Chapter 18

How Many Numbers are there in a Rational Numbers Interval? Constraints, Synthetic Models and the Effect of the Number Line

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The Set of Rational Numbers: An Expansion of the Natural Numbers Set?

A rational number is a number that can be expressed as the ratio of two integers, like 1/2, −2/3 etc. Alternatively, a rational number can be expressed either as a simple decimal or as a recurring decimal, like 0.23, −0.4, 0.3454545…, −0.333…, etc. It follows that all natural numbers are included in the rational numbers set, since 2, for instance, can be represented as 2/1 or 2.0. The same holds for any negative whole number, since −3, for instance, can be represented as −3/1 or −3.0. The rational numbers set is closed under subtraction and division, in the sense that the difference and the quotient of any rational number are rational numbers as well. This is a property that does not hold within the natural numbers set: For instance, the outcomes of $3 - 5$ or 3:5 are not natural numbers. The need for closure under subtraction and division offers a plausible explanation for the need to expand the set of natural numbers to the set of rational numbers. This expansion can be described linearly in the following way: The natural numbers set is expanded to the set of integers to comprise the negative whole numbers. The new set is closed under subtraction, e.g., $3 - 5 = -2$ is an integer number. The set of integers is then expanded to the set of rational numbers; the new set is closed under subtraction and division, e.g., $3 - 5$, 3:5 are rational numbers.

This characterization of the shift from natural to rational numbers may represent a good way to make a plausible summary of the historical development of the number concept. However, it does not accurately reflect the historical process. More importantly, it conceals the fact that the historical development of the rational number concept was far from being linear and smooth. There were harsh debates in the history of mathematics about the kind of mathematical constructs that can be considered as numbers and the kinds of operations that are legitimate (see also Merenluoto & Palonen, this volume). In a more general fashion, the shift from natural to rational numbers involved changes in the status and meaning

Reframing the Conceptual Change Approach in Learning and Instruction

ISBN: 0-08-045355-4

of the term "number" that cannot be accounted for in terms of the mere expansion of the natural number concept. For example, to accept negative numbers as *numbers* meant that the conceptualization of numbers as answers to the questions "How many?" and "How much?" for discrete and continuous quantities, respectively, had to be radically changed (Dunmore, 1992).

From a different perspective, the shift from accepting that $3 - 5$ is a legitimate operation to accepting that $3 - 5$ is a mathematical object can be viewed as an ontological shift during which a *process* now becomes a member of the category of *objects* (Sfard, 1991). The same holds for the shift in the conceptualization of the ratios of integers from relations between numbers to numbers.

What we are trying to say here is that the change from natural to rational numbers cannot be accounted for as a mere expansion of the natural numbers set, in the sense that there is more involved than just "adding" new numbers to it. Along with the changes mentioned above, the two sets also have radically different structures: The set of natural numbers is discrete, i.e., between two successive natural numbers there is no other natural number, whereas the set of rational numbers is dense, i.e., between any two, non-equal rational numbers, there are infinitely many numbers.

In this chapter, we investigate secondary and upper secondary students' understanding of this particular property of the rational numbers set, namely density. Understanding density is closely related to the development of the rational number concept. We argue that students' initial explanatory frameworks for number are tied around natural numbers and their basic properties, such as discreteness. When students are introduced to other kinds of numbers, like fractions and decimals, they have to deal with constructs that took millennia to develop. Although students do not have to recapitulate the historical growth of these mathematical ideas, they still face the challenging task to construct meaning for these new mathematical objects called "numbers" and, in particular, their notation. Moreover, while moving to a wider explanatory framework for number, students have to realize the confining role of the presuppositions pertaining to their initial framework, which are bound to interfere in the interpretation of new information about rational numbers. We argue that the shift to wider explanatory frameworks for number cannot be accomplished by a mere enrichment of the initial frameworks for number, but requires conceptual change. That is, it requires changes in the structure of the number concept, in the fundamental presuppositions within which it is embedded, as well as change in the contexts of its use.

Initial Explanatory Frameworks for Number

It has been argued that even before instruction children form a *principled understanding* of number that has its roots in the act of *counting* (Gelman, 2000; Gelman & Gallistel, 1978). During the first years of instruction, this initial understanding of number supports children to reason about natural numbers and to learn about their properties, to build strategies in relation to natural numbers operations, etc. For instance, the counting algorithm supports children to build the successor principle (e.g., that given a specific natural number, one can always find its next), which in turn may help them to infer that there are infinitely many natural numbers (Hartnett & Gelman, 1998). Addition and subtraction are conceptualized in terms of counting [see, for example, Resnick (1989) for a report of counting-based strategies], multiplication is viewed as repeated addition (e.g., Fischbein, Deri, Nello, & Marino,, 1985), and numbers can be ordered by means of their position on the count list.

