Quasi-linear PDEs (I)

Method of characteristics and introduction to shock waves

First-order PDEs and useful notions

Consider a **first-order linear PDE**, in two variables, which can generally be expressed as:

$$A(x,y)\frac{\partial u}{\partial x} + B(x,y)\frac{\partial u}{\partial y} + C_1(x,y)u = C_0(x,y)$$

This equation is called **homogeneous** if $C_0 \equiv 0$.

More generally, functions *A*, *B*, C_1 may depend on *u*; in this case, the first-order PDE of the form:

$$A(x,y,u)\frac{\partial u}{\partial x} + B(x,y,u)\frac{\partial u}{\partial y} = C(x,y,u)$$

is called **quasi-linear** (in two variables).

Remark:

Every linear PDE is also quasi-linear, because we can set:

$$C(x,y,u) = C_0(x,y) - C_1(x,y)u.$$

A prototypical -physically significant- example:

Transport (advection) equation

$$u_t + c u_x = 0$$

$$u_t + c(x,t) u_x = 0$$

Linear & homogeneous PDEs

$$u_t + c(x,t) u_x = h(x,t)$$

Linear & inhomogeneous PDE

$$u_t + c(u) u_x = 0$$

Quasi-linear PDE

Remark:

$$\left.\begin{array}{l} u_t + c(u) u_x = 0 \Rightarrow c'(u) u_t + c(u) c'(u) u_x = 0 \\ c'(u) u_t = c_t, \qquad c(u) c'(u) u_x = c c_x \end{array}\right\} \Rightarrow$$

$$c_t + c c_x = 0$$

Hopf (Riemann / inviscid Burgers) equation

Method of characteristics

Consider the Cauchy problem for the **quasi-linear PDE:**

$$a(x,t,u)\,u_t + b(x,t,u)\,u_x = c(x,t,u), \qquad u(x,0) = f(x)$$

- Introduce a curve **Γ** defined as: $\Gamma : \begin{cases} x = x(r), & x(0) = s \\ t = t(r), & t(0) = 0 \end{cases}$

- Assume that, on **Γ**: $u(x,t) = u(x(r), t(r))$, and differentiate wrt r:

$$\frac{du}{dr} = \frac{\partial u}{\partial t}\frac{dt}{dr} + \frac{\partial u}{\partial x}\frac{dx}{dr}$$

Next, require: $\frac{dt}{dr} = a, \quad \frac{dx}{dr} = b$

- Then, on **Γ**, the quasi-linear PDE is reduced to the ODE: $\frac{du}{dr} = c$

- The **equations:** $\frac{dt}{dr} = a, \quad \frac{dx}{dr} = b, \quad \frac{du}{dr} = c \Rightarrow \frac{dt}{a} = \frac{dx}{b} = \frac{du}{c}$

Lagrange-Charpit equations

are called the ***characteristics*** of the quasi-linear PDE.

Geometrical interpretation

Consider again the Cauchy problem for the **quasi-linear PDE:**

$$a(x,t,u)\,u_t + b(x,t,u)\,u_x = c(x,t,u), \qquad u(x,0) = f(x)$$

- Let $u(x,t)$ a **solution** of the PDE, and its **graph** $z = u(x,t)$, which is a **surface** in the **xtu-space.** The initial data which define a **curve** γ in the xt-plane (e.g., if $u(x,0)=f(x)$ then γ is the x-axis, etc) provides a **space curve Γ** that **lies on the graph**.

- Let $\mathbf{F}=(a, b, c)$ the vector field defined by PDE's coefficients
- The normal vector $\mathbf{N}$ to the surface z=u(x,t) is: $\mathbf{N}=(u_x, u_t, -1)$
- If $u(x,t)$ a **solution** of the PDE then

$$\mathbf{F}\cdot\mathbf{N} = au_t + bu_x - c = 0 \Leftrightarrow \mathbf{F} \perp \mathbf{N} \Leftrightarrow \mathbf{F} \text{ tangent to } z = u(x,t)$$

➨ **The graph of the solution can be constructed by finding the stream lines of $\mathbf{F}$ that pass through the initial curve Γ.**

1. Transport equation with const. velocity

- Simplest linear 1st-order problem: transport (advection) equation

$$u_t + c\,u_x = 0,$$
$$u(x,0) = F(x)$$

$\omega = ck,\ \omega''(k) = 0$

Non-dispersive system

➢ Method of characteristics

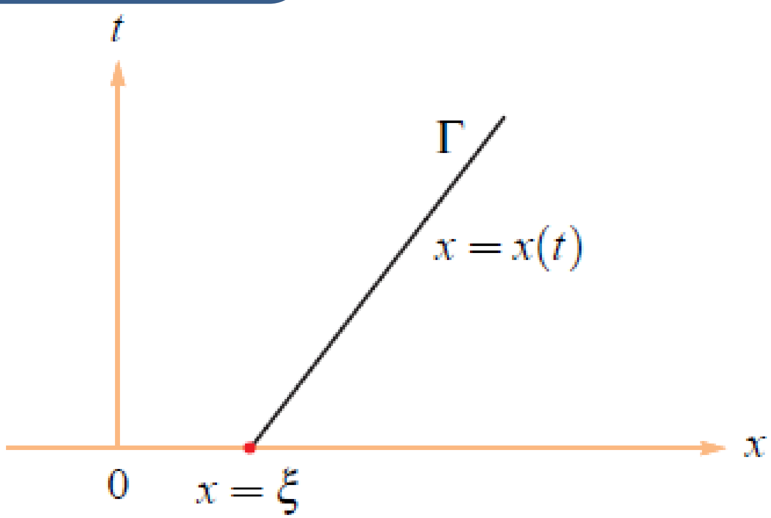
Reduce the problem to an ODE along some curve Γ: $x=x(t)$ such that $du/dt=0$

$$\frac{du(x(t),t)}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial t}\frac{dt}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = 0 \qquad \frac{dx}{dt} = c$$

