Chapter 19

Students' Interpretations of Literal Symbols in Algebra

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Large-scale research both in Great Britain (Hart, 1981) and in the United States (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1981) has shown that students have great difficulties in understanding algebra. Many students mention specifically the use of literal symbols as the origin of their difficulties, saying that they understood mathematics until literal symbols appeared (Sackur, 1995). In the present study, the conceptual change approach will be used as an explanatory framework for better understanding students' difficulties in interpreting the use of literal symbols as variables in algebra.

The Conceptual Change Approach

The conceptual change approach focuses on knowledge acquisition in specific domains and describes learning as a process that sometimes requires the significant reorganization of existing knowledge structures and not just their enrichment (Vosniadou, 1999; Vosniadou & Brewer, 1992). According to this approach, by the time formal education starts, students have already constructed a commonsense understanding of the world based on their everyday experiences and the influence of the culture. This prior knowledge can stand in the way of acquiring new knowledge when the new learning content is incompatible with what is already known. In these cases, the acquisition of new information requires conceptual change. Conceptual change is more difficult than learning that can be accomplished through enrichment. When students use additive mechanisms to assimilate incompatible information to what they already know (enrichment) they produce a type of misconception, which can be explained as "synthetic models" (Vosniadou, this volume).

Only recently, the conceptual change approach has been applied to the learning and teaching of mathematics (see Verschaffel & Vosniadou, 2004) with most of the relevant studies focusing on the development of the number concept.Prior research in the development of the number concept indicates that students' principle understanding of numbers is

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grounded on the natural numbers (Gelman, 2000). Many students even in secondary education tend to project their number concept grounded on natural numbers onto a non-natural number input. This seems to be one of the reasons why misconceptions and difficulties appear when numbers other than natural, such as rational numbers, are introduced in the mathematics curriculum. For example, many of the errors students make in the case of fractions can be interpreted to be caused by the application of properties of natural numbers to fractions (e.g., Gelman, 2000; Stafylidou & Vosniadou, 2004). Vamvakoussi and Vosniadou (2004) argue that the presupposition of discreteness, which is a property that characterizes the natural numbers, constrains students' understanding of density, which is a property of rational numbers. Other research also indicates that prior knowledge of natural numbers hinders students' understanding of the properties of rational numbers (Resnick et al., 1989). Students' prior experience based on calculating only with natural numbers is considered to be responsible for students' belief that "multiplication always makes the number larger". This belief in turn hinders students' understanding of calculations when real numbers are involved (Fischbein, Deri, Nello, & Marino, 1985).

The purpose of the present study is to examine students' difficulties to interpret the use of literal symbols in algebra. Literal symbols are used in many ways in algebra: They are used to stand for mathematical objects such as functions, matrixes, etc., but they are mostly used to represent the concept of *variable*. A variable is a mathematical entity that can be used to represent any number within a range of numbers and can stand on its own right in the algebraic formal system.

We hypothesized that students' prior knowledge about the way numbers are used in the context of arithmetic is likely to affect their interpretation of the use of literal symbols in algebra. Findings from previous research are consistent with this hypothesis. In the next session we discuss some of the most important relevant findings.

Research on Students' Interpretations of the Use of Literal Symbols in Algebra

Previous studies have demonstrated a series of misconceptions students have in relation to the use of literal symbols in algebra. For example, students often view literal symbols as labels for objects, i.e., they think that 'D' stands for David, 'h' for height, or they believe that 'y' — in the task "add 3 to 5y" — refers to anything with a 'y' like a yacht, a yoghurt or a yam. Alternatively, when students think of literal symbols as numbers they usually believe that they stand for a specific number only (Collis, 1975; Booth, 1984; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Kuchemann, 1978, 1981; Stacey & MacGregor, 1997). These misconceptions appear to be quite strong and difficult to change. Booth (1982, 1984) designed a teaching experiment specifically to encourage the acquisition of the interpretation of the literal symbol as generalized number, but found that students still faced great difficulties, even when specific instructions were given to them.