In terms of the conceptual change theoretical framework that we adopt (Vosniadou, this volume; Vosniadou & Verschaffel, 2004), this complex network of relations and operations constitutes young children's initial explanatory framework of number (see also Smith, Solomon, & Carey, 2005). The conceptual change framework predicts that when children will encounter numbers with different properties than natural numbers, their initial explanatory framework for number will stand in the way of further learning.

Indeed, there is a great deal of evidence showing that students at various levels of instruction make use of their knowledge of natural number to conceptualize rational numbers and make sense of decimal and fraction notation, often resulting in making systematic errors in ordering, operations, and notation of rational numbers (e.g., Fischbein et al., 1985; Moskal & Magone, 2000; Resnick et al., 1989; Stafylidou & Vosniadou, 2004; Yujing & Yong-Di, 2005). Many researchers attribute these difficulties to the constraints students' prior knowledge about natural numbers imposes on the development of the rational number concept. Such findings and interpretations are compatible with the conceptual change approach and constitute evidence of the confining role of students' initial explanatory frameworks for number. In this chapter, we will test the predictive power of this framework in relation to students' understanding of the dense structure of the rational numbers set, which has not been extensively investigated so far, as pointed out also by Smith et al. (2005).

Understanding the Structure of the Set of Rational Numbers: A Conceptual Change Approach

To understand the structure of the rational numbers set, students must (a) realize that discreteness is a property of natural numbers which is not preserved in the rational numbers set, and (b) be aware of the fact that the rational numbers set consists of elements that can be represented either as fractions or as decimals, yet remain invariant under different symbolic representations.

We assume that the discreteness of numbers is a *fundamental presupposition* (Vosniadou, 1994, 2001) of the initial explanatory framework of number, within which "orderability" is not differentiated from "nextness" (Greer, 2004). Research in the area of mathematics education has provided evidence that the idea of discreteness of numbers is a barrier to understanding the dense structure of the rational and real numbers set for students at various levels of instruction (Malara, 2001; Merenluoto & Lehtinen, 2002, 2004; Neumann, 1998) as well as for prospective teachers (Tirosh, Fischbein, Graeber, & Wilson, 1999).

Understanding that a rational number can be represented in multiple ways, but that its different symbolic representations refer to the same mathematical entity, requires the ability to move flexibly among different symbolic representations of rational numbers. This task, although easy for a mathematically versed person, is very challenging for the novices in mathematics. As Markovits and Sowder (1991) argue, one of the major difficulties faced by learners in mathematics is the use of different symbols to represent the same ideas. In the case of rational numbers notation, there is evidence to suggest that students interpret different symbolic representations to refer to different mathematical entities. This phenomenon has been noted by Khoury and Zazkis (1994) and reported by O'Connor (2001) as a fact noticed by mathematics teachers.

In our previous work (Vamvakoussi & Vosniadou, 2004), we found evidence that students treated different symbolic representations as if they were different numbers in their attempts to describe the structure of rational numbers intervals. For example, during an interview, a 9th grader stated that there are infinitely many numbers between 3/8 and 5/8 and she mentioned a number of fractions, all of which were equivalent to 4/8, such as 4.0/8, 8/16, etc., indicating her belief that these symbols represented different numbers. Another student said that there is no other number between 3/8 and 5/8 and he explained that 4/8 can be simplified to 1/2, and 1/2 is not between 3/8 and 5/8. This student, although he "knew" that 1/2 is equivalent to 4/8, failed to assign to it the properties of 4/8. In this sense, he considered it to be a different mathematical object.

The belief that different symbolic representations stand for different numbers may have another implication regarding students' thinking about the structure of the rational numbers set. We suggest that students may consider fractions and decimals to be different, unrelated "sets" of numbers. This assumption is compatible with evidence coming from research in various domains showing that novices tend to categorize objects on the basis of superficial characteristics (e.g., Chi, Feltovich, & Glaser, 1981). In the case of rational numbers, this tendency may be enhanced by the fact that there are considerable differences between the operations, as well as between the ordering of decimals and fractions. Thinking of the rational numbers set as consisting of different, unrelated sets of numbers is bound to constrain students' understanding of its dense structure. For instance, Neumann (1998) reports that 7th graders had difficulties accepting that there could be a fraction between 0.3 and 0.6. In a previous study, Vamvakoussi and Vosniadou (2004) found that students tend to think differently for fractions than for decimals with respect to their structure. For instance, a student stated that there is a finite number of numbers between two given decimals, but infinitely many numbers between two given fractions.