⇨ $\frac{dx}{dt} = c \;\Rightarrow\; x(t) = c\,t + \xi$

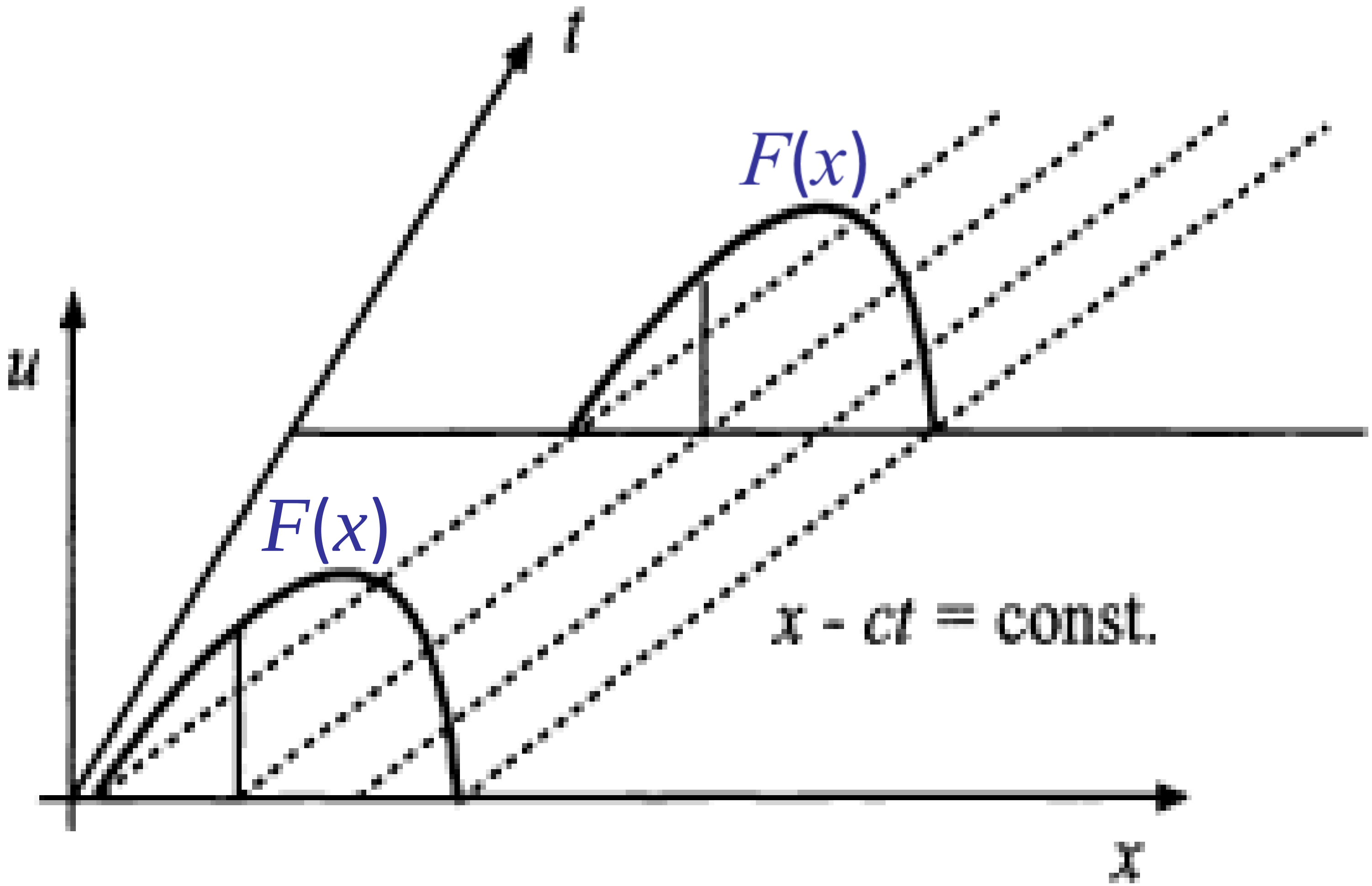
⇨ $\frac{du}{dt} = 0 \;\Rightarrow\; u(x,t) = u(\xi,0) = F(\xi)$

