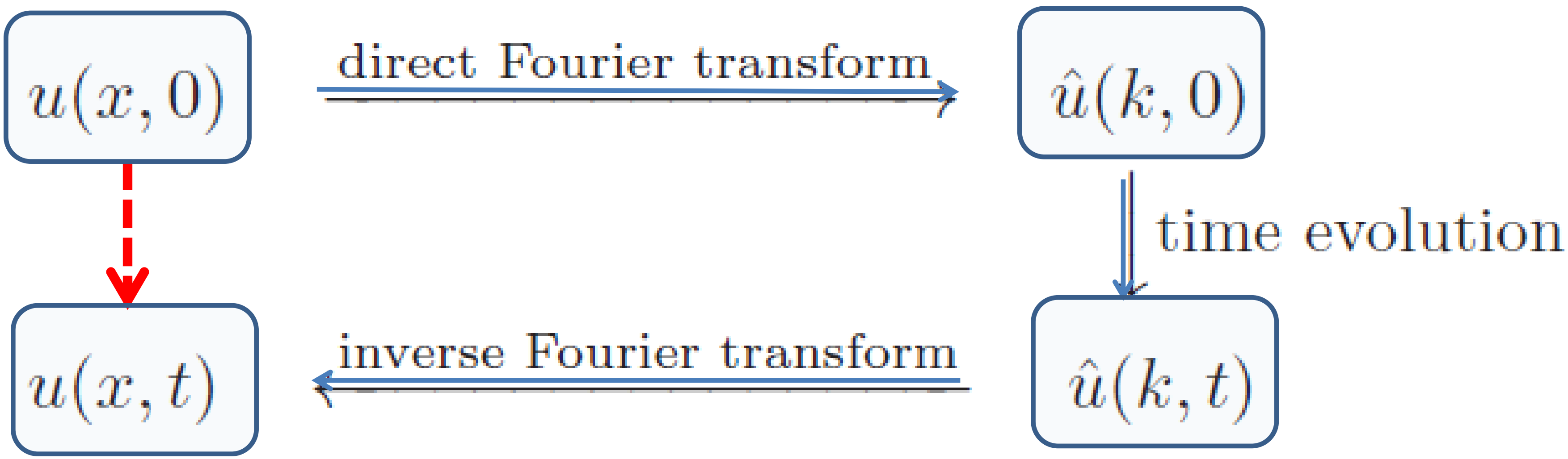
This equation gives: $\hat{u}(k,t) = \hat{u}(k,0)\exp[-i\omega(k)t]$

The Fourier Transform method (II)

3) Use the **Inverse Fourier Transform** and derive the solution:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{u}(k,t)\exp(ikx)dk = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{u}(k,0)\exp\{i[kx-\omega(k)t]\}dk.$$

The **Fourier Transform Method** can be described as follows:



The above scheme can be generalized to **certain nonlinear PDEs**

INVERSE SCATTERING TRANSFORM METHOD

Shallow WWs in the presence of dispersion

We will use the **Fourier Transform method** to solve the **IVP**:

$$u_t + u_{xxx} = 0$$
$$u(x,0) = u_0(x)$$

Linearized KdV equation

Taking into account that the **dispersion relation** is:

$$\omega(k) = -k^3$$

we find: $$u(x,t) = \int_{-\infty}^{\infty} A(k) \exp[i(kx + k^3 t)] \, \frac{dk}{2\pi}$$

where **Fourier amplitudes** are given by:

$$A(k) = \int_{-\infty}^{\infty} u_0(x') e^{-ikx'} \, dx'$$

The Green function

Substituting $A(k)$ into the solution, we obtain:

