Διάλεξη 5 14-3-24 (εργασίες) ...

Ξεκινάμε στις 2:15

- Από τι εξαρτάται η L
- Γενίκευση της στασιμοποίησης σωματηδοειδούς
- Προβλήματα στασιμοποίησης
- Ελευθερίες της L.

Εργασ. 4. Αν Δοκιμάσουμε ΜΗ φυσικ. διαδρ.

$$S_{I} = S_0\left(1 - \frac{x^2+y^2+2xy^2}{1-y^2}\right)$$

$S_0 = -\frac{1}{24} m g^2 T^3$ $\qquad x = \delta/H_0 \qquad y = \frac{\tau}{T/2}$

$$S_0 \le S_{II} \le S_{I}$$

$\alpha \underbrace{\left(\delta + \frac{1}{2} g \tau^2\right)}^2$

Αν δοκιμ. 2 φυσ. διαδρ.

$$S_{II} = S_0\left(1 - \frac{3}{4}\frac{(x+y^2)^2}{1-y^2}\right)$$

$$S = \int_{t_1}^{t_2} L(x,\dot{x},t)\,dt$$

έστω ότι $L(x,t) \xrightarrow{\text{στάσιμη}} \cancel{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)} - \frac{\partial L}{\partial x} = 0$

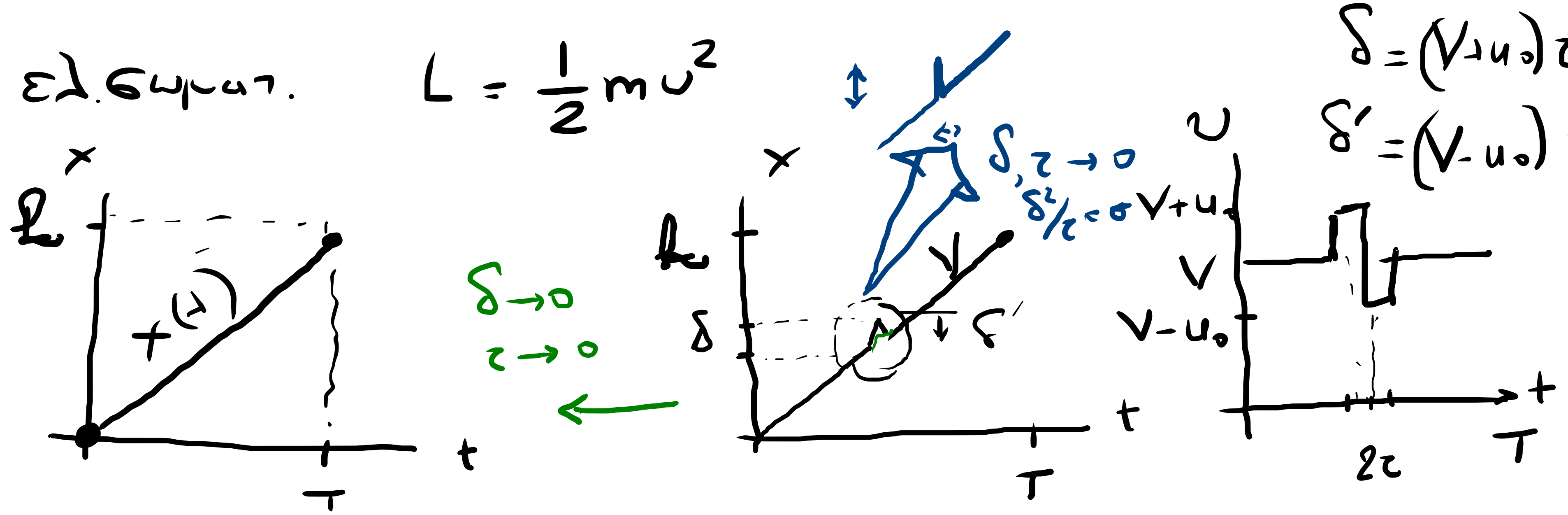
$L(\cancel{x}, t)$

γιατί L πρέπει να εξαρτ. ισαι από την $\dot{x}$

έστω ελ. σωματ. $L = \frac{1}{2} m u^2$

$\delta = (V+u_0)\tau$

$\delta' = (V-u_0)\tau$



$$L = \begin{cases} \frac{1}{2} m V^2 & \text{για} \quad t \notin [-\tau + t, t + \tau] \\ \frac{1}{2} m (V + u_0)^2 & t \in [-\tau + t, t] \\ \frac{1}{2} m (V - u_0)^2 & t \in [t, t + \tau] \end{cases}$$