General solution: $u(x,t) = F(x - c\,t)$



Solution of the transport equation

The IC, $F(x)$, is simply translated **without changing shape**



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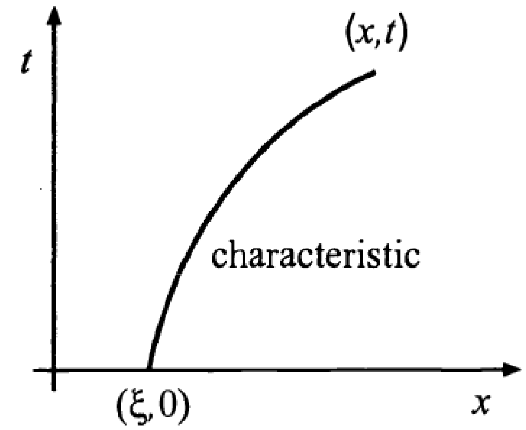
2. Transport equation with non const. velocity

Consider the Cauchy problem:

$$u_t + 2t\,u_x = 0 \quad (c = 2t)$$
$$u(x,0) = f(x) = \exp(-x^2)$$

On Γ: *x*=*x*(*t*) we have *du*/*dt*=0:

$$\frac{du}{dt} = \frac{\partial u}{\partial t}\frac{dt}{dt} + \frac{\partial u}{\partial x}\frac{dx}{dt} = u_t + \frac{dx}{dt}u_x = 0$$



This leads to: $\frac{dx}{dt} = 2t, \quad x(0) = \xi \quad \Rightarrow \quad x(t) = t^2 + \xi$

Also, on Γ:

$$\left.\begin{aligned} & du/dt = 0 \Rightarrow u(x,t) = \text{const.} \\ & t = 0: \quad u(x,t) = u(\xi,0) \\ & \text{IC}: u(x,0) = f(x) \Rightarrow u(\xi,0) = f(\xi) \end{aligned}\right\} \Rightarrow u(x,t) = f(\xi) = \exp[-(x-t^2)^2]$$

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3. A boundary-value problem

Consider the BVP: $\begin{cases} x^2 u_t + u_x + tu = 0, \quad x > 0, \quad t \in \mathrm{R} \\ \text{BC at } x = 0: \quad u(0,t) = f(t) \end{cases}$

On Γ: $t=t(x)$ we have: $\frac{du}{dx} = \frac{\partial u}{\partial t}\frac{dt}{dx} + \frac{\partial u}{\partial x}\frac{dx}{dx} = \frac{dt}{dx}u_t + u_x$

and we choose: $\frac{dt}{dx} = x^2 \Rightarrow t = \frac{1}{3}x^3 + \tau$ → our ξ in this case

On Γ: $t=t(x)$ we have: $\frac{du}{dx} = -tu \Rightarrow \frac{du}{dx} = -\left(\frac{1}{3}x^3 + \tau\right)u$ and thus:

$$\frac{du}{u} = -\left(\frac{1}{3}x^3 + \tau\right)dx \Rightarrow \int_{f(\tau)}^{u} \frac{du}{u} = \int_0^x -\left(\frac{1}{3}x^3 + \tau\right)dx \Rightarrow$$

$$u = f(\tau)\exp\left(-\frac{1}{12}x^4 - \tau x\right) = f\left(t - \frac{1}{3}x^3\right)\exp\left(-\frac{1}{4}x^4 - xt\right)$$

3. A nonlinear problem: Hopf equation

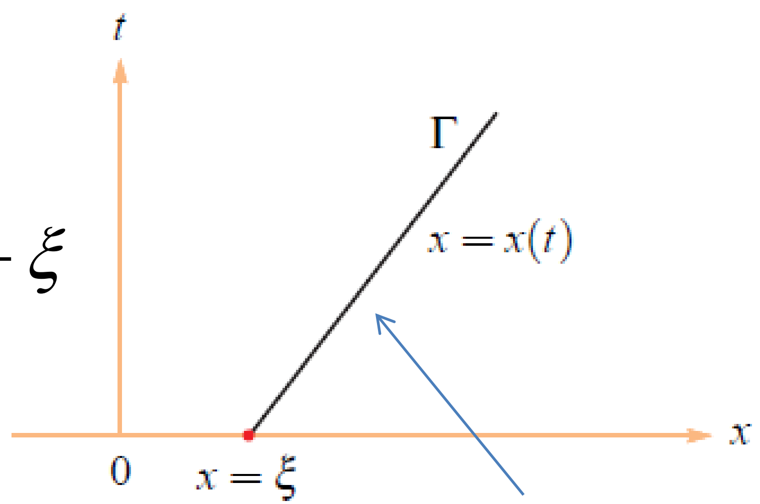
$$u_t + u u_x = 0, \quad u(x,0) = f(x), \quad x \in \mathrm{R}, \ t > 0$$

On Γ: $x = x(t)$ we have: $du/dt = 0$: $\frac{du}{dt} = \frac{\partial u}{\partial t}\frac{dt}{dt} + \frac{\partial u}{\partial x}\frac{dx}{dt} = u_t + \frac{dx}{dt} u_x = 0$

As before:

$$du/dt = 0 \Rightarrow u(x,t) = \text{const.}$$

$$dx/dt = u, \quad x(0) = \xi \quad \Rightarrow \quad x(t) = ut + \xi$$



Characteristics are again straight lines

Since:

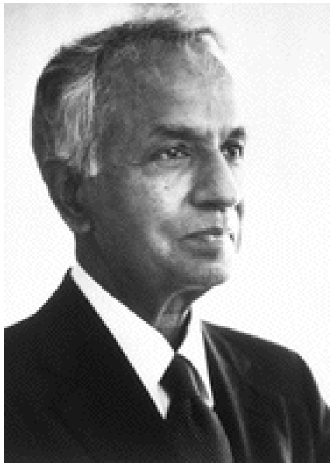
$$u(x,0) = f(x) \Rightarrow u(\xi,0) = f(\xi)$$

$$\left.\begin{aligned} u(x,t) = u(\xi,0) = f(\xi) \\ \xi = x - ut \end{aligned}\right\} \Rightarrow \boxed{u = f(x - ut)}$$

Implicit solution of the Hopf equation

An explicit solution of the Hopf equation

S. Chandrasekhar found (1943) an **explicit solution** of the following IVP:

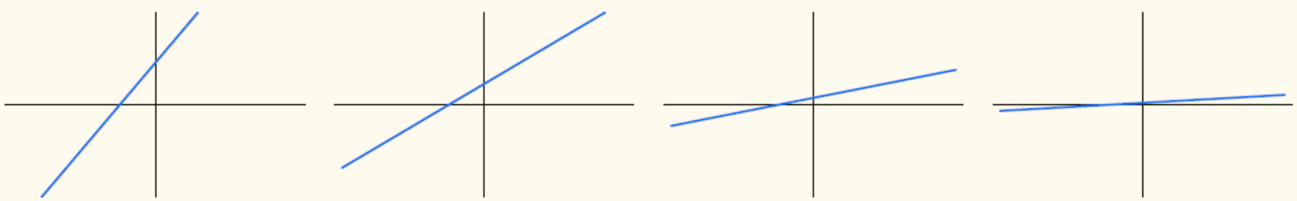


$$u_t + u\,u_x = 0, \qquad u(x,0) = f(x) = ax + b$$

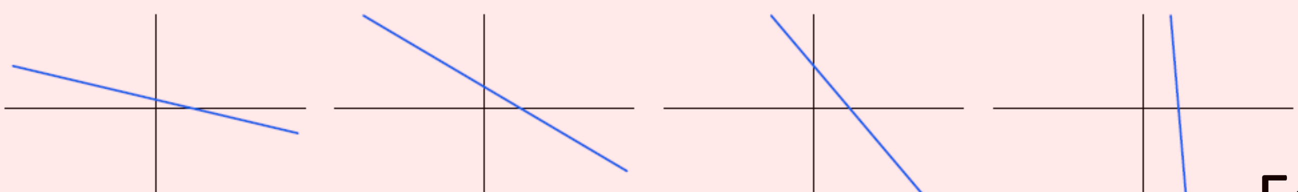
We have found: $u = f(x - ut)$

and thus: $u = a(x - ut) + b \Rightarrow$ $$u = \frac{ax + b}{1 + at}$$

$a > 0$: the solution **flattens** as $t \to \infty$



Rarefaction wave

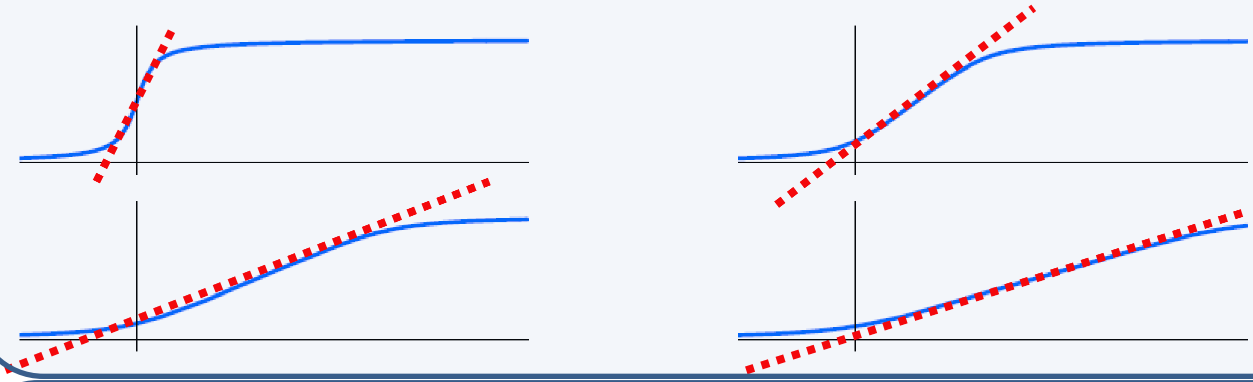


Shock wave

For $t = t_B = -1/a$ the solution **blows up**

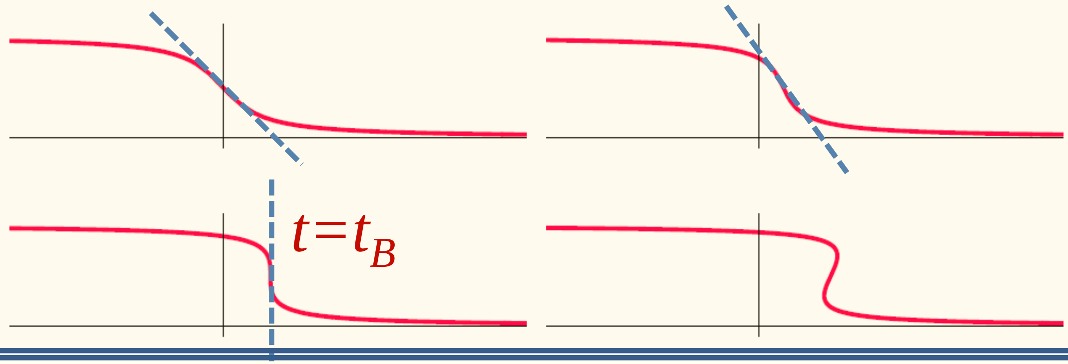
$a < 0$: the solution **steepens** as $t \to \infty$