Another difficulty students appear to have with the use of literal symbols in algebra is known as 'the lack of closure' error (Collis, 1975), which refer to students' unwillingness to accept algebraic expressions that contain literal symbols as final answers. When presented with tasks such as "write the number which is twice as big as x", students are unwilling to accept "$2x$" as a final answer (Booth, 1984; Firth, 1975), but insist on replacing it with a specific number. Based on such findings Booth (1984) suggested that students consider mathematics to be an empirical subject, which requires the production of numerical answers only.

The 'concatenation problem' presents another difficulty that students have with the use of literal symbols in algebra. In arithmetic, concatenation denotes implicit addition, as both in place-value numeration, e.g., 27 equals 20 plus 7, and in mixed-numeral notation, e.g., 2 ½ means 2 plus ½ (Matz, 1980). On the contrary, in algebra, concatenation denotes multiplication; $2a$ means 2 times a. When asked to substitute 2 for a in $3a$, students tend to interpret concatenation as it is used in arithmetic responding with 32. It is only when asked specifically to respond as 'in algebra' that some students reply with '3 times 2' (Chalouh & Herscovics, 1988).

Students' tendency to think that literal symbols can stand for objects, names of objects, or specific numbers only was originally explained in terms of Piagetian theory. In other words, it was argued that the students had not yet reached the stage of formal operations and, therefore, they were not capable of acquiring the concept of variable (e.g., Collis, 1975; Kuchemann, 1978, 1981). Other researchers noticed that there might be an interaction between students' knowledge of arithmetic and their attempts to learn new content in algebra. For example, Booth (1984, 1988) suggested that students' difficulties in algebra may be partly due to their problematic knowledge in arithmetic. As the students construct their algebraic notions on the basis of their experience in arithmetic, erroneous arithmetical knowledge can be transferred in algebra. On the contrary, Matz (1980) argued that students' difficulties with algebra are not necessarily due to problematic knowledge of arithmetic, but rather reflect inappropriate use of the properties of arithmetic to interpret a new field in mathematics.

A similar, albeit more elaborated account than that of Matz's was offered by Kieran (1992). According to Kieran, most of the problems students have in their introduction to algebra arise because of the shift to a set of conventions different from those used in arithmetic. For example, in arithmetic, letters can be used as labels: 'm' can be used to denote meters, monkeys, etc., and 12 m can mean 12 meters, that is, 12 times 1 meter. But in algebra, 12 m can mean 12 times the number of meters (Kieran, 1990). Algebra uses many of the symbols used in arithmetic, such as the equal sign or the addition and subtraction signs, but in different ways. For example, the equal sign is used in arithmetic as the 'enter command' for the result of a calculation to be announced. Kieran (1981) argues that the belief of novices in algebra that the equal sign is a "do something signal" rather than a symbol of the equivalence between the left and right sides of an equation is demonstrated by their initial reluctance to accept statements such as $4 + 3 = 6 + 1$. AQ2

Both Matz and Kieran suggest that students' prior experience with arithmetic and the fact that symbols have different roles in arithmetic compared to algebra can explain some of students' difficulties in algebra. This view is highly compatible with the conceptual change approach. The conceptual change approach provides an explanatory framework that can account for the previous findings and also make new and meaningful predictions regarding students' difficulties with the use of literal symbols in algebra.

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The Conceptual Change Approach and the Use of Literal Symbols in Algebra

When students are introduced to algebra, they face the difficult task of (a) assigning meaning to new symbols and (b) assigning new meaning to old symbols, which were used in the context of arithmetic. According to Resnick (1987), when students connect an algebraic expression[1] which contains literal symbols with the 'world of numbers', they give a referential meaning to the algebraic expression which can affect their performance in various mathematical tasks. We suggest that a number of misconceptions that students have with the use of literal symbols could be explained to result from the inappropriate transfer of prior knowledge about numbers in the context of arithmetic, for the interpretation of literal symbols in algebra. Because students' explanatory frameworks of number are initially tied around natural numbers (Gelman & Gallistel, 1978; Gelman, 2000), we will start by making explicit, in Table 19.1, some of the important differences between the use of natural numbers in the context of arithmetic and the use of literal symbols as variables in the context of algebra.

