Quasi-linear PDEs (III)

Conservation laws, shock dynamics and weak solutions

Conservation laws

A conservation law has the general form:

$$\frac{\partial}{\partial t}T(x,t)+\frac{\partial}{\partial x}X(x,t)=0$$

where ***T*(*x*,*t*)** is the **density of the conserved quantity** and ***X*(*x*,*t*)** is the **associated flux.**

Integrating with respect to *x*, the conservation law can be expressed in the following integral form:

$$\frac{\partial}{\partial t}\int_{x_1}^{x_2}T(x,t)\,dx=-X(x,t)\Big|_{x=x_1}^{x_2}$$

i.e., the rate of change of the density over an interval depends only on the flux through its end points

If $X(x_2,t)=X(x_1,t)$ then $\int_a^b T(x,t)\,dx=\text{const.}$ conserved quantity

Examples

1) Linearized KdV equation: $u_t + u_{xxx} = 0 \quad (1)$

This equation is already in the form of conservation law:

$$(u)_t + (u_{xx})_x = 0, \quad T_1 = u, \quad X_1 = u_{xx}$$

$$\int_{-\infty}^{+\infty} T_1 dx = \int_{-\infty}^{+\infty} u\, dx = \text{mass} = \text{const.};$$

Also, from **Parseval's theorem**, linear dispersive equations with real $\omega(k)$ **conserve the energy** **[Can you show this?]**

Indeed, multiplying (1) by u we find:

$$(u^2)_t + \left(uu_{xx} - \frac{1}{2}u_x^2 \right)_x = 0, \quad T_2 = u^2, \quad X_2 = uu_{xx} - \frac{1}{2}u_x^2$$

$$\int_{-\infty}^{+\infty} T_2 dx = \int_{-\infty}^{+\infty} u^2 dx = \text{energy} = \text{const.}$$

Examples – cont.

2) Schrödinger equation: $$i\psi_t + \frac{1}{2}\psi_{xx} - V(x)\psi = 0 \quad (2)$$

Multiplying (2) by $\overline{\psi}$, its c.c. by ψ, and adding, we find:

$$(|\psi|^2)_t + \left(\frac{i}{2}(\overline{\psi}\psi_x - \psi\overline{\psi}_x)\right)_x = 0, \quad T = |\psi|^2,$$

$$\int_{-\infty}^{+\infty} |\psi|^2 dx = \text{const.},$$

$$X = \frac{i}{2}(\overline{\psi}\psi_x - \psi\overline{\psi}_x) = \text{current}$$

3) Hopf equation: $$u_t + uu_x = 0, \quad (3)$$

$$(u)_t + \left(\frac{1}{2}u^2\right)_x = 0, \quad T = u, \;\; X = \frac{1}{2}u^2,$$

Additional comments on the Hopf equation

Consider the Hopf equation: $u_t + uu_x = 0, \quad u(x,0) = f(x)$

and assume that $u(x,t)$ represents a density; then:

$$M_{x_1,x_2} = \int_{x_1}^{x_2} u(x,t)\,dx :$$ total mass in $x \in [x_1, x_2]$

Using the conservation law and its integral form we obtain:

$$(u)_t + \left(\frac{1}{2}u^2\right)_x = 0 \Rightarrow \frac{d}{dt}\int_{x_1}^{x_2} u\,dx = \frac{dM_{x_1,x_2}}{dt} = \frac{1}{2}u^2(x_1,t) - \frac{1}{2}u^2(x_2,t)$$

$\frac{dM_{x_1,x_2}}{dt}$: rate of change of mass

$\frac{1}{2}u^2(x_1,t) - \frac{1}{2}u^2(x_2,t)$: net mass / flux through the endpoints

E.g., in the traffic flow problem: ***u(x,t)* = traffic density**;

Rate of change of # of vehicles in $[x_1,x_2]$=

{# of vehicles entering x_1} - {# of vehicles leaving at x_2}

Additional comments on the Hopf equation

Assume that the initial data, $f(x)$, has a finite mass:

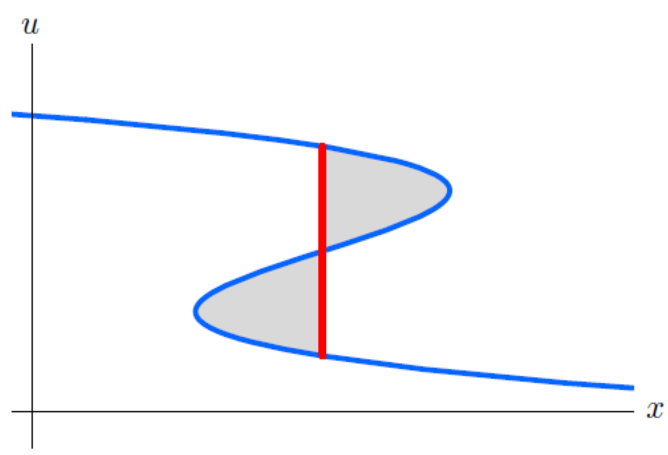
$$u(x,0) = f(x), \quad \left| \int_{-\infty}^{+\infty} f(x,t)\,dx \right| < \infty$$

i.e., $f(x) \to 0$ rapidly as x $\to \infty$. Then,

$$\int_{-\infty}^{+\infty} u(x,t)\,dx = \int_{-\infty}^{+\infty} u(x,0)\,dx = \int_{-\infty}^{+\infty} f(x)\,dx = \text{const.}$$

i.e., mass remains constant - and is equal to its initial value

What happens in the case of a shock, where u(x,t) displays a **discontinuity**?



Equal Area Rule: **it ensures that the total mass of the shock wave solution remains constant**

Shock waves and conservation laws

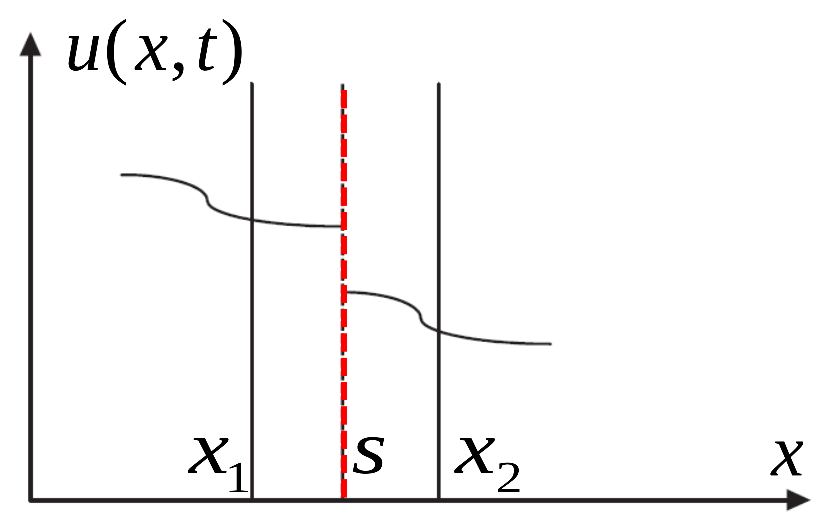
- We wish to take the solution further in time, beyond $t = t_B$, *but* ***do not*** **want multi-valued solutions** for **physical** and **mathematical** reasons

- Discontinuities, i.e., shocks, can be introduced by considering the Hopf equation as coming from the above mentioned **conservation law** and its corresponding **integral form**:

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0 \Rightarrow \frac{d}{dt}\int_{x_1}^{x_2} u(x,t)dx + \frac{1}{2}\left[u^2(x_2,t) - u^2(x_1,t)\right] = 0$$

This equation can support a shock wave since it is an **integral relation**.