What can we learn from the explicit solution?



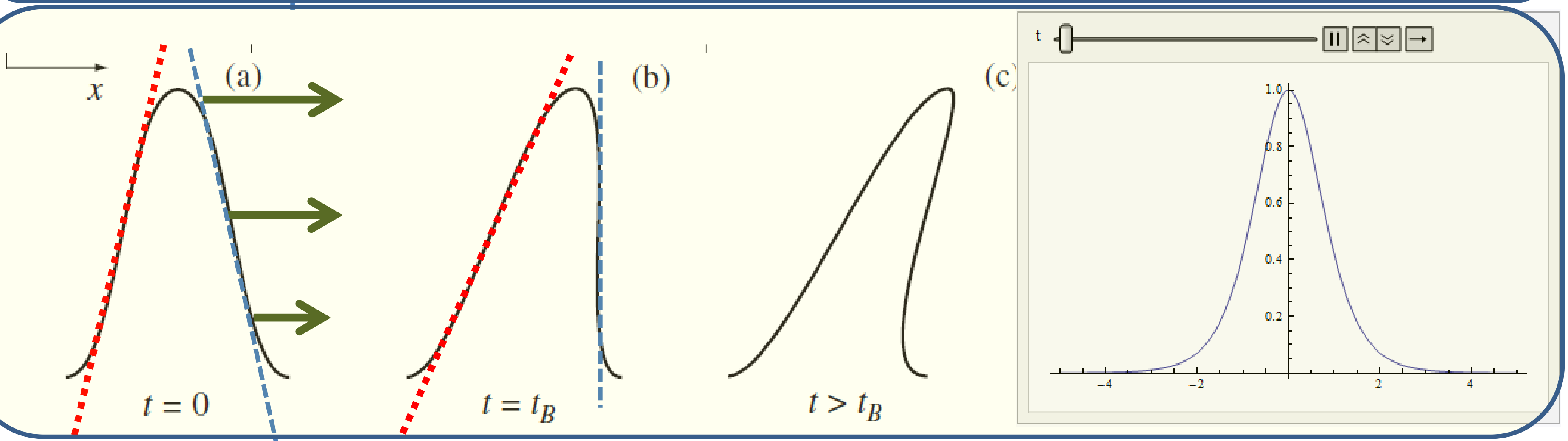
Positive slope

Rarefaction wave



Negative slope

Shock wave



Breaking time

Consider the general problem: $u_t + u\,u_x = 0,\ u(x,0) = f(x)$

We have found: $\left.\begin{aligned} x &= ut + \xi \\ u &= f(\xi) \end{aligned}\right\} \Rightarrow \underline{x = \xi + f(\xi)t \quad (1)}$

We wish to determine the **breaking time t_B** occurring when the profile of the solution develops an infinite slope:

$$\left.\begin{aligned} & u_x = \frac{\partial u}{\partial x} = \frac{du}{d\xi}\frac{\partial \xi}{\partial x} = f'(\xi)\xi_x \\ & (1),\ \partial_x :\ 1 = \xi_x + f'(\xi)t\,\xi_x \Rightarrow \xi_x = \frac{1}{1 + f'(\xi)t} \end{aligned}\right\} \Rightarrow \boxed{u_x = \frac{f'(\xi)}{1 + f'(\xi)t}}$$

If $f'(\xi) > 0$ $\forall x$ then the solution is **finite** $\forall t$ $\rightarrow$ rarefaction wave

If $f'(\xi) < 0$ the solution breaks up at the **earliest critical time**:

$$\boxed{t_B = \min_{\xi>0}\{-1/f'(\xi) \mid f'(\xi) < 0\}}$$

What happens before the breaking time t_B

Using $x = \xi + f(\xi)t \quad (1)$ we found: $u_x = \dfrac{f'(\xi)}{1+f'(\xi)t} \quad (2)$

Similarly:

$$\left.\begin{aligned} &u_t = \frac{\partial u}{\partial t} = \frac{du}{d\xi}\frac{\partial \xi}{\partial t} = f'(\xi)\xi_t \\ &(1),\ \partial_t:\ 0 = \xi_t + f'(\xi)t\,\xi_t + f(\xi) \Rightarrow \xi_t = -\frac{f(\xi)}{1+f'(\xi)t} \end{aligned}\right\} \Rightarrow$$

$$u_t = -\frac{f(\xi)f'(\xi)}{1+f'(\xi)t} \quad (3)$$

Hence, from (2)-(3), and for $t < t_B$, the solution remains **single-valued** and satisfies the Hopf equation:

$$u_t + uu_x = -\frac{f(\xi)f'(\xi)}{1+f'(\xi)t} + f(\xi)\frac{f'(\xi)}{1+f'(\xi)t} = 0 \quad ✓$$