Table 19.1: Differences between the natural numbers in the context of arithmetic and literal symbols in the context of algebra.

	Natural Numbers in arithmetic	**Literal symbols as variables in algebra**
Form	1, 2, 3, …	a, b, x, y, …
Sign	Actual sign (positive)	Phenomenal sign (positive or negative)
Symbolic Representation	Each number in the natural number set has a unique symbolic representation — different symbols correspond to different numbers.	Each literal symbol corresponds to a range of real numbers — different symbols could stand for the same number.
Ordering —Density	Natural numbers can be ordered by means of their position on the count list. There is always a successor or a preceding number. There are no numbers between two subsequent numbers.	Literal symbols in algebra cannot be ordered by means of their position on the alphabet. There is no such thing as a successor or preceding literal symbols.
Relationship to the unit	The unit is the smallest natural number.	There is no smallest number that can be substituted for a variable, unless otherwise specified.

[1] An algebraic expression is a mathematical object that contains one or more variables together with other symbols, such as numbers or signs.

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As we can see in Table 19.1, natural numbers are expressed as a series of digits (1, 2, 37, etc.), whereas variables are expressed as letters of the alphabet (*a*, *b*, *x*, *y*, etc.). Natural numbers are positive numbers and as such they have no sign attached to them. On the other hand, variables may have a *phenomenal sign*, which is the intuitively obvious sign that the variable appears to have as a superficial characteristic of its form. Mathematically speaking, the variable does not have a specific sign of its own. The values of a variable are determined when specific numbers are substituted for the literal symbol. Variables can stand for either positive or negative numbers independently of the phenomenal sign that is attached to them. For example, the variable which is represented by literal symbol 'x' can stand for positive and also for negative numbers; '$-x$' can stand for positive or negative numbers as well [this happens because $-(-3)=3$].

In the natural number set, every natural number has a unique symbolic representation and different symbols represent different numbers. For example, the symbol of the natural number '2' stands only for the number '2'. On the other hand, in the algebraic notation, a literal symbol could stand for a range of real numbers. For example, the literal symbol 'x' could stand for any real number such as 2, 4.555, ½, etc., and different literal symbols could stand for the same number. The arithmetical value of a literal symbol is determined by the number that it represents and for this reason literal symbols cannot be ordered without reference to the numbers they stand for. For the same reason, a literal symbol does not have a lowest arithmetical value unless the range of numbers that it represents has a lower limit. Because literal symbols can stand for a range of real numbers they do not have any special relation with the unit.

The conceptual change perspective predicts that incompatibility between the use of literal symbols in algebra and students' prior knowledge about numbers, in particular natural numbers, could result in errors. These errors could be explained to originate in students' tendency to use their prior experience with numbers in the context of arithmetic to interpret literal symbols in algebra.

Findings of prior studies on students' difficulties with the use of literal symbols in algebra are consistent with this view. For example, some students believe that when the literal symbol changes, then the value that it represents also changes (Booth, 1984; Kuchemann, 1981; Wagner, 1981). These students explain that '$x + y + z$' can never equal '$x + p + z$' because 'different literal symbols must correspond to different numbers'. They are unwilling to accept that different symbols could stand for the same value. However, this belief is applicable to natural numbers, where each number has a unique symbolic representation and where different symbols stand for different numbers.