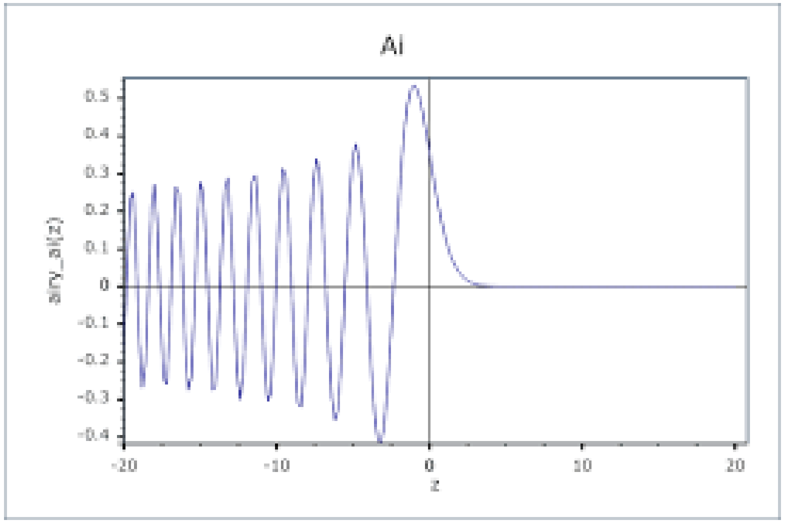
$$u(x,t) = \int_{-\infty}^{\infty} u_0(x')G(x-x',t)\,dx',$$

where $G(x,t)$ is the **Green function** of the linearized KdV:

$$G(x,t) = \int_{-\infty}^{\infty} e^{i(kx+k^3t)}\,\frac{dk}{2\pi} = \frac{1}{\pi}\int_{0}^{\infty} \cos(kx+k^3t)\,dk.$$

Then, taking into regard that the **Airy function** is defined as:

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_{0}^{\infty} \cos\left(\frac{1}{3}k^3 + xk\right) dk$$



we can express $G(x,t)$ as:

$$G(x,t) = \frac{1}{(3t)^{1/3}}\,\mathrm{Ai}\left(\frac{x}{(3t)^{1/3}}\right)$$

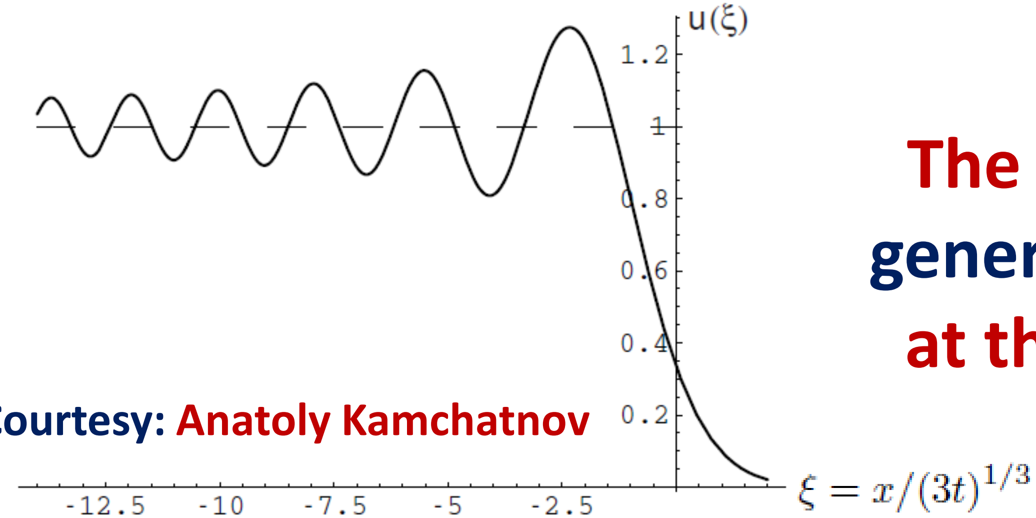
The effect of dispersion (I)
Decay of a step-like pulse

Consider an initial condition in the form of a **step-like pulse**:

$$u_0(x) = \begin{cases} 1, & x \leq 0, \\ 0, & x > 0. \end{cases}$$

Then, the pulse profile, at time t, is given by:

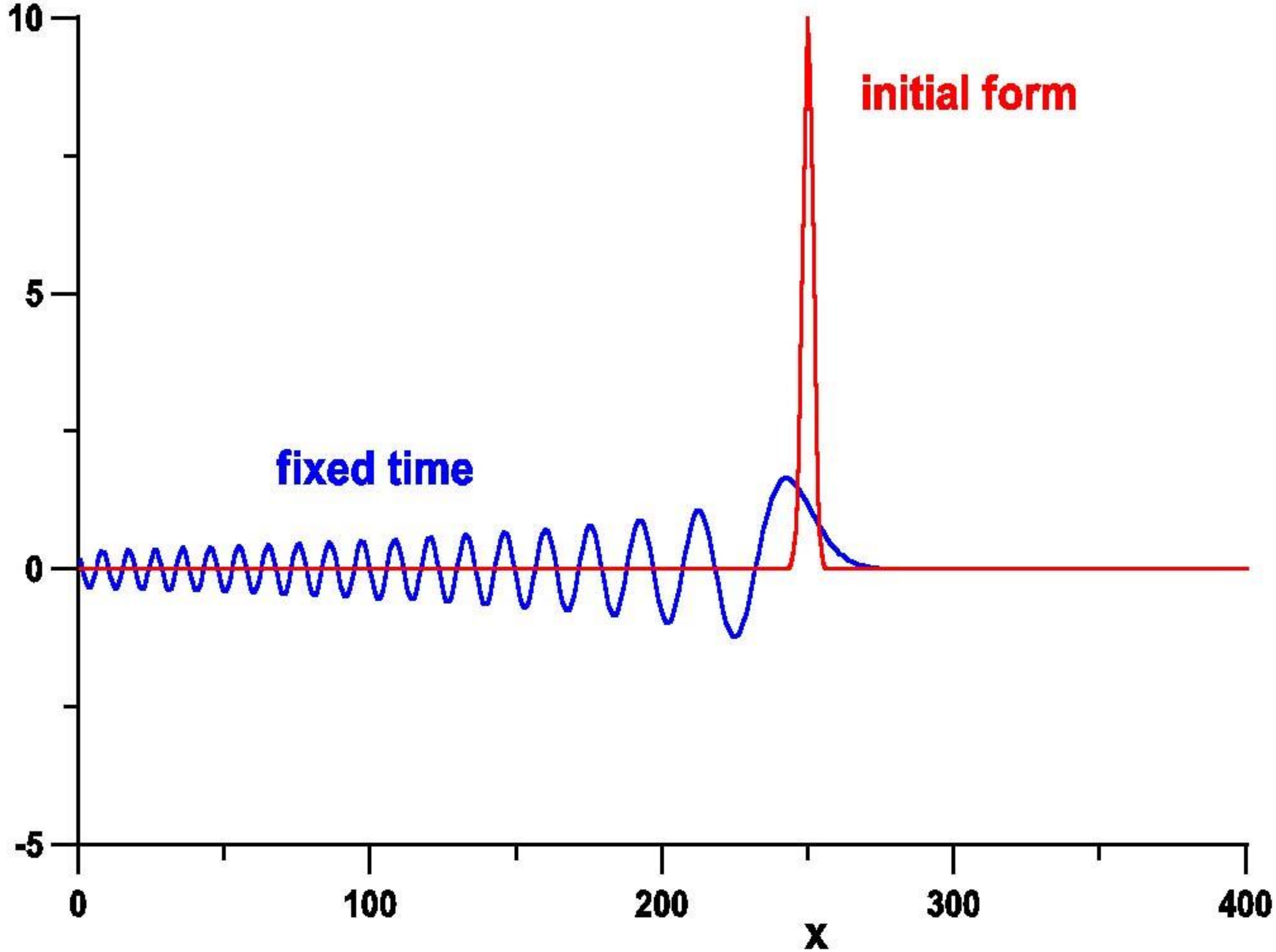
$$u(x,t) = \frac{1}{(3t)^{1/3}} \int_{-\infty}^{0} \mathrm{Ai}\left(\frac{x - x'}{(3t)^{1/3}}\right) dx' = \int_{x/(3t)^{1/3}}^{\infty} \mathrm{Ai}(z)\, dz$$