What happens for times $t \geq t_B$:
Characteristics intersect

The slope of a characteristic passing (x^0, t^0) is: $\dfrac{dx}{dt} = c(f(x^0))$

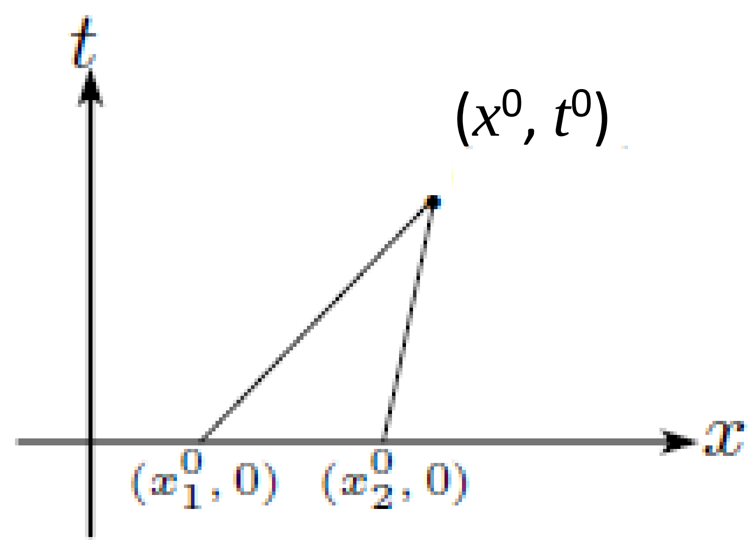
Here, $c = c(u)$ is a **strictly increasing** function:

$$u_t + c(u)u_x = 0,\ \ c(u) = u \Rightarrow c'(u) = 1 > 0$$

Let f be a **strictly decreasing** function; then, $c(f)$ is also **strictly decreasing:**

$$x_1^0 < x_2^0 \to c(f(x_1^0)) > c(f(x_2^0))$$

and, hence, characteristics passing through $(x_1^0, 0),\ (x_2^0, 0)$ **intersect!**

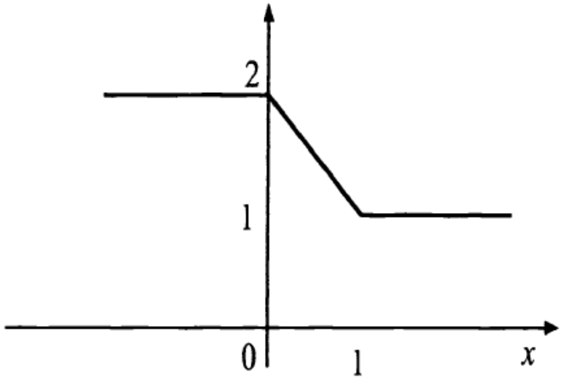


At the intersection point, the solution $u(x,t)$ becomes **multi-valued** because it takes both values $f(x_1^0),\ f(x_2^0)$

An example

Consider the IVP: $u_t + u u_x = 0, \quad u(x,0) = f(x), \quad x \in \mathrm{R}$

$$f(x) = \begin{cases} 2, & x < 0 \\ 2 - x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$



We have found: $\left.\begin{array}{l} x = ut + \xi \\ u = f(\xi) \end{array}\right\} \Rightarrow x = \xi + f(\xi)t$

Characteristics:

$$f(\xi) = \begin{cases} 2, & \xi < 0 \\ 2 - \xi, & 0 \le \xi \le 1 \\ 1, & \xi > 1 \end{cases}$$