There is also evidence that some students associate literal symbols with natural numbers, in the sense that they respond as if there is a correspondence between the linear ordering of the alphabet and that of the natural numbers system (Booth, 1984; Stacey & MacGregor, 1997; Wagner, 1981). For example, they tend to assign the numerical value 8 to the literal symbol 'h' (used to represent a boy's height), because 'h' is the eighth letter in the alphabet. Or they say that $10 + h = 18$, and $10 + h = R$, because the tenth letter after 'h' in the alphabet is 'R'.

In this study, we hypothesized that prior experience with numbers, in particular with natural numbers, would result in a strong tendency on the part of the students to interpret literal symbols as standing mostly for natural numbers. We also hypothesized that it would

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be difficult for students to understand that variables have a phenomenal sign, which is not the actual sign of the values they represent. This hypothesis is based on the fact that in the context of arithmetic, no sign implies positive value. This is a characteristic of natural numbers that holds for all positive numbers. When students are introduced to negative numbers they learn that the presence of the negative sign means 'negative value'. We thus predicted that students would tend to interpret 'x' to stand for positive numbers and '$-x$' to stand for negative numbers, a phenomenon also noted by researchers such as Chiarugi, Fracassina, and Furinghetti (1990) and Vlassis (2004). These hypotheses were investigated in a series of empirical studies.

Previous work by Christou and Vosniadou (2005) investigated some of the above-mentioned hypotheses. They gave 8th- and 9th-grade students a questionnaire (Questionnaire A, QR/A), which asked them to write down the numbers they thought could be assigned to algebraic expressions such as 'a', '$-b$', '$4g$', 'a/b', 'd+d+d', etc. The results showed that only about one-third of the students gave the mathematically correct response, namely that 'all numbers can be assigned to each algebraic expression'. When asked, for example, to write down numbers they thought could be assigned to the algebraic expression 'a', 66% of the students responded only with natural numbers. Natural numbers were mostly used in the remaining questions as well. In most of their responses students substituted only natural numbers for the literal symbols of the given algebraic expression and maintained the form of the algebraic expression: fraction-like in the case of 'a/b', multiples of 4 in the case of '$4g$', natural numbers larger than 3 in the case of '$k + 3$', etc. When asked to write down numbers that can be assigned to '$-b$', 72% of the students responded only with negative whole numbers (-1, -2, -3, etc.).

Again, we interpreted these responses to reflect students' tendency to substitute only natural numbers for the literal symbol 'b' and to maintain the phenomenal negative sign of the given algebraic expression '$-b$'. Very few students answered the question by providing numbers other than natural numbers, such as decimals, fractions, negatives, or real numbers. Students in this questionnaire were affected by the phenomenal sign of the algebraic expressions in the sense that they maintained it when they substituted numbers for the literal symbols. The majority of the students assigned only positive numbers to the positive-like algebraic expressions and negative numbers to the negative-like algebraic expression '$-b$'. It could, therefore, be objected that the students responded with the first numbers that came to their mind, in full knowledge that their answer was correct, since all values can be assigned to any algebraic expression. However, these responses differ in important ways from the responses expected from a mathematically sophisticated participant. AQ3

In order to further explore this possible hypothesis, we designed a second open-ended questionnaire (Questionnaire B, QR/B) in which the students were asked to write down numbers that they thought *could not* be assigned to a set of algebraic expressions. The set of algebraic expressions used was the same as the one used in QR/A. Unlike QR/A, in QR/B there is only one correct response — namely, that "all numbers can be assigned to each algebraic expression" or that "there are no such numbers that cannot be assigned to each algebraic expression".

The results obtained in QR/B showed that again only about one-third of the students gave this mathematically correct response. About half of the students said that negative whole numbers (-1, -2, -3, etc.) could not be assigned to the algebraic expression '*a*' and that natural numbers (1, 2, 3, etc.) could not be assigned to '$-b$'. Similarly in the remaining algebraic expressions, '4*g*', '*a*/*b*', '*d*+*d*+*d*', and '$k + 3$', the students tended to respond by replacing literal symbols only with negative whole numbers, while maintaining the form of the algebraic expression.