Assume that between two points x_1 *and* x_2 *we have a discontinuity* that can change in time, $x = s(t)$



Rankine-Hugoniot (RH) condition (I)

Then, we rewrite the integral form of the conservation law as:

$$\frac{d}{dt}\left[\int_{x_1}^{s(t)} u(x,t)dx + \int_{s(t)}^{x_2} u(x,t)dx\right] + \frac{1}{2}\left[u^2(x_2,t) - u^2(x_1,t)\right] = 0$$

Let: $\begin{cases} x_1 = s(t) - \varepsilon \\ x_2 = s(t) + \varepsilon \end{cases}$ with $\varepsilon > 0$.

Next, using **Leibnitz integral rule** we obtain:

$$u(s,t)\frac{ds}{dt} - u(s-\varepsilon,t)\frac{ds}{dt} + \int_{s-\varepsilon}^{s} u_t(x,t)dt$$

$$+ u(s+\varepsilon,t)\frac{ds}{dt} - u(s,t)\frac{ds}{dt} + \int_{s}^{s+\varepsilon} u_t(x,t)dt$$

$$+ \frac{1}{2}\left[u^2(s+\varepsilon,t) - u^2(s-\varepsilon,t)\right] = 0$$

Rankine-Hugoniot (RH) condition (II)

Thus: $$\frac{ds}{dt}\left[u(s+\varepsilon,t)-u(xs-\varepsilon,t)\right]+\int_{s-\varepsilon}^{s} u_t dt+\int_{s}^{s+\varepsilon} u_t dt$$
$$+\frac{1}{2}\left[u^2(s+\varepsilon,t)-u^2(s-\varepsilon,t)\right]=0$$

In the limit $\varepsilon \to 0$:
$$\begin{cases} x_1 = s-\varepsilon \to x^-, \quad x_2 = s+\varepsilon \to x^+ \\ \int_{s-\varepsilon}^{s} u_t dt+\int_{s}^{s+\varepsilon} u_t dt = \int_{s-\varepsilon}^{s+\varepsilon} u_t dt \to 0 \end{cases}$$

Hence:

$$\frac{ds}{dt}\left[u(x^-,t)-u(x^+,t)\right]+\frac{1}{2}\left[u^2(x^+,t)-u^2(x^-,t)\right]=0 \Rightarrow$$

$$\frac{ds}{dt}=\frac{1}{2}\left[u(x^+,t)+u(x^-,t)\right]=\frac{1}{2}(u^+ + u^-), \quad u^{\pm}=u(x^{\pm},t)$$

speed of the shock *in terms of the jump discontinuities*

Generalization of the RH condition

Consider the Hopf equation: $u_t + c(u)u_x = 0$

and assume that F is the **antiderivative** of $c(u)$, i.e., $F'(u) = c(u)$

Then, the Hopf equation can be written as the conservation law:

$$u_t + \frac{\partial}{\partial x}\big(F(u)\big) = 0$$

In this case, the **Rankine-Hugoniot condition** take the form:

$$\frac{ds}{dt} = \frac{F(u_+,t) - F(u_-,t)}{u_+ - u_-}$$

or: $s'(t)\big[u\big] = \big[F(u)\big]$ **Jump condition**

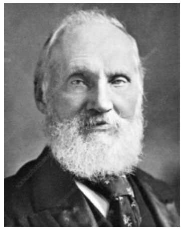
where $[g]$ denotes the jump of g along a discontinuity

Rankine-Hugoniot condition - history

- They are named in recognition of work carried out by **William Rankine (Scottish)** in **1870** and **Pierre Henri Hugoniot (French)** in **1887**

- They were originally found by **Sir George Stokes (Irish)** in **1848** and **Bernhard Riemann (German)** in **1860**

- **Lord Kelvin (Irish)** and **Lord Rayleigh (English)** criticized Stokes' work.



Kelvin

Rayleigh

W.Rankine

P. Hugoniot

G. Stokes

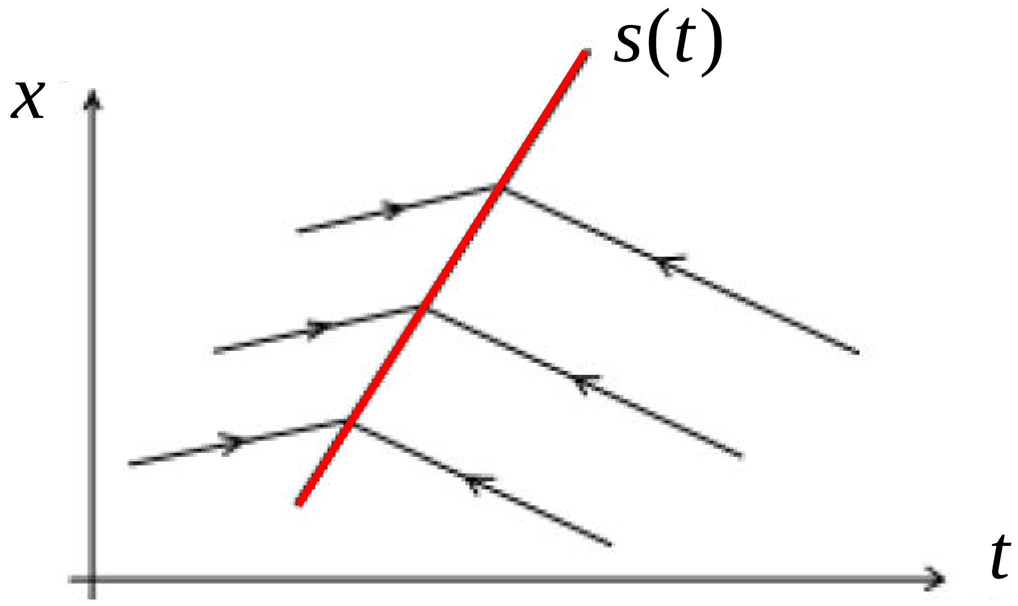
B. Riemann

- **Stokes** thought that he was **wrong** and deleted that part when his collected works were published. **The missing part was restored in 1966**

RH condition and weak solutions (I)

- The **RH condition** (also referred to as shock condition or jump condition), are used to **avoid multi-valued behavior** in the solution, which would otherwise occur after characteristics cross

- We are interested in finding **weak** (or **generalized**) **solutions** featuring only **essential discontinuities**: the discontinuity curve $x = s(t)$ is essential if characteristics from each side of $x = s(t)$ intersect on $s(t)$ as t grows.