Courtesy: Anatoly Kamchatnov

The dispersion leads to generation of oscillations at the edge of the pulse

The effect of dispersion (II)
Decay of a Gaussian pulse



Courtesy: Dmitry Pelinovsky

Exercises

1) Consider the **Klein-Gordon (KG)** equation:

$$u_{tt} - c^2 u_{xx} + m^2 u(x,t) = 0$$

which is a relativistic wave equation occurring in the description of weak interaction (in this case, m is the boson mass), in plasmas (with m being the plasma frequency), in waveguides (in this case, m plays the role of the cutoff frequency of the waveguide), etc.

a) Derive the dispersion relation $\omega=\omega(k)$ and plot it. Then, identify **band**s and **gaps**, where propagation may, or not, be possible.

b) Determine the **phase and group velocity, v_p and v_g**. Show that $v_p > c$ and $v_g < c$, and provide a *geometrical interpretation* of this result, using the plot of $\omega=\omega(k)$. Show that $v_p v_g = c^2$.

Exercises (cont.)

c) Use the Fourier Transform method to solve the initial value problem for the KG equation:

$$u_{tt} - c^2 u_{xx} + m^2 u(x,t) = 0,$$
$$u(x,0) = f(x), \qquad u_t(x,0) = g(x).$$

Show that in the special case: $u(x,0) = \delta(x)$ and $u_t(x,0) = 0$, one obtains:

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos(kx) \cos\left[\omega(k)t\right] dk.$$

Exercises (cont.)

2) Quasi-1D Bose-Einstein condensates (BECs) are described by the **Gross-Pitaevskii** equation (GPE):

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\psi_{xx} + g|\psi|^2\psi,$$

where $\hbar$ is the Planck's constant, m is the atomic mass, and g is the interaction coefficient. Let $g > 0$ (for repulsive interactions).

a) Show that the GPE possesses the solution (BEC's ground state):

$$\psi = \sqrt{\rho_0}\exp(-i\mu t/\hbar)$$

where μ is the chemical potential. Show that: $\mu = \rho_0 g$.

Exercises (cont.)

b) We seek the dispersion relation for linear waves propagating on top of the ground state. To derive this, follow this procedure:

- Substitute into the GPE the ansatz:

$$\psi = \left(\sqrt{\rho_0} + \rho\right)\exp\left(-i(\mu/\hbar)t + i\varphi\right)$$

where $\rho = \rho(x,t)$ and $\varphi = \varphi(x,t)$ are small perturbations (ρ, $\varphi << 1$).

- Separate real and imaginary parts, and linearize the resulting two equations with respect to ρ, φ. Show that, this way, the following linear system can be derived:

$$\rho_t = -(\hbar^2/2m)\sqrt{\rho_0}\,\varphi_{xx}, \quad \hbar\sqrt{\rho_0}\,\varphi_t + 2\mu\rho = (\hbar^2/2m)\rho_{xx}$$

Exercises (cont.)

- Seek solutions of the above system in the form:

$$\rho = \tilde{\rho} \exp[i(kx - \omega t)], \quad \varphi = \tilde{\varphi} \exp[i(kx - \omega t)]$$

and derive a linear homogeneous system for the unknown amplitudes $\tilde{\rho}, \tilde{\varphi}$. This will lead to the desired, so-called, **Bogoliubov dispersion relation**.

c) Derive the (same of course!) dispersion relation as follows: Eliminate ρ, from the above mentioned linear system for ρ and φ; derive a PDE for φ, and show that it has the form of a linearized Boussinesq equation, namely:

$$u_{tt} - c^2 u_{xx} + \beta u_{xxxx} = 0, \quad \beta > 0$$

Then determine the dispersion relation of this model.

Exercises (cont.)

d) Once the dispersion relation is found, consider right-going waves, and show:

- For long waves, the dispersion relation reduces to $\omega = kc$. Determine c (this is usually called the "speed of sound").
- In the absence of interactions (g=0), the dispersion relation reduces to the so-called de Broglie form: $E=p^2/2m$, where $E = \hbar\omega, \quad p = \hbar k.$