For example, the majority of the students gave numbers such as $(-1) + (-1) + (-1)$, or $(-2) + (-2) + (-2)$, as numbers that could not be assigned to the algebraic expression '*d*+*d*+*d*', and numbers such as $(-2)/(-3)$, and $(-3)/(-4)$ as numbers that could not be assigned to the fraction-like algebraic expression '*a*/*b*'. We suggest that these students interpreted the literal symbols to stand only for natural numbers and thus they thought that the additive inverses of natural numbers (the negative whole numbers) could not be substituted for the literal symbols. In QR/B students appeared again to be sensitive to the phenomenal sign of the algebraic expressions which they seem to have interpreted as the actual sign of the values that they represented. The majority of the students responded to the questions by using numbers whose actual sign was the opposite of the phenomenal sign of the algebraic expressions used. They said that negative numbers could not be assigned to the positive-like algebraic expressions and that positive numbers could not be assigned to the negative-like algebraic expression '$-b$'.

In order to further examine students' tendencies to maintain or change the phenomenal sign of the given algebraic expression as a function of the questionnaire type, students' responses were subjected to a one-way ANOVA. Responses that maintained the phenomenal sign were marked as 1, responses that changed the phenomenal sign were marked as 2, and mathematically correct responses were marked as 3. The results showed main effects for questionnaire type [$F(1, 281) = 6.126$, $p < 0.05$], which were due to the fact that students maintained the phenomenal sign in QR/A but changed it in QR/B. We interpreted students' sensitivity to the phenomenal sign of the algebraic expression to be intricately related to their belief that literal symbols in algebra stand only for natural numbers. Students think of '$-7x$', for example, as always negative and '$7x$' as always positive because they tend to think of the literal symbol '*x*' as only standing for natural numbers.

In another ANOVA we examined the effect of students' tendency to substitute only natural numbers vs non-natural numbers for the literal symbols themselves, independently of the sign of the algebraic expression, in the two questionnaires. Responses that substituted literal symbols only with natural numbers were marked as 1, responses that used non-natural numbers were marked as 2, and mathematically correct responses were marked as 3. The results showed no significant differences between the two questionnaires. In both questionnaires, students tended to substitute mostly natural numbers for the literal symbols themselves and appeared unwilling to also present any fractions, decimal numbers, or real numbers under any condition. This finding was consistent with the theoretical hypothesis of the research, namely, that students tend to consider literal symbols in algebra to stand for natural numbers only.

A possible criticism of our experiment could be that the students provided natural numbers not because they thought that these are the only 'correct' substitutions for the literal symbols, but because these are the most common numbers, used both at school and in everyday situation. In school mathematics, natural numbers are used in most of the pro blems students are asked to solve and the solution to these problems, most of the time, involves natural numbers only. Thus, it could be argued that students responded with natural numbers because they thought that this was what they should do and not because they did not know that it is possible to substitute the literal symbols with non-natural numbers.

In order to further explore this possibility we designed another study in which we used a forced-choice questionnaire. The advantage of a forced-choice questionnaire, in comparison to the open-ended ones used earlier, is that it presents students with specific alternatives that can include both natural and non-natural numbers. It can thus provide a more rigorous test of the hypothesis that students interpret literal symbols in algebra to stand only for natural numbers.

The Present Study

In this study, we constructed a forced-choice questionnaire (Questionnaire C, QR/C) that presented students with a set of specific alternatives for the same algebraic expressions used in Questionnaires A and B described earlier. These alternatives included both natural and non-natural numbers such as negative integers, positive and negative fractions, and positive and negative decimals. The correct response, namely that 'all numbers can be assigned', was one of the alternatives. The students were asked to choose the alternatives that they thought could not be assigned to the given algebraic expression. We used the negative substitution form (could not be assigned) because it is only in this condition that we can say with certainty that only the mathematically correct response applies. In the positive substitution condition all responses can be considered to be technically correct.