To summarize, children's initial explanatory frameworks of number are tied around natural numbers. In elementary school, children are introduced to new "kinds" of numbers, i.e., fractions and decimals and later on to negative numbers as well. Accepting that these novel constructs are full-fledged numbers is a major conceptual shift *per se* (see, for example, Sfard, 1991). But let us focus on the expansion of numbers, from a learner's point of view. To master the rational number concept (at the level required in school context), students must move from their initial to a wider explanatory framework for number, realizing that certain presuppositions that held before, such as the discreteness of numbers, are only valid within specific contexts. In developing this wider framework, it does not suffice to accept that the term "number" refers equally well to natural numbers, fractions, and decimals; one must also realize that fractions and decimals (simple and recurring),[1] despite

[1] Here and in the following we will use the term "decimal number" with reference only to the decimal numbers that belong to the rational numbers set.

their differences in notation, ordering, operations, and contexts of use, are alternative representations of rational numbers and not different kinds of numbers. This requires the ability on the part of the learner to take different perspectives on fractions and decimals, according to the context of use, while being able to think of them as interchangeable representations of the same mathematical entities, namely rational numbers. According to the conceptual change framework, the construction of this wider explanatory framework for number, accompanied by the deposition of natural number of its privileged position and the development of different perspectives with respect to rational number notation, cannot but be difficult, gradual, and time-consuming; moreover, in the process, misconceptions will appear. In particular, it is predicted that students are bound to generate *synthetic models* of the structure of the set of rational numbers, which are caused when learners assimilate aspects of the new, incompatible information in their existing knowledge.

While examining the development of the number concept, it is essential to take into consideration the artifacts and tools available in school context to support rational and real number reasoning. Indeed, there is a general agreement that external representations have the potential to facilitate students to grasp mathematical ideas. With respect to numbers, students are presented with external representations, such as Venn diagrams and the number line. In particular, the number line, being itself continuous, appears to be a good external representation to convey the idea of density of numbers. However, some mathematics educators argue that the meaning of a mathematical idea is not necessarily carried by a more concrete representation (e.g., Clements & McMillen, 1996). This view has its counterpart in the conceptual change literature, where there is an ongoing discussion about the effect of external representations, such as the globe or the map, on students' reasoning and understanding — in this case, about the shape of the Earth (Ivarsson, Schoultz, & Säljö, 2002; Schoultz, Säljö, & Wyndhamn, 2001; Vosniadou, Skopeliti, & Ikospentaki, 2005). According to Vosniadou and her colleagues, the effect of external representations should be analyzed within a constructivist framework, where the external representations are themselves interpreted on the basis of students' prior knowledge. Among the aims of this study was to investigate the effect of the number line on students' reasoning about the density of rational numbers.

The Present Study

In the present study we investigated 9th and 11th graders' understanding of the dense structure of the rational numbers set and the effect of the number line on their reasoning.

Understanding density is not an explicit goal of the Greek secondary school mathematics curriculum. Still, students by grade 8 have already been taught everything they are supposed to know about real numbers, including ordering, turning a decimal into a fraction and *vice versa*. They also have dealt with tasks that implicitly refer to density. For instance, in the 7th grade book, students are asked to find a number between two fractions with common denominators and successive numerators. Eighth graders are explicitly taught how to approximate $\sqrt{2}$ with two rational numbers. Moreover, in grade 10, students are introduced, through Venn diagrams, to the interrelations between the various subsets of the real numbers. They also review everything they are supposed to know about real numbers, including properties, operations, and ordering. From grade 8 and on, students use the real number line to represent real numbers. The mathematics teachers in the schools where this study was conducted told us that they had mentioned to the students on several occasions that there are infinitely many numbers in an interval.

We assumed that the idea of discreteness, being a fundamental presupposition of initial explanatory frameworks of number, is a major barrier on students' understanding of the dense structure of the set of rational numbers. We also assumed that new knowledge of fractions and decimals is used by students in their attempts to describe rational number intervals, but it is constrained by their belief that different symbolic representations refer to different numbers, and its implications that were discussed in the introduction.
More specifically, we expected that:

- The performance of 11th graders would be better than 9th graders'. Still, the answer that there is a finite number of numbers in a rational number interval would appear frequently in both age groups.
- The presence of the number line would have a helpful but limited effect on students' performance in tasks related to density.
- Students would perform better in forced-choice items, than in open-ended items.
- Understanding the structure of the set of rational numbers would be a slow and gradual process, and not an "all or nothing" situation: We assumed that there would be intermediate levels of understanding and that students at these levels would form synthetic models of the structure of rational numbers intervals. We expected these models to reflect the presupposition of discreteness and the belief that different symbolic representations stand for different numbers.

Method

Participants

The participants of this study were 301 students, of which 164 were 9th graders and 137 were 11th graders (approximate age 15 and 17 years, respectively). They came from five different schools in the Athens area. Almost half of our participants were girls.