RH condition and weak solutions (II)

To be more specific, consider the Hopf equation: $u_t + \left(F(u)\right)_x = 0$

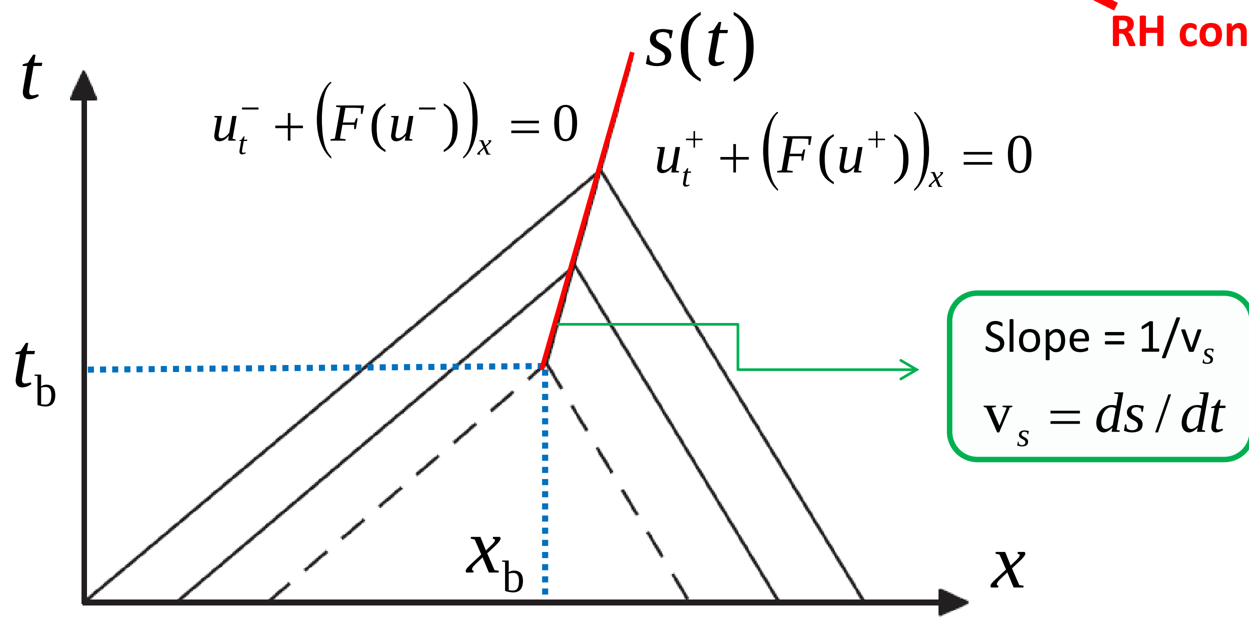
Then, a **weak solution** of the Hopf equation, **valid for $t \geq t_b$**, is:

$$u = \begin{cases} u^-, & \text{for } (x,t) \text{ on the } \textit{left} \text{ of } s(t) \\ u^+ & \text{for } (x,t) \text{ on the } \textit{right} \text{ of } s(t) \end{cases}$$

with $s(t)$ satisfying the ODE:

$$\frac{ds}{dt} = \frac{F(u^+,t) - F(u^-,t)}{u^+ - u^-}, \quad s(t_b) = x_b$$

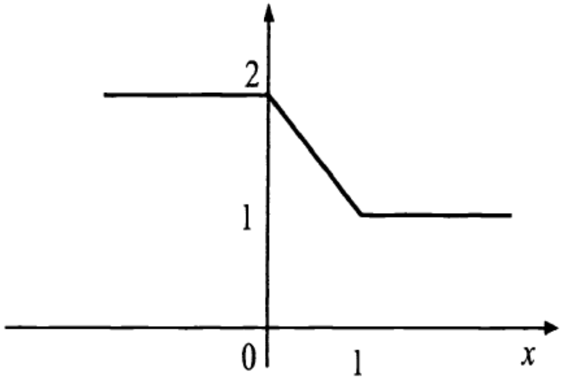
RH condition



Revisiting a previous example

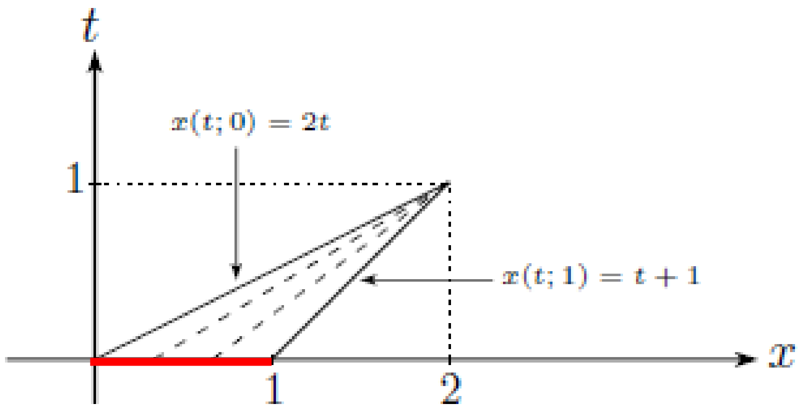
Consider the IVP: $u_t + u\,u_x = 0, \quad u(x,0) = f(x), \quad x \in \mathrm{R}$

$$f(x) = \begin{cases} 2, & x < 0 \\ 2 - x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$



Recall that $t_b = 1$ (and $x_b = 2$) and the solution reads:

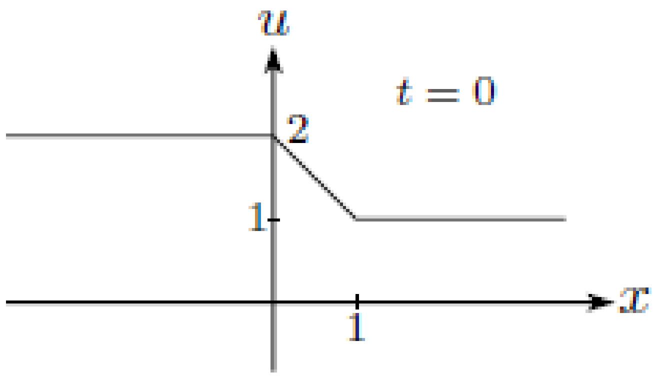
$$u(x,t) = \begin{cases} 2, & x < 2t \\ \dfrac{2-x}{1-t}, & 2t \le x \le t+1 \\ 1, & x > t+1 \end{cases}$$