$$S = \frac{1}{2} m V^2 T + \frac{1}{2} m u_0^2 \, 2\tau = S_0 + m\sigma \neq S_0$$

$$u_0^2 \tau = \sigma \neq 0 \qquad \delta, \tau \to 0$$

$$\downarrow$$

$$(V + u_0)\tau = \delta \Rightarrow u_0 = \frac{\delta}{\tau} - V \Rightarrow \left(\frac{\delta}{\tau} - V\right)^2 \tau = \sigma$$

$$\tau = \frac{\delta^2}{\sigma} \quad \delta, \tau \to 0 \qquad \frac{\delta^2}{\tau} - 2V\delta + V^2 \tau$$

$\delta$ στασιμοποίηση για $L(x, \dot{x}, t)$

συσ. σύστ. ήταν ένα σύστ. με 1 βαθμ. ελευθ.

π.χ. 1 σωμ. που κινείται σε 1-D ή 1 εκκρεμ. ή 1 α.τ. σε 1D

Έστω $L(x_1, x_2, \ldots, x_n, \dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n, t)$

στασιμ. αυτής της $S = \int_{t_1}^{t_2} L\, dt \longrightarrow$

$$x_1(t) = x_{10}(t) + \varepsilon \xi_1(t)$$
$$x_2(t) = x_{20}(t) + \varepsilon \xi_2(t)$$
$$\vdots$$
$$x_n(t) = x_{n0}(t) + \varepsilon \xi_n(t)$$

$$S_0 = \int L(x_{10}, x_{20}, \ldots, x_{n0}, \dot{x}_{10}, \dot{x}_{20}, \ldots, \dot{x}_{n0}, t)\,dt$$

φυσ. κίνησης

$x_{10}(t), x_{20}(t), \ldots, x_{n0}(t)$

η φυσική εξέλιξη του

συστήματος

$$S[\vec{x} = \vec{x}_0 + \varepsilon\vec{\xi}\,] = S_0 + \underbrace{\Big[\varepsilon \int_{t_1}^{t_2} \Big( \frac{\partial L}{\partial x_1}\Big|_{\vec{x}_0, \dot{\vec{x}}_0, t} \xi_1 + \frac{\partial L}{\partial x_2}\Big|_{\vec{x}_0, \dot{\vec{x}}_0, t} \xi_2 + \cdots}_{0}$$

$$+ \frac{\partial L}{\partial x_n}\Big|_{\vec{x}_0, \dot{\vec{x}}_0, t} \xi_n + \frac{\partial L}{\partial \dot{x}_1}\Big|_{\vec{x}_0, \dot{\vec{x}}_0, t} \dot{\xi}_1 + \cdots$$

$$+ \frac{\partial L}{\partial \dot{x}_n}\Big|_{\vec{x}_0, \dot{\vec{x}}_0, t} \dot{\xi}_n \Big) dt \Big]$$

$$+ \varepsilon^2 \cdots$$

$$\frac{\partial L}{\partial \dot{x}_m}\dot{\xi}_m = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_m}\xi_m\right) - \xi_m \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_m}\right)$$

$$S = S_0 + \varepsilon\left[\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial x_i}\Big|_{\vec{x}_0,\dot{\vec{x}}_0,t}\xi_i - \xi_i\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\Big|_{\vec{x}_0,\dot{\vec{x}}_0,t}\right)\right)dt + \int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\Big|_{\vec{x}_0,\dot{\vec{x}}_0,t}\xi_i\right)dt\right] + \varepsilon^2 \ldots$$

χρησ. την αθρ. συμβ του Einstein

στη συνεχ. θα παραλείψω το που υπολογίζονται οι $\frac{\partial L}{\partial \ldots}$

$$\left[\int \quad + \int \quad\right] = 0 \quad \forall\, \xi_i(t) : \xi_i(t_1) = \xi_i(t_2) = 0 \quad \forall i$$

$$\int_a^b df(t) = f(b) - f(a)$$

$$\int_{t_1}^{t_2} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\xi_i\right)dt \underset{\text{αθρ. σύμβ}}{\overset{\text{όχι}}{=}} \left.\frac{\partial L}{\partial \dot{x}_i}\xi_i\right|_{t_1}^{t_2} = 0$$