Materials

We designed two types of questionnaires: an open-ended (QT_1) and a forced-choice one (QT_2). Both types of questionnaires had two parts, each consisting of three questions. One part made use of the number line while the other did not. Tables 18.1 and 18.2 show the items in QT_1 and QT_2, respectively. All questions had a common, general form — indicated as GF on the tables — which focuses on the number of numbers in a given rational numbers interval.[2]

[2] The number of numbers in an interval is not the only aspect of the property of density; there is also the question of whether there is a "next number" to a given rational number. However, we chose to focus on this particular aspect of the property, which we considered to be appropriate for our participants' age and level of instruction.

Table 18.1: The open-ended questionnaire (QT_1).

Items without the number line	
GF	Are there any numbers that are greater than *a* and, at the same time, less than *b*? If yes, define how many/which these numbers are If no, explain why
$\mathbf{Q_1}$	$a = 0.005$, $b = 0.006$
$\mathbf{Q_2}$	$a = 3/8$, $b = 5/8$
$\mathbf{Q_3}$	$a = 0.001$, $b = 0.001$
Items with the number line (Number line present and number *a* already placed on the number line)	
GF	Place *b* on the number line Are there any numbers that are greater than *a* and, at the same time, less than *b*? If yes, define how many/which these numbers are If no, explain why
$\mathbf{Q_4}$	$a = 0.01$, $b = 0.02$
$\mathbf{Q_5}$	$a = 1/3$, $b = 2/3$
$\mathbf{Q_6}$	$a = 0.01$, $b = 0.1$

Procedure

The students in each class were equally divided between the QT_1 and the QT_2 condition. Half of them received the items with the number line first (NL_1), whereas the other half received first the items without the number line (NL_2). In both cases, the first part of the questionnaire was withdrawn, before the second part was administered. Students had one school hour (about 45 minutes) to answer all six questions.

Results

Students' responses to each of the six questions were marked as "Finite," "Undefined," and "Infinite." "Finite" refers to the response type "There is finite number of numbers," while "Infinite" refers to the response type "There are infinitely many numbers." The answer "Undefined" appeared only in the open-ended questionnaires when a student selected the answer "Yes" to claim that there are numbers in a given interval but did not give any further information regarding the number of numbers. Table 18.3 shows the percentage of every type of answer, in the total of the answers given in all six questions, as a function of grade [9th (GR1), 11th (GR2)] and type of questionnaire (QT_1, QT_2).

Table 18.2: The forced choice questionnaire (QT_2).

Items without the number line

GF How many numbers are there, that are greater than *a* and, at the same time, less than *b*?

i. There is no such number
ii. There are the following numbers:
iii. There are infinitely many decimals/fractions
iv. There are infinitely many numbers: simple decimals, decimals with infinitely many decimal digits, fractions, square roots
v. None of the above. I believe that …

Q_1 $a = 0.005$, $b = 0.006$
ii. 0.0051, 0.0052, 0.0053, 0.0054, 0.0055, 0.0056, 0.0057, 0.0058, 0.0059

Q_2 $a = 3/8$, $b = 5/8$
i*. There is only one number, namely 4/8
ii*. There are infinitely many fractions, all equivalent to 4/8, e.g., 8/16

Q_3 $a = 0.001$, $b = 0.01$
ii. a. 0.002, 0.003, 0.004, 0.005, 0.006, 0.007, 0.008, 0.009
ii. b. 0.0011, 0.0012, 0.0013,…, 0.0097, 0.0098, 0.0099

Items with the number line
(Number line present and number *a* already placed on the number line)

GF Place *b* on the number line.
How many numbers are there, that are greater than *a* and, at the same time, less than *b*?

i. There is no such number
ii. There are the following numbers:
iii. There are infinitely many decimals (or fractions)
iv. There are infinitely many numbers: simple decimals, decimals with infinitely many decimal digits, fractions, square roots
v. None of the above. I believe that …

Q_4 $a = 0.01$, $b = 0.02$
ii. 0.11, 0.12, 0.13, 0.14, 0.15, 0.16, 0.17, 0.18, 0.19

Q_5 $a = 1/3$, $b = 2/3$

ii. $\frac{1.1}{3}, \frac{1.2}{3}, \frac{1.3}{3}, \frac{1.4}{3}, \frac{1.5}{3}, \frac{1.6}{3}, \frac{1.7}{3}, \frac{1.8}{3}, \frac{1.9}{3}$

Q_6 $a = 0.01$, $b = 0.1$
ii. a. 0.02, 0.03, 0.04, 0.05, 0.06, 0.07, 0.08, 0.09
ii. b. 0.011, 0.012, 0.013,..., 0.020, 0.021,..., 0.099

Table 18.3: Percentage of answer types, in the total of the answers given in all six questions, as a function of the type of questionnaire and of grade.