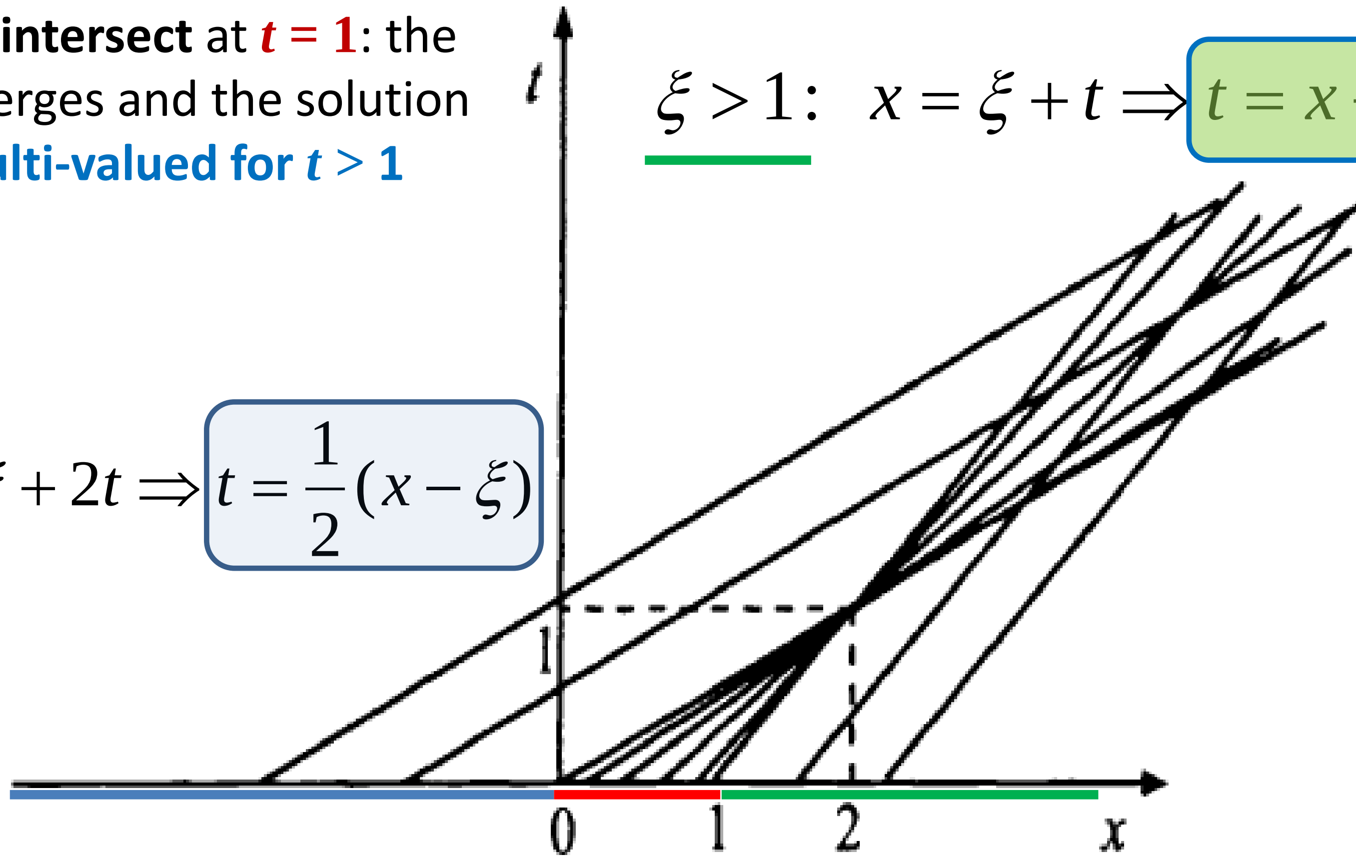
$$x = \begin{cases} \xi + 2t, & \xi < 0 \\ \xi + (2 - \xi)t, & 0 \le \xi \le 1 \\ \xi + t, & \xi > 1 \end{cases}$$

An example (cont.) - characteristics

Characteristics intersect at ***t*** **= 1**: the **shock wave** emerges and the solution becomes **multi-valued for *t* > 1**

$$\xi > 1: \quad x = \xi + t \Rightarrow t = x - \xi$$

$$\xi < 0: \quad x = \xi + 2t \Rightarrow t = \frac{1}{2}(x - \xi)$$



$$0 \le \xi \le 1: \quad x = \xi + (2 - \xi)t \Rightarrow t = \frac{x - \xi}{2 - \xi}$$

An example (cont.) – breaking time

How to determine the **breaking time t_B**:

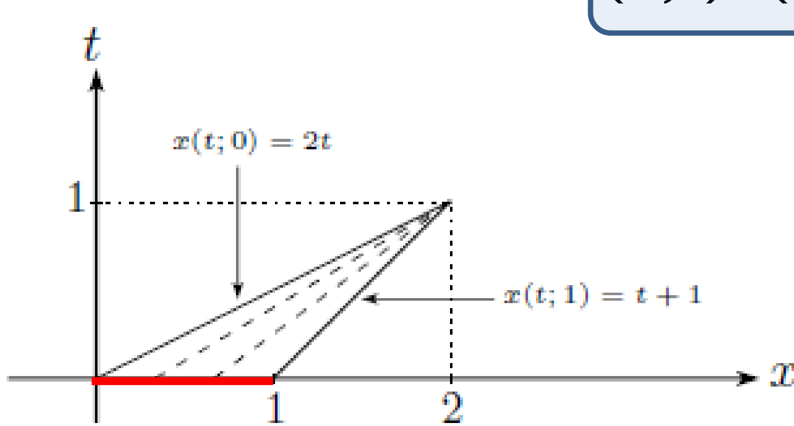
Recall that: $t_B = \min_\xi\{-1/f'(\xi) | f'(\xi) < 0\}$

Here, we have: $f(\xi) = \begin{cases} 2, & \xi < 0 \\ 2-\xi, & 0 \le \xi \le 1 \\ 1, & \xi > 1 \end{cases}$ and hence $t_B = 1$