αφού $\xi_i(t_1) = \xi_i(t_2) = 0$

$$0 = \int_{t_1}^{t_2} \xi_i \underbrace{\left[\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right)\right]}_{(\vec{F}(t))_i} dt \quad \forall\, \xi_i(t)$$

$$\int_{t_1}^{t_2} \vec{\xi}(t) \cdot \vec{F}(t)\, dt = 0$$

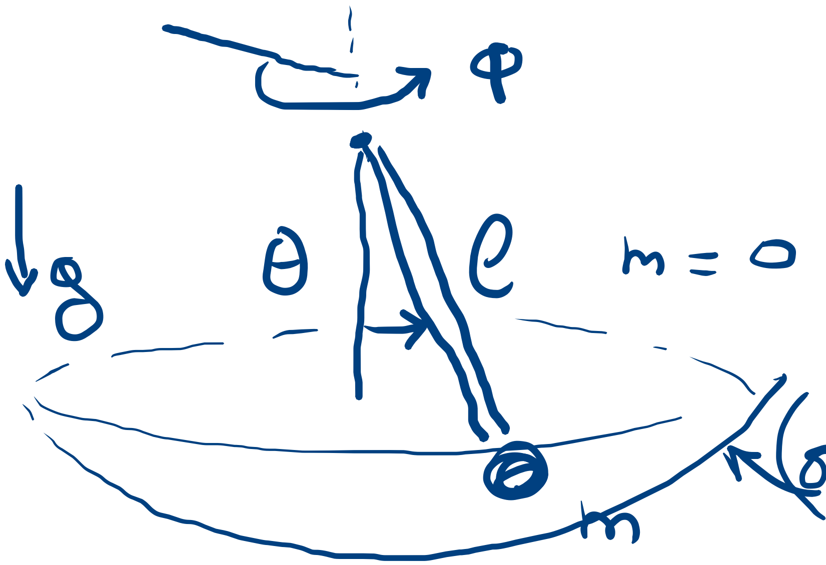
$$\forall\, \vec{\xi} \Rightarrow$$

$$\vec{F}(t) = \vec{0}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \qquad \forall i$$

σύστημα διαφ. εξισ. Euler-Lagrange εξισώσεις

Σφαιρικό εκκρεμές (κίνηση στις 3 διαστάσεις)




$$L(\theta, \varphi, \dot{\theta}, \dot{\varphi}, t)$$

$$L = \frac{1}{2} m \left[ (\ell \dot{\theta})^2 + (\ell \dot{\varphi} \sin\theta)^2 \right] - mg\ell(1-\cos\theta)$$

$$\frac{\partial L}{\partial \theta} = \frac{1}{\cancel{2}} m \ell^2 \cancel{2} \dot{\varphi}^2 \sin\theta \cos\theta - mg\ell \sin\theta \quad (1)$$

$$\frac{\partial L}{\partial \dot{\theta}} = m\ell^2 \dot{\theta} \; (2) \qquad \frac{\partial L}{\partial \dot{\varphi}} = m\ell^2 \sin^2\theta \, \dot{\varphi} \; (3) \qquad \frac{\partial L}{\partial \varphi} = 0 \; (4)$$

$$\text{E-L}_{\varphi}\ ^{(3,4)}: \quad \frac{d}{dt}\left(m\ell^2\sin^2\theta\,\dot\varphi\right) - 0 = 0$$

$$m\ell^2\sin^2\theta\,\dot\varphi = \sigma = \text{const.} \qquad (5)$$

$$\to \varphi(t)$$

$$\text{E-L}_{\theta}: (1,2) \quad \frac{d}{dt}\left(m\ell^2\dot\theta\right) - \left[m\ell^2\dot\varphi^2\sin\theta\cos\theta - mg\ell\sin\theta\right] = 0$$