	Open-Ended (QT_1)		**Forced-Choice (QT_2)**	
	9th grade, (GR_1) $N = 498$ (83×6)	**11th grade (GR_2), $N = 396$ (66×6)**	**9th grade (GR_1) $N = 486$ (81×6)**	**11th grade (GR_2) $N = 426$ (71×6)**
Finite	65%	41.7%	52%	34.5%
Infinite	11.9%	26%	42.7%	61.3%
Undefined	20.4%	28.2%	–	–
No answer	2.8%	4%	5.3%	4.2%

The Presupposition of Discreteness

As shown in Table 18.3, the answer "Finite" was the dominant answer for 9th graders, in QT_1 and QT_2. The same holds for 11th graders, but only in QT_1. In QT_2, 11th graders gave the "Infinite" answer more often. Still, about one-third of the answers were "Finite." This result shows that the presupposition of discreteness was strong in the 9th graders and remained strong in the 11th graders too.

Differences by Age

In order to calculate each student's overall performance in QT_2, we scored the "Finite" and "Infinite" answers as 1 and 2, respectively, and calculated the sum of the scores in all six questions. In QT_1, we scored the "Finite," "Undefined," and "Infinite" answers as 1, 2, and 3, respectively, and calculated the sum of the scores in all six questions. We scored the answer "Undefined" higher than the "Finite" and lower than the "Infinite" answer, respectively, assuming that:

- a student who could not define the number of numbers in an interval, yet took their existence for granted, should be scored higher than a student who answered that there are no numbers in the same interval.
- a student who was able to characterize the number of numbers in an interval as "infinitely many" should be scored higher than a student who did not use this expression.

Under both conditions (QT_1, QT_2), either when a student gave no answer or when he gave an irrelevant answer, s/he was scored with 0.

We compared 9th and 11th graders' performance in QT_1 by performing a Mann–Whitney test, which showed that 11th graders' overall performance was significantly better than 9th graders' ($z = -2.862$, $p < .05$). A second Mann–Whitney test showed that 11th graders' overall performance was significantly better than 9th graders' ($z = -2.537$, $p < .05$) in QT_2 as well.

Effects of the Number Line

Examining the influence of the number line, we first tested for possible order effects. We compared the performance of students who received the items with the number line first, to the performance of those who received first the items without the number line (NL_1 vs. NL_2). We performed four independent Mann–Whitney tests ($GR_i \times QT_j$, $i, j = 1, 2$) that showed no significant order effect. As a result, the order of presentation of the number line was not taken into consideration in subsequent statistical analyses.

In order to test for possible effects of the presence of the number line on students' performance, we performed four independent Wilcoxon signed ranks tests ($GR_i \times QT_j$, $i, j = 1, 2$). We found no significant difference in 11th graders' performance in the presence of the number line either in QT_1 or in QT_2. However, the presence of the number line improved significantly the performance of 9th graders, in both QT_1 ($z = -2.467$, $p < .05$) and QT_2 ($z \times -3.395$, $p = .001$).

In order to examine further the effect of the number line on students' performance, we separated the students into three categories: those who scored higher in the items with the number line, those who scored lower in these items, and the remaining students who scored the same in the items with and without the number line. Table 18.4 shows the number and percentage of students in each category for each grade and type of questionnaire.

As can be seen in Table 18.4, there are a considerable number of students whose performance was not affected at all by the presence of the number line. Moreover, the number of 11th graders who scored higher in the questions with the number line is close to the number of those who scored lower in these questions. Taking a closer look at our data we find that in QT_1, only 3 of the 21 9th graders whose overall performance was better for questions with the number line had actually moved from "Finite" to "Infinite" answers; in addition, only one of them gave the "Infinite" answer consistently in all questions with the number line. In QT_2, 20 of the 31 9th graders who performed better for questions with the number line moved from "Finite" to "Infinite" answers. Only seven of these students gave

Table 18.4: Number and percentage of students in categories formed with respect to their performance with and without the number line, as a function of grade and type of questionnaire.

	Open-Ended (QT_1)		**Forced-Choice (QT_2)**	
Performance	**9th grade (GR_1), $N = 83$**	**11th grade (GR_2) $N = 66$**	**9th grade (GR_1), $N = 81$**	**11th grade (GR_2), $N = 71$**
Better with the number line	21 (25.3%)	16 (24.2%)	31 (38.3%)	16 (22.5%)
Worse with the number line	6 (7.2%)	20 (30.3%)	8 (9.9%)	10 (14.1%)
No effect	56 (67.5%)	30 (45.5%)	42 (51.8%)	45 (63.4%)

the "Infinite" answer consistently for all questions with the number line. In conclusion, it appears that the presence of the number line had a limited effect on students' performance.