If students indeed interpret literal symbols to stand only for natural numbers, they should exclude some numbers from the given set, such as fractions, decimals, etc., depending on the form of the given algebraic expression. For example, given the algebraic expression 'a/b', they should think that only positive fractions could be assigned to it, and thus that whole numbers or even decimal numbers could not be assigned. Alternatively, we would expect that in the case of '$-b$', they would tend to exclude all the positive numbers of the given set of alternatives.

Method

Participants

Thirty-four children participated in this study. There were 8th and 9th graders (mean age 14.5 years old) from two middle-class high schools in Athens. All of them completed the forced-choice questionnaire (QR/C).

Materials

QR/C consisted of the following six algebraic expressions: Q1: a, Q2: $-b$, Q3: $4g$, Q4: a/b, Q5: $d+d+d$, and Q6: $k + 3$. For each algebraic expression, the students were asked

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to choose from a given set of alternative numbers those they thought could not be assigned to them. The set of alternatives consisted of 11 number choices, which included positive and negative fractions, positive and negative decimals, positive and negative integers. The twelfth alternative was always the correct response, namely, that all numbers can be assigned to each algebraic expression. An example is shown in Table 19.2.

Procedure

The following instructions were given to the students: "In algebra, we use literal symbols (such as *a*, *b*, *x*, *y*, etc.) mostly to stand for numbers. In this questionnaire we use such symbols. Read the following questions carefully. If you think there are some numbers among the given alternatives that cannot be assigned to the given algebraic expressions, please place a circle around them. You may choose more than one numbers if you wish". Students completed the questionnaire in the presence of one of the experimenters and their mathematics teacher in their classroom.

Results

Based on the findings from our previous studies, we created three main categories of responses namely 'NN/1', 'NN/2', and 'Phenomenal sign'. The category 'NN/1' attempted to capture all responses that reflected the belief that literal symbols stand for natural numbers only (NN belief). For example, students who chose all numbers from the given alternatives except the positive fractions for the algebraic expression '*a*/*b*', would be placed in this category.

The category 'NN/2' captured responses that included some but not all alternatives predicted by the NN belief. For example, in the case of '*a*/*b*', students could choose all the integers of the given set of alternatives. These responses would be placed in this category. In the 'Phenomenal sign' category we placed student's responses that included all numbers with the opposite sign from the phenomenal one. For example, in this category we placed

Table 19.2: An example of the way in which questions were posed in the forced choice questionnaire.

Are there some numbers among the following alternatives that you think cannot be assigned to 4*g*?

a) 6	e) 6.74	i) $-\frac{2}{3}$
b) 2	f) $\frac{5}{7}$	j) 8
c) -0.25	g) 8	k) 2.333
d) –3	h) 4	l) No, all numbers can be assigned to it.

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students' responses only with negative numbers in the case of '*a*/*b*', or only with positive numbers in the case of '$-b$'.

There were three additional categories of responses: 'Non-systematic', 'No response', and 'Correct'. The 'Non-systematic' category was used for all non-systematic responses. The 'No response' category included null responses, and the 'Correct' category represented the correct alternative.

All responses were categorized by one of the experimenters, and a second rater scored half of the responses using the same criteria. The agreement of the categorization was 96%. All disagreements were discussed until consensus was achieved. Tables 19.3, and 19.4 in more detail, presented the frequencies and the percentages of each category of responses.

Table 19.3 presents the total percentage of students' responses to all the questions for each response category. Only 18.6% of the students' responses represented the 'Correct response despite the fact that it was an explicit alternative in all questions. One-third of students' responses (30.3%) were affected by the NN belief in the strict (22%, NN/1) or in a more differentiated way (8.3%, NN/2). One-fourth of students' responses (25.4%) were affected by the phenomenal sign of the algebraic expressions. There was a large percentage of responses in the non-systematic category (16.1%) that could be explained by the complexity and counterintuitiveness of the questions in QR/C, the fact that they were expressed in the negative form, and, finally, the forced choice nature of the questionnaire. Previous studies have also shown that non-systematic responses appear more frequently in forced-choice questionnaires (see Vosniadou, Skopeliti, & Ikospentaki, 2004) than in open-ended ones.