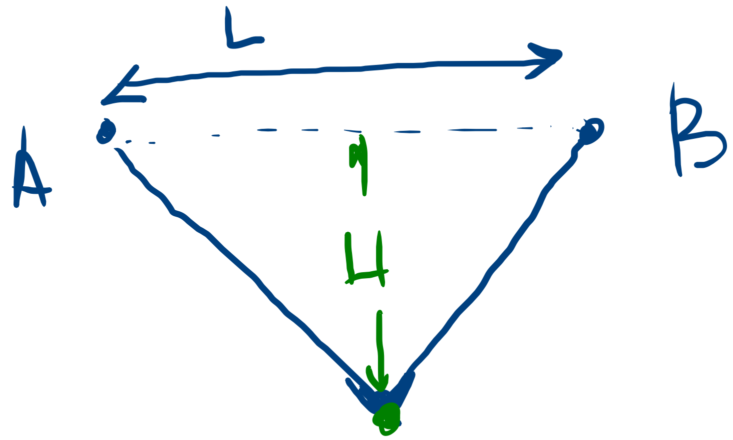
$$\cancel{m}\ell^2\ddot\theta = \cancel{m}\left(\ell^2\dot\varphi^2\sin\theta\cos\theta - g\ell\sin\theta\right)$$

$$\theta(t)$$

$$\ddot\theta = \left(\dot\varphi^2\cos\theta - \left(g/\ell\right)\right)\sin\theta$$

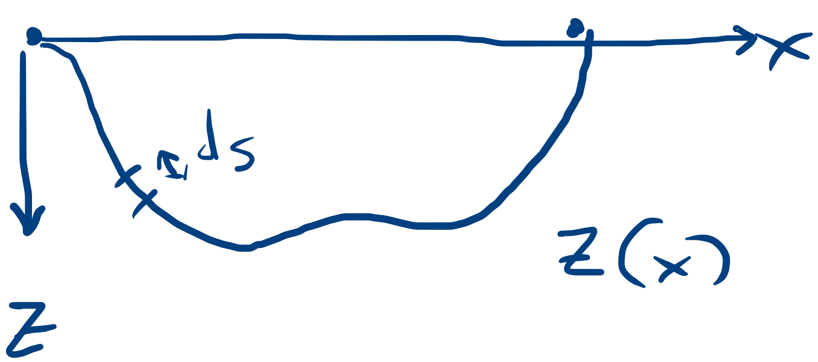
$$= \left[\left(\frac{\sigma}{m\ell^2\sin^2\theta}\right)^2\cos\theta - g/\ell\right]\sin\theta$$

Πρόβλημα βραχυστόχρονου




$\downarrow g$

? H ώστε $t_{A\to B} = \min$




$$T_{A\to B} = \int_A^B dt = \int_A^B \frac{ds}{v}$$

$$ds = \sqrt{dz^2 + dx^2}$$

$$z(0) = z(L) = 0$$

$$v = \sqrt{2gz} \leftarrow 0 = \frac{1}{2}mv^2 - mgz$$

$T_{A\to B}$ = συναρτησοειδές της $z(x)$ με δοσμένα άκρα

$$T_{A\to B}[z(x)] = \int_0^L \underbrace{\frac{dx\sqrt{1+(z')^2}}{\sqrt{2gz}}}_{f(z,z')}$$

| | φυσ. προβ. | βραχ κινημ |
|---|---|---|
| ολοκλ. ως προς | $t$ | $x$ |
| | $L(x,\dot{x},t)$ | $f(z,z',\cancel{x})$ |
| | $S$ | $T$ |

χρησ. της εξ. Ε-L

$$\frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) - \frac{\partial f}{\partial z} = 0$$

$$\Downarrow$$

$$f(z,z') = \sqrt{\frac{1+(z')^2}{z}} \qquad \frac{\partial f}{\partial z'} = \frac{z'}{\sqrt{z(1+(z')^2)}} \qquad \frac{\partial f}{\partial z} = -\frac{1}{2}\frac{\sqrt{1+(z')^2}}{z^{3/2}}$$

$$\frac{d}{dx}\left(\frac{z'}{\sqrt{z(1+(z')^2)}}\right) + \frac{1}{2}\frac{\sqrt{1+(z')^2}}{z^{3/2}} = 0$$

$$T = \frac{1}{\sqrt{2g}} \int_0^L \sqrt{\frac{1+(z')^2}{z}}\, dx$$

$$\sqrt{dx^2+dz^2} = dx\sqrt{1+(z')^2}$$

$$= dz \sqrt{1+\left(\frac{dx}{dz}\right)^2}$$

$$\int_{\text{"}0\text{"}}^{\text{"}0\text{"}} dz \underbrace{\sqrt{\frac{1+(x')^2}{z}}}_{F} \qquad x(z)$$