Effect of Questionnaire Type

We categorized all students into three groups on the basis of their responses to all six questions. More specifically, we placed in the first group students who answered only "Finite." In the third group, we placed students who answered only "Infinite." In the second, intermediate group, we placed all remaining students. The first, second, and third group will be referred to as "Finiteness," "Mixed," and "Infinity" category, respectively.

A χ^2 test showed that the effect of the questionnaire type was statistically significant (df(2), $p = .001$). More specifically, students who filled in the open-ended questionnaires were found more often in the category "Finiteness" and less often in the other two categories, as compared to students who filled in the forced-choice questionnaires.

Intermediate Levels of Understanding

In our attempts to examine intermediate levels of understanding of the structure of the rational numbers set, we refined our categorization further and identified two subcategories for the category "Finiteness": *Discreteness* and *Refined Discreteness*, both in QT_1 and in QT_2. In the *Discreteness* category we placed students who considered the given numbers to be successive, e.g., 0.005 to be immediately next to 0.006, in all six questions. In the *Refined Discreteness* category, we placed students who did not consider the given numbers to be successive, in at least one out of six questions. These students still answered that there is a finite number of numbers in the interval, e.g., 0.0051, 0.0052, …, 0.0059 between 0.005 and 0.006.

In the case of forced-choice questionnaires, we were able to define two sub-categories for the category "Infinity": *Constrained Density* and *Density*. Students who answered that there are infinitely many numbers of the same symbolic representation in the given interval, for at least one out of the six questions, were placed in the "Constrained Density" category. Students in this category were reluctant to accept, for example, that there may be decimals between two fractions or *vice versa*. In the "Density" category, we placed the students who answered that there are infinitely many numbers, regardless of their symbolic representation, in all six questions. Almost half of the students of the "Infinity" category were placed in the "Constrained Density" category.

Table 18.5 summarizes the percentage of students placed in the *Discreteness, Refined Discreteness, Mixed*, *Constrained Density,* and *Density* categories, for both grades and types of questionnaire.

As shown in Table 18.5, there were three intermediate levels of understanding, between the naïve level of *Discreteness* and the more sophisticated level of *Density.* Students in the *Refined Discreteness* category were constrained by the presupposition of discreteness. Students in the *Constrained Density* category had overcome the barrier of discreteness, yet, they were constrained by their belief that different symbolic representations stand for different numbers, in the sense that they tend to group numbers according to their symbolic representation. Finally, students in the *Mixed* category, who answered that there are infinitely

Table 18.5: Percentage of students placed in the five categories, as a function of type of questionnaire and of grade.

Category	**Open-Ended Questionnaire**		**Forced-Choice Questionnaire**	
	9th grade (N = 83)	**11th grade (N = 66)**	**9th grade (N = 81)**	**11th grade (N = 71)**
Discreteness	30%	12.1%	4.9%	4.2%
Refined discreteness	16.9%	7.6%	30.9%	16.9%
Mixed	43.4%	66.7%	40.7%	42.3%
Constrained density	9.6%	13.6%	12.3%	15.5%
Density			11.1%	21.1%

many numbers in some, but not all questions, are still constrained by the idea of discreteness and are also affected by the symbolic representation of numbers. Indeed, more than half (53.9%) of the students in the "Mixed" category referred consistently to intervals with different structure, according to the symbolic representation of the first and the last number. For instance, some answered that there are infinitely many numbers between two decimals but a finite number of numbers between two fractions, or *vice versa*.

The belief that different symbolic representations refer to different numbers was expressed also in a more explicit way. More specifically, in the case of forced-choice questionnaires, 12 out of 81 (15%) 11th graders and 14 out of 71 (19.70%) 9th graders answered that there are infinitely many fractions between 3/8 and 5/8, all equivalent to 4/8. We also noted that in the case of open-ended questionnaires, 8% of the students in the Discreteness category (total 149 students), answered that there is no number between 3/8 and 5/8, probably because they simplified 4/8 to 1/2 and decided that it is not between 3/8 and 5/8.

Discussion

Our results agree with previous findings showing that the idea of discreteness is a barrier to the understanding of the dense structure of the rational numbers set (Malara, 2001; Merenluoto & Lehtinen, 2002, 2004; Neumann, 1998; Tirosh et al., 1999). They also agree with findings showing that realizing that rational numbers remain invariant under different symbolic representations is a major conceptual difficulty for students (e.g., Khoury & Zazkis, 1994). These results are in full agreement with the predictions made within the conceptual change framework adopted.