Alternatively, recall that we found: $x = \xi + (2-\xi)t, \quad 0 \le \xi \le 1$

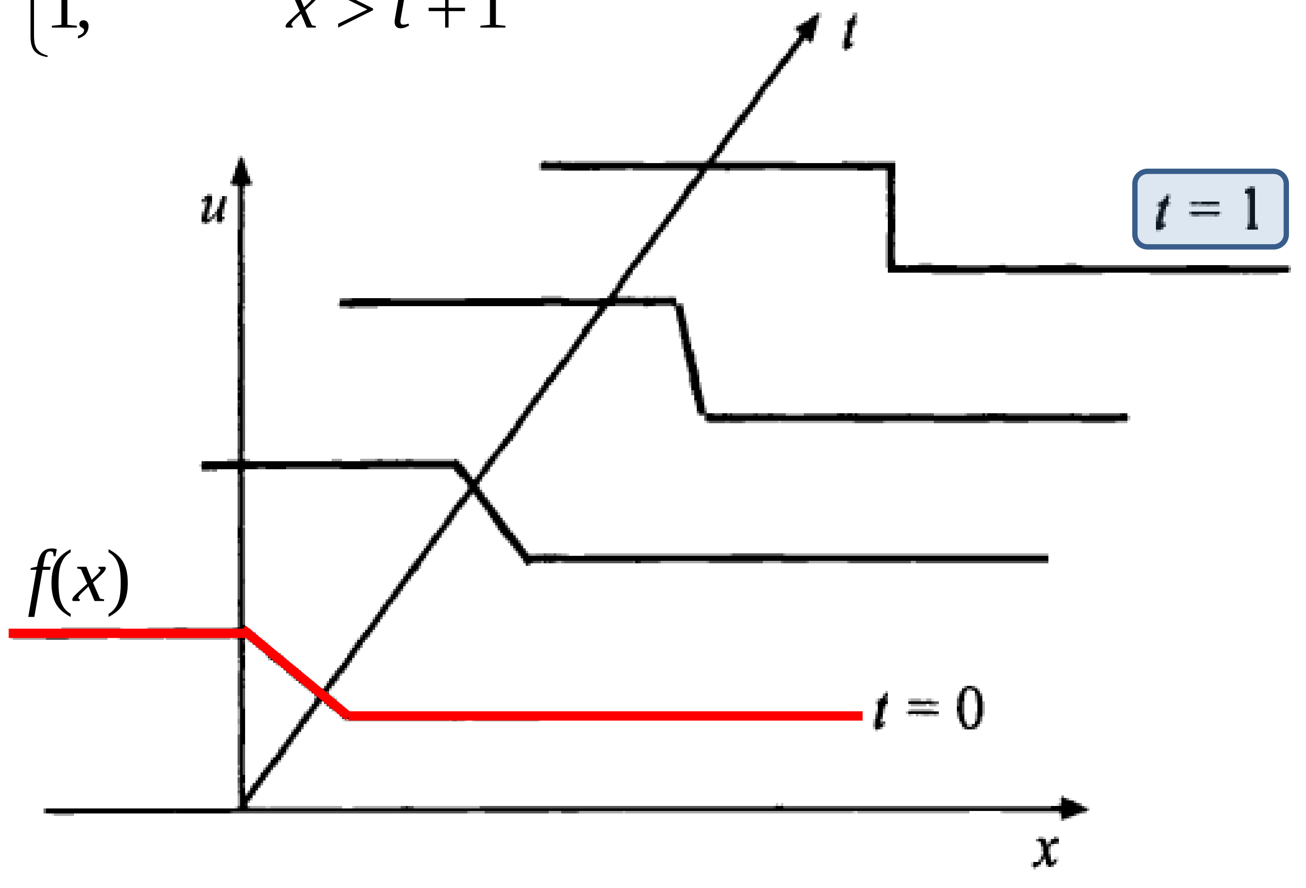
and so $\left.\begin{aligned} \xi = 0: &\quad x = 2t \\ \xi = 1: &\quad x = 1+t \end{aligned}\right\} \Rightarrow$ characteristics intersect at $(x,t) = (2,1)$

All other characteristics in this interval pass $(x,t) = (2,1)$



An example (cont.) – shock formation

Solution: $u(x,t) = \begin{cases} 2, & x < 2t \\ \dfrac{2-x}{1-t}, & 2t \le x \le t+1 \\ 1, & x > t+1 \end{cases}$



Exercises – linear problems

1) Use the method of characteristics to solve the IVPs:

1.1) $$u_t + u_x + u = 0, \quad u(x,0) = f(x) = \cos x, \quad x \in \mathrm{R}$$

1.2) $$u_t + 2xt\,u_x = u, \quad u(x,0) = f(x) = x, \quad x \in \mathrm{R}$$

2) Use the method of characteristics to show that the solution of the IVP:

$$u_t + cu_x = h(x,t), \quad u(x,0) = f(x), \quad x \in \mathrm{R}$$

is: $$u(x,t) = f(x-ct) + \int_0^t h(x - c(t-t'), t')\,dt'$$

In all cases confirm, by direct substitution, that the solution you found satisfies the corresponding PDE.

Exercises – nonlinear problems

3) Use the method of characteristics to solve the Hopf equation:

$$u_t + uu_x = 0, \quad u(x,0) = f(x)$$

in the following cases:

3.1) $$f(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

3.2) $$f(x) = \begin{cases} 0, & x < 0 \\ -x, & 0 \le x \le 1 \\ -1, & x > 1 \end{cases}$$

In both cases:

a) Draw the characteristics in the xt-plane

b) Write down the solution and draw some characteristic snapshots of the solution at different time instants

c) Determine the breaking time (when relevant)