$$\frac{d}{dz}\left(\frac{\partial F}{\partial x'}\right) - \cancel{\frac{\partial F}{\partial x}}^{\,=0} = 0 \Rightarrow \frac{\partial F}{\partial x'} = \text{קבוע}.$$

$$x'^2 = C^2 z\,(1+x'^2). \quad \leftarrow \quad \frac{x'}{\sqrt{1+(x')^2}\,\sqrt{z}} = \text{קבוע} = C$$

$$x'^2\left(1-c^2 z\right) = c^2 z$$

$$x'^2 = \frac{c^2 z}{1-c^2 z}$$

$$x' = \sqrt{\frac{c^2 z}{1-c^2 z}} \leftarrow x(z)$$

$$c^2 z = \sin^2\theta$$

$$\frac{dx}{dz} = \frac{\sin\theta}{\cos\theta}$$

$$d\left(c^2 z\right) = 2\sin\theta\cos\theta\, d\theta$$

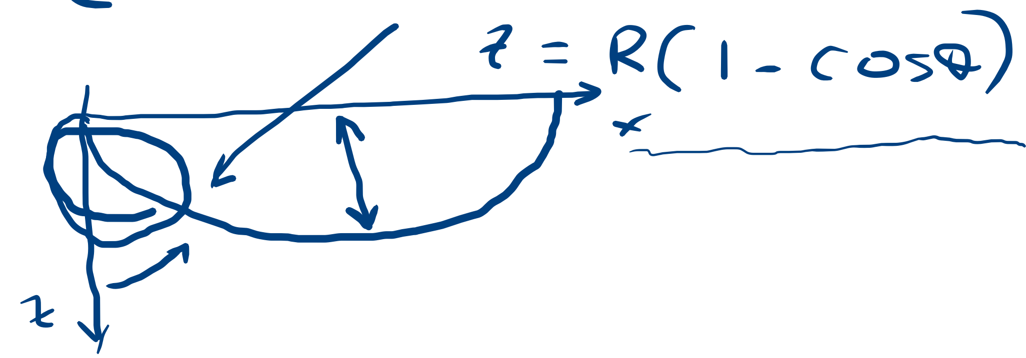
$$\frac{dx}{\frac{2\sin\theta\cos\theta\, d\theta}{c^2}} = \tan\theta$$

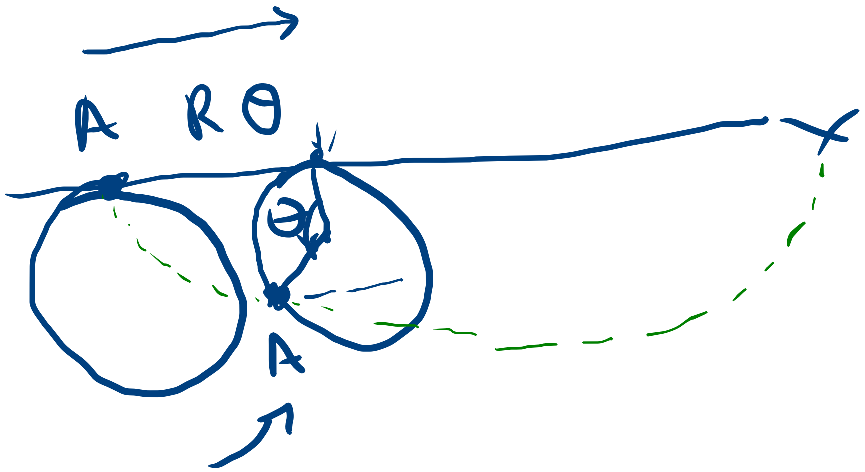
$$\frac{dx}{d\theta} = \frac{2}{c^2}\sin^2\theta$$

$$x = \frac{2}{c^2}\int \sin^2\theta\, d\theta \dashrightarrow$$

$$x = R(\theta - \sin\theta)$$

$$z = R(1-\cos\theta)$$






$$x_A = R\theta - R\sin\theta$$

$$z_A = R - R\cos\theta$$

$$2\pi R = L_{AB} \Rightarrow R = \frac{L_{AB}}{2\pi}.$$