Understanding density appears to be a slow and gradual process, in the course of which students are bound to generate synthetic models of the rational numbers intervals. Indeed, with the exception of students in the Density category, our participants gave alternative accounts of the structure of the given intervals. These accounts reflected the enrichment of an initial explanatory framework of numbers with information about fractions and decimals. However, within this wider explanatory framework, these new "kinds of numbers" were considered as different, unrelated sorts of numbers; in addition, the presupposition of discreteness appeared to play a confining role. In this sense, the accounts generated by our participants can be interpreted as synthetic models of the structure of the rational numbers intervals. In Table 18.6 there is a more detailed description of the synthetic models generated by our participants.

Table 18.6: Students' accounts of the structure of rational numbers intervals: some examples.

Category	**Description**	**Example**
Discreteness	Intervals that preserve the discrete structure of natural numbers	(0.005–0.006)
	The initial numbers are considered successive	(1/3–2/3)
Refined discreteness	Intervals that preserve the discrete structure of natural numbers	(0.0051, 0.0052, 0.0053,…,0.0059, 0.006)
	The initial numbers are not considered successive	(3.0/8, 3.1/8,…, 4.0/8,…, 5/8)
Within the mixed category	Different structure, according to the symbolic representation of the first and last numbers	(Decimal, *infinitely many decimals*, decimal) and (fraction, *finite number of fractions,* fraction). Or *vice versa*
	Intervals that contain "infinitely many" equivalent numbers	(3/8, 4/8, 4.0/8, 8/16,…, 5/8)
Constrained density	Intervals that contain infinitely many numbers of the same symbolic representation	(Decimal, *infinitely many decimals,* decimal) or (fractions, *infinitely many fractions,* fractions)
Density	Any interval contains infinitely many numbers, regardless of their symbolic representations	

Students in the *Discreteness* category treated the given numbers as if they were successive, thus preserving the discrete structure of the set of natural numbers.

Students in the *Refined Discreteness* category also preserved the discrete structure of natural numbers. However, they made use of new knowledge about rational numbers to give a more sophisticated answer. For instance, they appealed to the fact that adding one decimal digit to 0.005, one gets a number that is bigger than 0.005 and still smaller than 0.006, to conclude that there are (a finite number of) numbers between 0.005 and 0.006. Yet, their commitment to the idea of discreteness prevented them from using this knowledge efficiently and infer that there can be even more numbers in the given interval. Students who completed the forced-choice questionnaires were placed more often in the Refined Discreteness category than in the Discreteness category. Apparently, while these students were facilitated by the presence of more sophisticated answers, they were not able to overcome the presupposition of discreteness and chose the answer "Infinite." This result, along with the finding that students gave more sophisticated answers in QT_1, as compared to QT_2, is in accordance with findings from the conceptual change research, which show that students perform better in forced-choice tasks (Vosniadou, Skopeliti, & Ikospentaki,, 2004).

Students in the *Constrained Density* category were able to answer that there are infinitely many numbers in all given intervals, but were reluctant to accept that these numbers can have various symbolic representations. This type of synthetic model reveals students' disposition to group numbers according to their symbolic representation, which reflects the belief that different symbolic representations stand for different numbers.

Among the students in the *Mixed Category*, those who answered differently, according to the symbolic representation of the first and last number of the interval, generated synthetic models of the structure of rational numbers intervals, for example discrete structure between fractions and dense structure between decimals. We suggest that these students were also constrained by their belief that different symbolic representations refer to different numbers, with different properties. This suggestion is based on findings from our previous work (Vamvakoussi & Vosniadou, 2004). For example, a 9th grader who participated in that study claimed that there are infinitely many numbers between 3/8 and 5/8, yet there is a finite number of numbers between 0.001 and 0.01. When asked to consider both his answers, he explained that if one turns the same decimals into fractions, one can find more, infinitely many numbers between them.

The belief that the different symbolic representations of a number refer to different numbers is explicitly expressed in the case of the students who answered that there are "infinitely many" numbers between 3/8 and 5/8, all being fractions equivalent to 4/8. We believe that the same belief drove some students to answer that there is no other number between these fractions. Based on the findings of a previous study (Vamvakoussi & Vosniadou, 2004) we suggest that these students converted 4/8 into 1/2, and then assumed that 1/2 is not between 3/8 and 5/8, based on the difference between the symbolic representations. If this is the case, these students, although they may know that 1/2 is equivalent to 4/8, fail to assign to it the properties of 4/8. In this sense, they consider it to be a different mathematical entity.