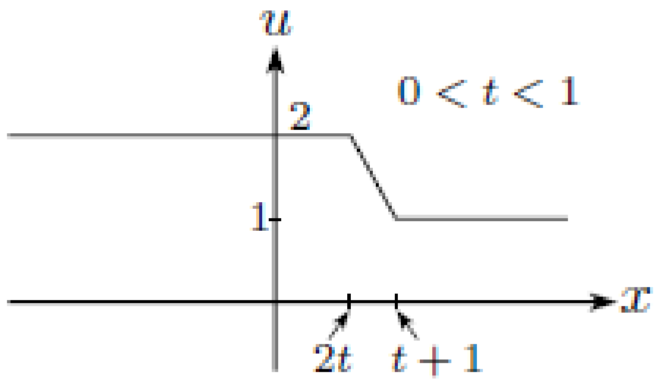


Hence, at $t = t_b = 1$ the solution becomes discontinuous

Revisiting a previous example – cont.

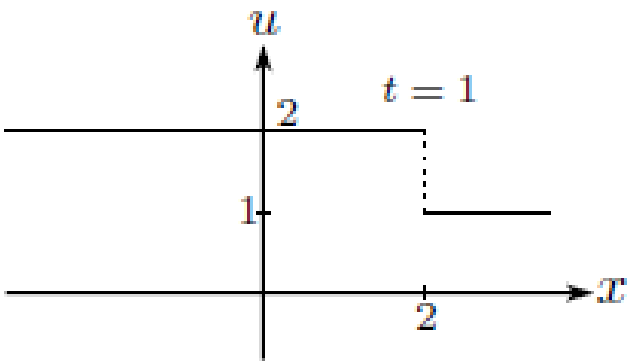
In particular, the solution has the form:





and for $t = t_b = 1$ the solution reads:

$$u(x, t=1) = \begin{cases} 2, & x < 2 \\ 1, & x > 2 \end{cases}$$



Hence, for $t > t_b = 1$, no classical solution exists

Nevertheless, we can construct a **weak solution** as follows

Constructing the weak solution

Observe that: $u = \begin{cases} u^- = 2, & \text{for } (x,t) \text{ on the } \textit{left} \text{ of } s(t) \\ u^+ = 1 & \text{for } (x,t) \text{ on the } \textit{right} \text{ of } s(t) \end{cases}$

with $s(t)$ satisfying the ODE: $\dfrac{ds}{dt} = \dfrac{F(u^+,t) - F(u^-,t)}{u^+ - u^-}, \quad s(t_b) = x_b$

Here: $F(u) = \dfrac{1}{2}u^2$

$$\frac{ds}{dt} = \frac{F(u^+,t) - F(u^-,t)}{u^+ - u^-}$$

$$\Rightarrow \frac{ds}{dt} = \frac{1}{2}(u^+ + u^-) = \frac{1}{2}(1+2) = \frac{3}{2}$$

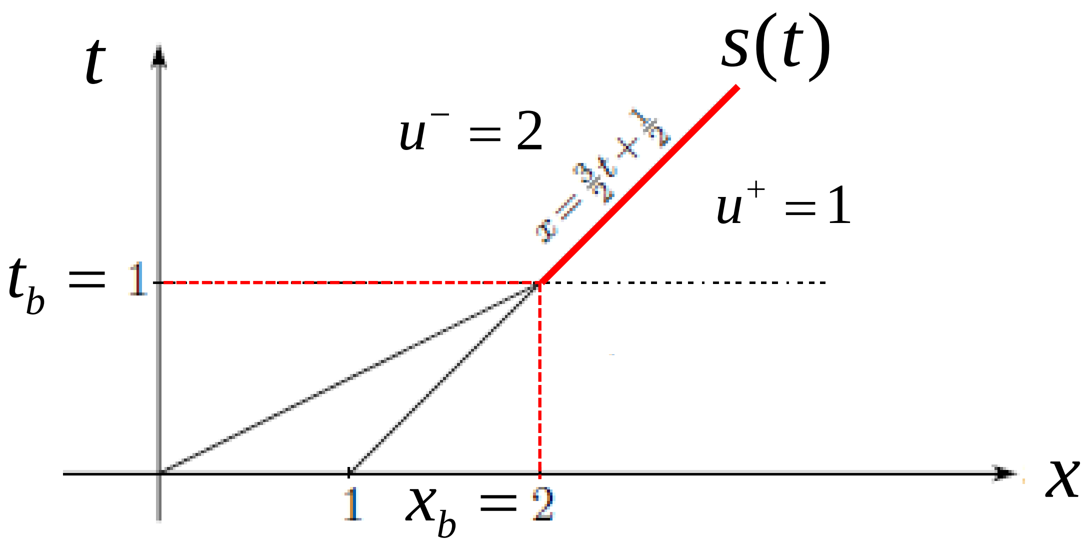
RH condition for the usual Hopf equation