Table 19.4 presents in greater detail the frequencies and percentages for students' responses in each category for the 6 algebraic expressions, together with examples of the type of numbers/responses for each category. In the case of the algebraic expression '*a*', only about one-third of the students (29.4%) responded by selecting the correct response, namely that 'all values can be assigned to it'. The majority of the responses reflected the belief that the literal symbol '*a*' stands only for positive numbers (38.2%). Another 20.5% of the responses indicated that numbers other than natural numbers could not be assigned to '*a*'. In the remaining algebraic expressions, such as '4*g*', '$k + 3$', or '*a*/*b*', students' responses appeared to be slightly different. The majority of the students were affected

Table 19.3: Percentages of students' responses to all questions in the forced choice questionnaire.

Questionnaire C (choose, from the given set of numbers, those that you think cannot be assigned to the given algebraic expressions)

Categories	**Correct**	**NN/1**	**NN/2**	**Phenomenal Sign**	**Non-systematic**	**No Response**
Mean percentage	18.6%	22% 30.3%	8.3%	25.4%	16.1%	9.3%

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more by the NN belief than by the phenomenal sign. For example, in the case of '*a*/*b*', 32.3% of the responses were placed in the category NN/1, and 11.7% in the category 'Phenomenal sign'. In the remaining algebraic expressions the results were similar with the exception of '$-b$'. In this case, the results were quite different because of the presence of the negative sign. About half of the students' responses (52.9%) indicated that students interpreted this expression as standing for negative numbers only.

Table 19.4: Frequencies, percentages, and examples of students' responses in all categories of responses.

Questionnaire C (choose, from the given set of numbers, those that you think cannot be assigned to the given algebraic expressions)

Questions	**Correct**	**NN/1**	**NN/2**	**Phenomenal Sign**	**Non-systematic**	**No Response**
a		All but NN	Decimals, fractions	Negatives		−
	10 (29.4%)	7 (20.5%)	3 (8.8%)	13 (38.2%)	1 (2.9%)	
$-b$		All but negative wholes	Natural numbers	Positives		
	6 (17.6%)	2 (5.9%)	2 (5.9%)	18 (52.9%)	5 (14.7%)	1 (2.9%)
Q3: $4g$		All but NN		Negatives		
	7 (20.5%)	9 (26.5%)	−	5 (14.7%)	10 (29.4%)	3 (8.8%)
Q4: $\frac{a}{b}$		All but positive fractions	Integers	Negatives		
	6 (17.6%)	11 (32.3%)	2 (5.9%)	4 (11.7%)	5 (14.7%)	6 (17.6%)
Q5: $d+d+d$		All but NN	Fractions	Negatives		
	4 (11.7%)	10 (29.4%)	3 (8.8%)	6 (17.6%)	6 (17.6%)	5 (14.7%)
Q6: $\kappa + 3$		All but NN	Fractions	Negatives		
	5 (14.7%)	6 (17.6%)	7 (20.5%)	6 (17.6%)	6 (17.6%)	4 (11.7%)

Discussion

The results from the present study are consistent with the findings from our previous study (Christou & Vosniadou, 2005), and support the hypothesis of the conceptual change approach that students' interpretation of the use of literal symbols in algebra is strongly influenced by their experience with numbers, in particular natural numbers, in the context of arithmetic. The conclusion was based on two sources of evidence: first, students tended to substitute only natural numbers for the literal symbols of the algebraic expressions, and second, students interpreted the phenomenal sign of the algebraic expressions as the actual one of the numbers that they represent. This was the case, despite the fact that students were taught in school that each literal symbol corresponds to a range of real numbers.