Finally, our findings with respect to the number line support the view that the meaning of a mathematical idea is not necessarily carried by the mere presence of a more concrete representation (e.g., Clements & McMillen, 1996). Our results indicate that the number line, despite being continuous, does not facilitate considerably students' reasoning about the infinity of numbers in a rational number interval. This is because the number line improved significantly only the performance of 9th graders, and not of the 11th graders. Second, even in this case, the 9th graders who scored higher in the presence of the number line did not answer consistently that there are infinitely many numbers when the number line was present; in fact, some of them did not change any of their answers from "Finite" to "Infinite." Third, the order of appearance of the number did not have a significant effect on students' performance.

It is important to note that the effect of the number line was not always positive, especially with respect to 11th graders. This result, although not statistically significant, is interesting and somewhat surprising: Taking a second look at the questions with the number line we noticed that they might be considered easier than the questions without the number line. Indeed, the decimals involved in the first case have fewer decimal digits than those that appear in the second case. Although further research is needed to clarify this issue, we believe that this result might be a side effect of what Greek students are taught in upper secondary school about the line in the context of Euclidean geometry. More specifically, in 10th grade, students are introduced to the notion of a line as a set of points, which is different from the "holistic" line (Nunez & Lakoff, 1998) and may convey the idea that lines are discrete in nature. If the number line is interpreted on the basis of students' thinking about the geometrical line, as Vosniadou et al. (2005) would suggest, then this could account to some extent for the decrease of 11th graders' performance in the presence of the number line.

The findings of this study support the hypothesis that understanding the dense structure of the rational numbers set requires a number of changes in students' explanatory frameworks for number. These changes involve objectifying rational numbers, conceptualizing natural numbers as a subset of rational numbers, and assigning proper meaning to fractional and decimal notation, which involves abandoning the belief that different symbolic representations refer to different numbers; most importantly, students must stop thinking about rational numbers in terms of natural numbers and their properties. These changes cannot be accomplished by mere enrichment of the initial explanatory frameworks of number. Our view is very similar to the view of Smith et al. (2005), who argue that the shift in younger children's thinking about numbers is explained in terms of conceptual change. We fully agree that

> "conceptual changes involve differentiations and coalescences such that the extension of a concept and its relations to other concepts are qualitatively different after the change than before it." (Smith et al., 2005, p. 133)

However, we do not believe that these differentiations and coalescences "commit the child to concepts that would be incoherent in each side of the divide" (Smith et al., 2005, p. 133). We want to stress that we do not see the antecedent and subsequent states as "incommensurable," in the sense that after the change children think of numbers in terms of rational numbers. Rather, we argue for the ability on part of the learner to move flexibly between natural and rational number reasoning, according to the context of use. Learners who have not made the transition to the wider explanatory framework for number are clearly not in a position to entertain both perspectives. However, as it has been nicely illustrated in Merenluoto and Palonen (this volume), mathematically versed persons not only do they show this ability, but they are also fully aware of their different modes of thinking about numbers.

Conceptual change in the number concept cannot but be gradual and time consuming and they become even more difficult if the design of instruction builds on the presupposition that learning is accomplished through additive mechanisms. To give an example, in the Greek 8th grade mathematics textbook, which is based on a procedure-oriented curriculum, the set of rational numbers is presented in the following way: "All the numbers that we know, namely, natural numbers, decimals, and fractions, together with the respective negative numbers, constitute the set of rational numbers." This "definition" certainly builds on students' prior knowledge about numbers, but at the same time, enhances students' initial tendency to group numbers on the basis of their symbolic representations and presents the rational numbers set as consisting of different, unrelated sorts of numbers; in a more general fashion, it presents the rational numbers set as an expansion of the natural numbers set, in the course of which "new" numbers are being added to it. The fact that students are explicitly taught how to turn a decimal into a fraction and *vice versa* does not necessarily support them in accomplishing the corresponding conceptual knowledge, as suggested by the results of our study.

The rational number concept is notoriously difficult for students to develop and this fact has been amply demonstrated in numerous studies. We suggest that the conceptual change approach could make a useful theoretical tool in the attempt to synthesize such widespread findings. In a more general fashion, it might be useful to look at mathematics curricula through the lenses of the conceptual change approach and reconsider the sequence of appearance of certain mathematical concepts (see also Vosniadou & Vamvakoussi, 2006). According to Resnick (2006), the answer to the question "Is there a 'best' developmental sequence for teaching mathematical concepts that will maximize positive effects of prior mathematical learning and minimize interference?" is yet to come. Resnick also points out that students should not be the only targets of conceptual change interventions — teachers should also be taken into consideration. The first step towards this direction would probably be to inform teachers about the issue of conceptual change and stress the fact that an expansion of a concept from a mathematics point of view may not correspond to an enrichment of the prior knowledge of the learner.

Acknowledgments

The present study was funded through the program EPEAEK II in the framework of the project "Pythagoras — Support of University Research Groups" with 75% contribution from European Social Funds and 25% contribution from National Funds.

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