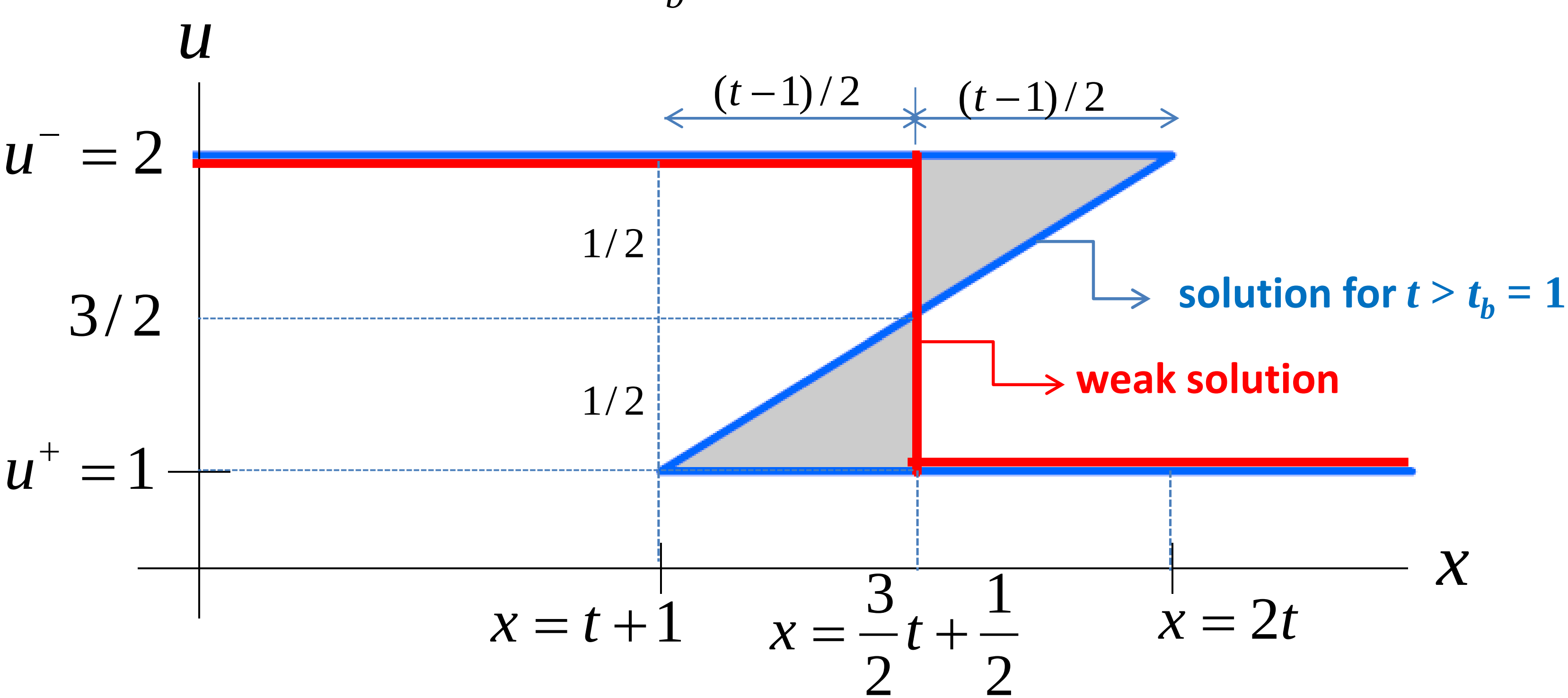
Hence: $\dfrac{ds}{dt} = \dfrac{3}{2} \Rightarrow s(t) = \dfrac{3}{2}t + x_0$

$s(t_b) = x_b \Rightarrow s(1) = 2$

$$\Rightarrow s(t) = \frac{3}{2}t + \frac{1}{2}$$

The *xt*-plane and the weak solution





The generalized solution

- Concluding, a **generalized solution** of the considered IVP is:

$$u(x,t) = \begin{cases} u_{classical}(x,t), & \text{for} \quad t < t_b = 1 \\ u_{weak}(x,t) & \text{for} \quad t \geq t_b = 1 \end{cases}$$

- The classical solution for $t < 1$ reads:

$$u_{classical}(x,t) = \begin{cases} 2, & x < 2t \\ \dfrac{2-x}{1-t}, & 2t \leq x \leq t+1 \\ 1, & x > t+1 \end{cases}$$

- The weak solution for $t \geq 1$ reads:

$$u_{weak}(x,t) = \begin{cases} 2, \ \text{for} & x < \dfrac{3}{2}t + \dfrac{1}{2} \\ 1, \ \text{for} & x > \dfrac{3}{2}t + \dfrac{1}{2} \end{cases}$$

Riemann problems

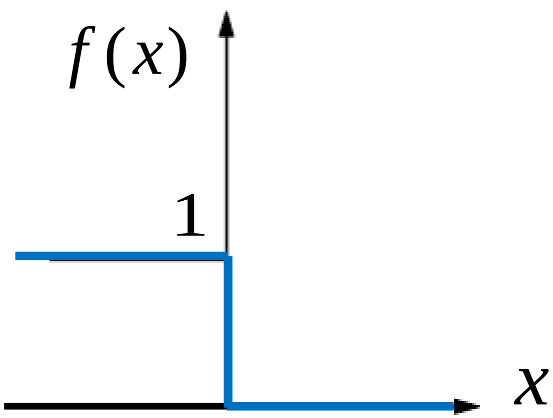
- A **Riemann problem** is an initial value problem for a hyperbolic PDE (or a system thereof) in which the initial data is **piecewise constant with a discontinuity**

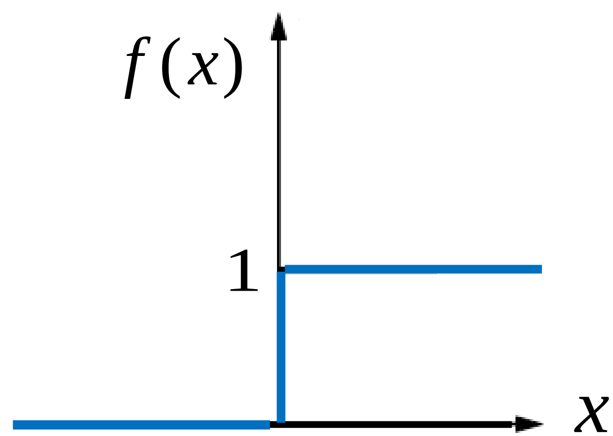
Examples

Consider the IVP: $u_t + u\,u_x = 0, \quad u(x,0) = f(x), \quad x \in R, \quad t > 0$

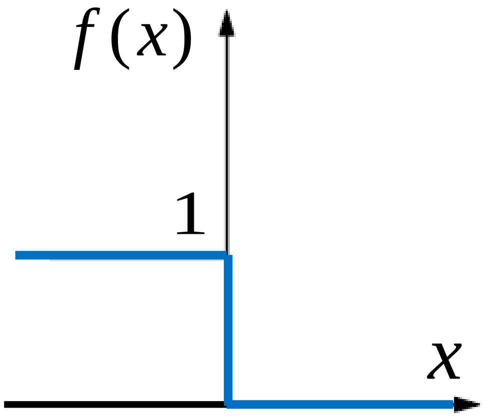
with the following sets of discontinuous initial data:

$$\text{(I)} \quad f(x) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases} \qquad\qquad \text{(II)} \quad f(x) = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$$





Riemann problem (I)

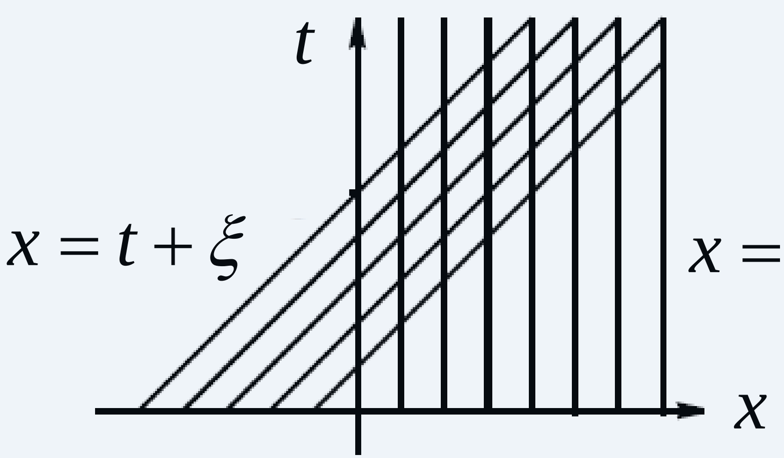


In this case, the initial data is already the form of a shock wave. Indeed:

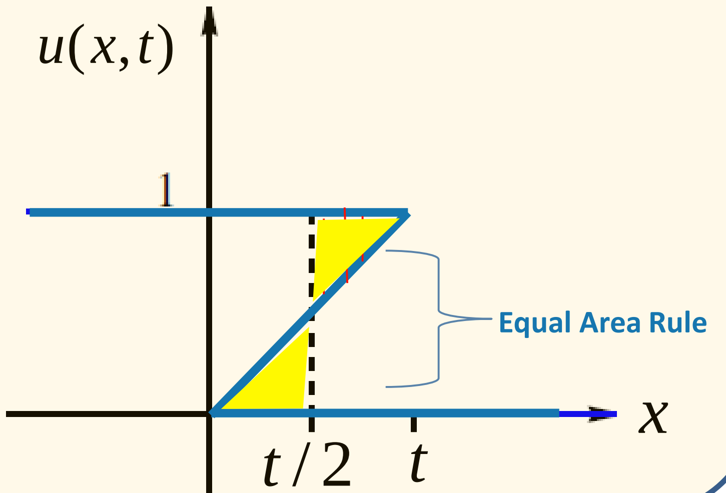
$$\frac{dx}{dt} = u,\ x(0) = \xi; \quad \frac{du}{dt} = 0,\ u(0) = f(\xi)$$

lead to: $x = f(\xi)t + \xi;\ \ u(x,t) = f(\xi) = \begin{cases} 1, & \xi < 0 \text{ or } x < t \\ 0, & \xi > 0 \text{ or } x > 0 \end{cases}$

Characteristics intersect for $\mathbf{0} < \boldsymbol{x} < \boldsymbol{t}$



Form of $u(x,t)$



Riemann problem (I) – cont.

Weak solution: $u = \begin{cases} u^- = 1, \text{ for } (x,t) \text{ on the } \textit{left} \text{ of } s(t) \\ u^+ = 0 \text{ for } (x,t) \text{ on the } \textit{right} \text{ of } s(t) \end{cases}$

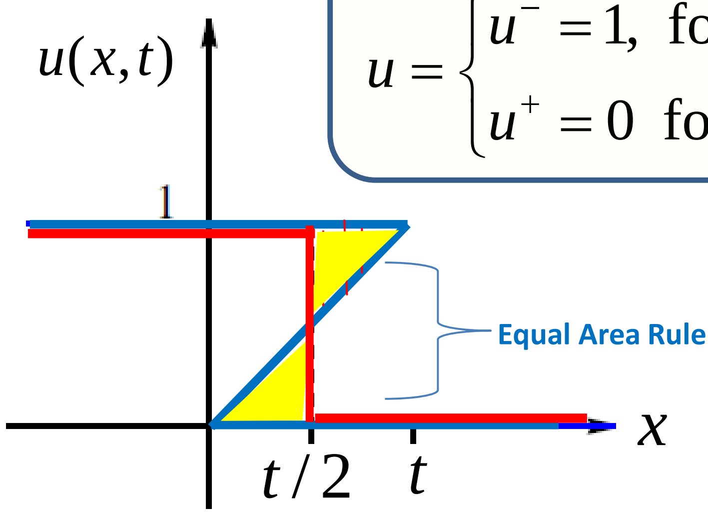
with $s(t)$ satisfying the ODE: $\frac{ds}{dt} = \frac{1}{2}(u^+ + u^-)$, $s(0) = 0$

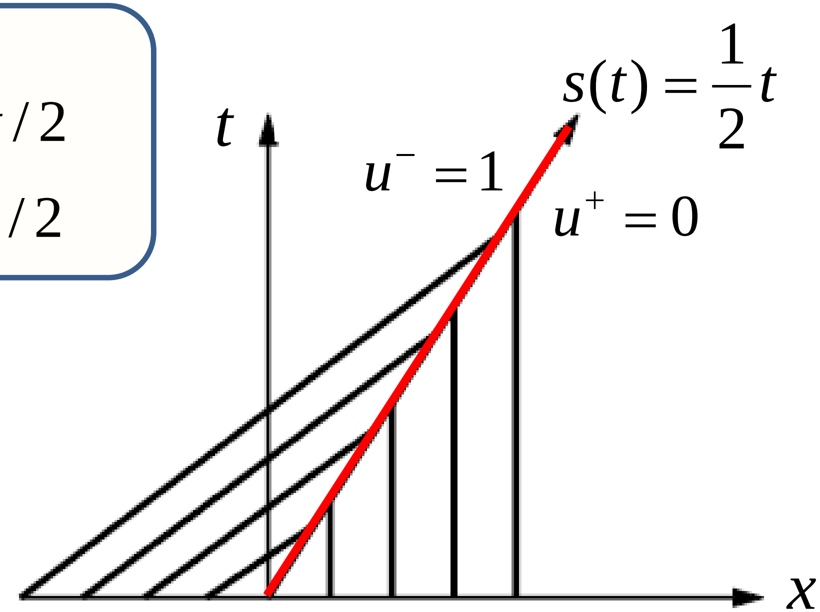
RH condition

$$\frac{ds}{dt} = \frac{1}{2}(u^+ + u^-) = \frac{1}{2}(0+1) = \frac{1}{2} \Rightarrow s(t) = \frac{1}{2}t$$

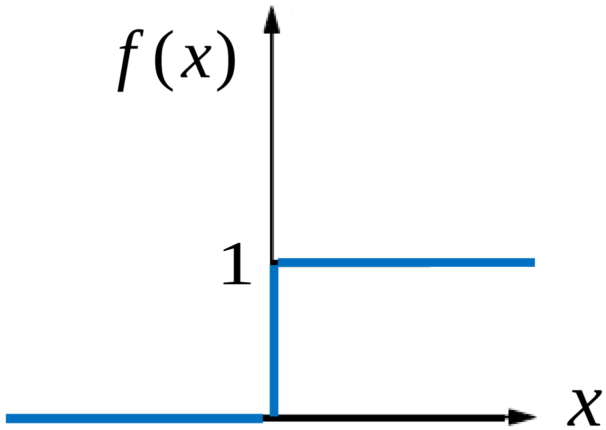
weak solution

$$u = \begin{cases} u^- = 1, \text{ for } x < t/2 \\ u^+ = 0 \text{ for } x > t/2 \end{cases}$$





Riemann problem (II)



In this case, we again have:

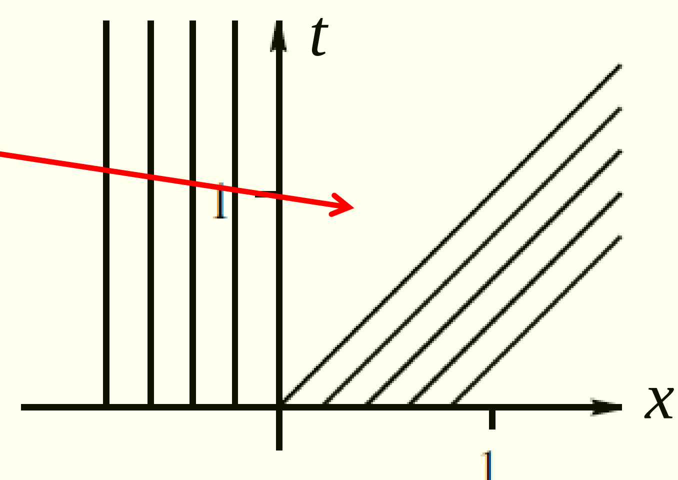
$$\frac{dx}{dt} = u,\ x(0) = \xi; \quad \frac{du}{dt} = 0,\ u(0) = f(\xi)$$

and the characteristics are of the form:

$$x = f(\xi)t + \xi; \ \ u(x,t) = f(\xi) = \begin{cases} 0, & \xi < 0 \\ 1, & \xi > 0 \end{cases} \Rightarrow x = \begin{cases} \xi, & \xi < 0 \\ t + \xi, & \xi < 0 \end{cases}$$

However, we still have a problem: there exists a region on which there is not enough information!

No crossing of characteristics



Question:

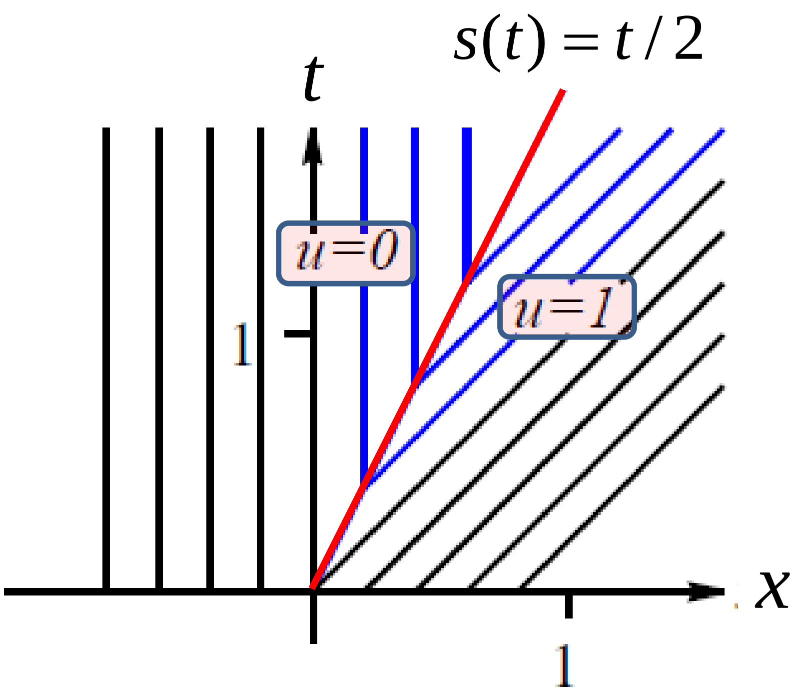
How should we define the solution in this region?

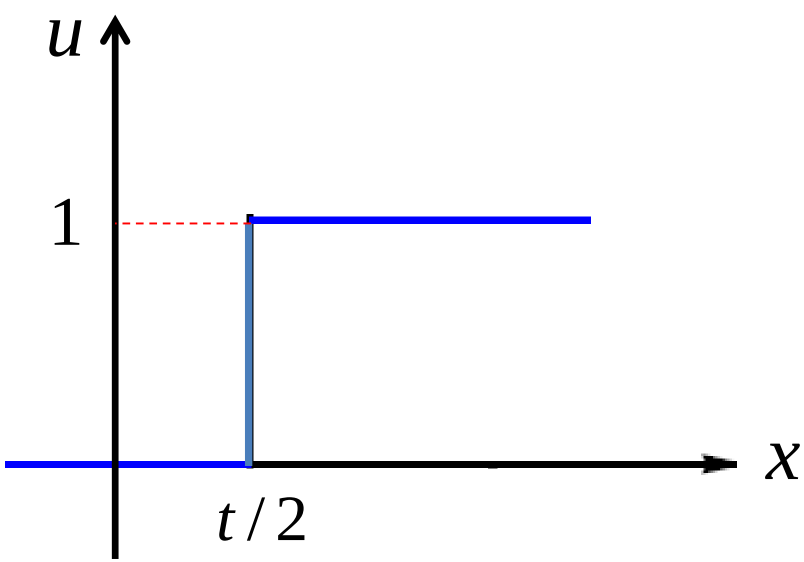
Riemann problem (II) – cont.

One possibility:

A possible **weak solution** of the problem is: $u = \begin{cases} 0, & \text{for } x < t/2 \\ 1, & \text{for } x > t/2 \end{cases}$

This is composed by classical solutions on either side of the **curve of discontinuity $s(t)=t/2$**. Furthermore, this solution satisfies the **RH condition** along the curve of discontinuity.





Riemann problem (II) – cont.

Another possibility:

Another possible **weak solution** of the problem can be found upon filling the wedge $0 < x < t$ with another, similarity solution, of the Hopf equation. Such a **similarity solution** is of the form:

$$u(x,t) = t^n g(\eta), \;\; \eta = x t^m$$

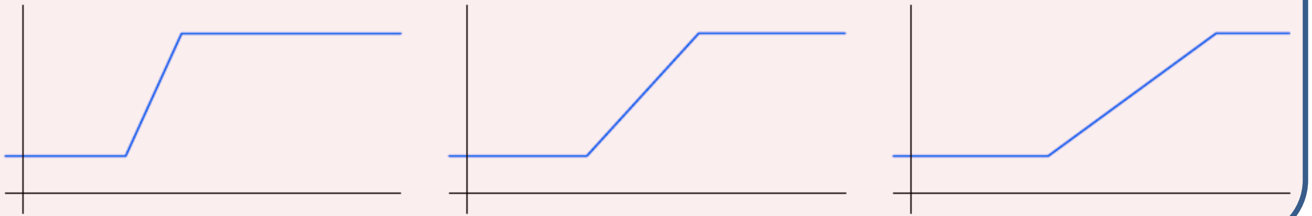
Observing that the Hopf equation $u_t + u\,u_x = 0$ is invariant under the transformations $x \mapsto \lambda x, \;\; t \mapsto \lambda t$, a simple calculation leads to:

$$n = 0, \quad m = -1, \quad g = \frac{x}{t}$$

Hence, another solution of the problem is the continuous function:

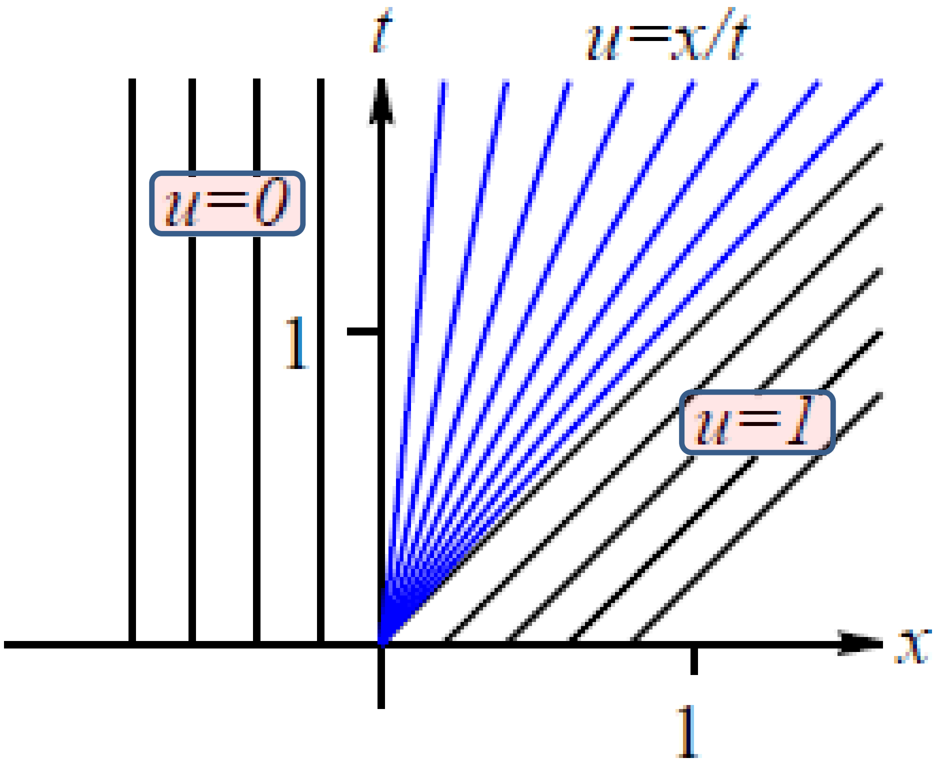
$$u = \begin{cases} 0, \;\; \text{for} \;\; x < 0 \\ \dfrac{x}{t}, \;\; \text{for} \;\; 0 < x < t \\ 1, \;\; \text{for} \;\; x > t \end{cases}$$

This **weak solution** is called a **rarefaction wave**



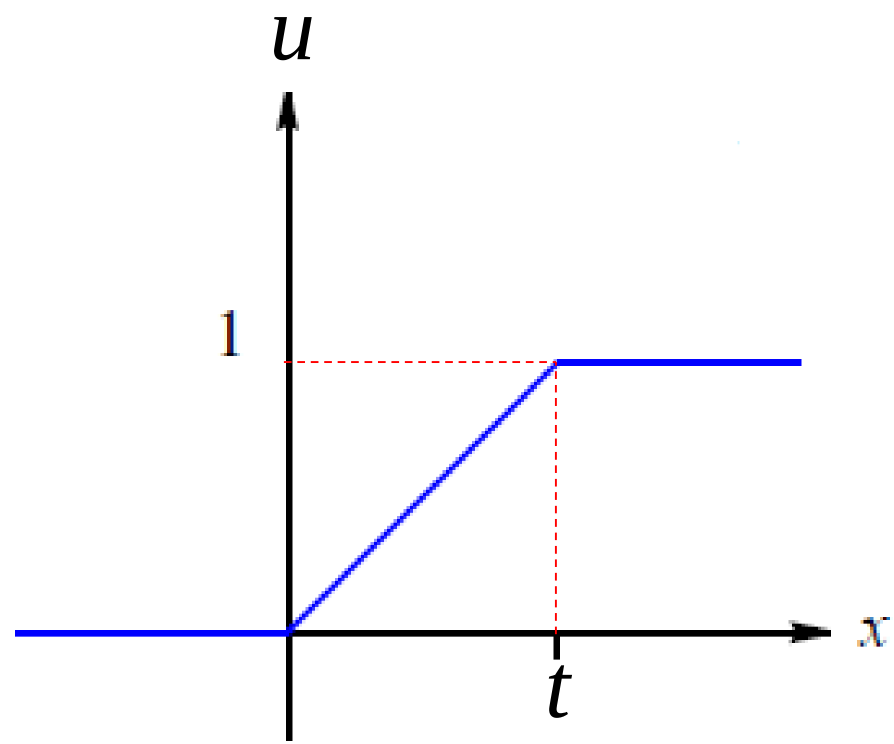
Riemann problem (II) – cont.

xt-plane and characteristics



Notice the **"fan"** at $x = 0$

The form of the solution



Recall: $u = \begin{cases} 0, & \text{for } x < 0 \\ \dfrac{x}{t}, & \text{for } 0 < x < t \\ 1, & \text{for } x > t \end{cases}$

The Lax entropy condition

There exist at least **two solutions** of the Riemann problem (II), with the same initial data and hence **the solution is not unique**.

Question: **which one of these solutions is physically meaningful?**

For our initial data, **the wave is higher to the right**. Consequently, we expect the part of the wave to the right to move faster. Hence, physically, we do not want to allow for the 1st solution. Instead, **we accept the 2nd one as a physically more realistic solution.**

It can be proved that there exists a unique weak discontinuous solution of the Cauchy problem which satisfies the following inequality on the curve of discontinuity:

$c(u^-) > s'(t) > c(u^+)$ **Lax entropy condition**	**Criterion for a unique weak discontinuous solution**

The Lax entropy condition – cont.

$$c(u^-) > s'(t) > c(u^+)$$

- This condition states that **the wave speed just behind the shock is greater than the wave speed just ahead of it.** In other words, **the wave behind the shock catches up to the wave ahead of it.**
- This entropy criterion is a special case of the **second law of thermodynamics: entropy *increases across a shock.***
- **Geometrically,** Lax entropy condition can be stated as follows: The characteristics originating on either side of the discontinuity curve, when continued in the direction of increasing t, intersect the curve of discontinuity.
- *For every nonlinear IVP there exists a unique weak solution defined* $\forall t \geq 0$ *with only shock as a discontinuity.* The **proof** is fairly difficult and was provided by **Lax (1973).**