Students interpreted algebraic expressions such as '$k + 3$' or '$d + d + d$' to stand mostly for positive numbers and believed that negative numbers cannot be assigned to them. The greatest influence of the phenomenal sign appeared in the case of the negative-like algebraic expression '$-b$', which the majority of the students interpreted as standing for negative numbers only.

This finding is consistent with results from Vlassis' (2004) research with polynomials, where students appeared to consider the minus sign at the beginning of a polynomial as the sign of a negative number. We interpret students' belief that the phenomenal sign of the algebraic expression is the actual one of the numbers that it represents to originate from prior experience with arithmetic. In the context of arithmetic, numbers with "no sign" means numbers with "positive value", whereas numbers with "negative sign" means numbers with "negative value". The transfer of this knowledge in the area of algebra causes a misconception, which is strong even in the case of the older students and constrains the acquisition of more advanced mathematical concepts such as, for example, the absolute value of a number. For any real number a, the absolute value of a, denoted $|a|$, is always a positive number, so $|a|$ is equal to a, if $a \geq 0$ or to $-a$, if $a < 0$. As the students are affected by the phenomenal sign of the algebraic expressions they do not think of '$-a$' as a symbol that could possibly stand for a positive number when 'a' stands for a negative one (see Chiarugi et al., 1990). In order for students to understand that the phenomenal sign of an algebraic expression is not the actual one of the numbers that it represents, they need to reorganize their prior knowledge about numbers as shaped in the context of arithmetic.

Furthermore, students' prior experience with numbers in the context of arithmetic constrains their understanding of the generalized nature of a literal symbol, i.e., as a variable that stands for a range of real numbers. The present findings agree with the previous research which shows that the initial understanding of number as natural number may hinder the acquisition of more advanced mathematical concepts, as in the case of fractions, rational numbers, or algebraic rules, etc. (Gelman, 2000; Resnick et al., 1989; Stafylidou & Vosniadou, 2004; Vamvakoussi & Vosniadou, 2004).

Resnick (1987) has argued that algebraic expressions can take their meaning from their position in the formal system of algebra. Over and above, there is also a referential meaning, which algebraic expressions take either from the situations in which relations among quantities and actions upon quantities play a role or from its connection with the 'world of numbers'. She noted that student's capability to assign a referential meaning to the algebraic expressions affects their performance in algebraic transformation tasks. Focusing on

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the referential meaning that can be assigned to the algebraic expressions with its linkage to the world of numbers, we found that many students tend to think that literal symbols represent only natural numbers and as a consequence they have a very restricted range of numbers from which the algebraic expressions take their referential meaning. This affects students' performance in many mathematical tasks such as, for example, when students have to estimate the sign or the value of an algebraic expression in situations where the monotony of a function is tested, or in the case of a radicand.

We believe that we have provided some evidence that the conceptual change theoretical framework can help us systematize students' errors in interpreting the use of literal symbols in algebra. The results of the present study can also provide useful information for the design of curricula and for instruction. It is important for teachers of algebra as well as for the curriculum designers to be familiar with students' beliefs and the possible reasons for their mistakes when they use literal symbols in algebra, as well as in other domains for example physics, chemistry, etc. (see, e.g., Heck, 2001; Sherin, 2006). Greer (1994, 2005) has suggested various devices for expanding arithmetic operations beyond natural numbers. He proposed to give students mathematical tasks, which use non-natural numbers as factors, such as the equation $2.67x^2 - 3.86x - 12.23 = 0$, as this could help them extend their conceptual fields beyond the natural numbers. Some researchers investigate the implications of introducing algebraic thinking in elementary school (Carraher, Schliemann, & Brizuela, 2001). Of course, more detailed empirical research is needed to further investigate students' difficulties and the effect of specific instructional innovations before introducing them in schools.

AQ4

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