4 Taylor

$f(x) = x^4 + x^3 + x^2 + x + 1 \qquad f(0) = 1$

$f'(x) = 4x^3 + 3x^2 + 2x + 1 \qquad f'(0) = 1$

$f''(x) = 12x^2 + 6x + 2 \qquad f''(0) = 2$

$f'''(x) = 24x + 6 \qquad f'''(0) = 6$

$f^{(4)}(x) = 24 \qquad f^{(4)}(0) = 24, \quad f^{(k)}(0) = 0 \quad k \geq 5$

$f^{(5)}(x) = 0, \quad f^{(k)}(x) = 0$ για κάθε $k \geq 5$

$$T_n(f,0;x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(n)}(0)}{n!} x^k}$$

$$= \begin{cases} n \leq 4 & 1 + x + \dots + x^n \\ n > 4 & 1 + x + x^2 + x^3 + x^4 \end{cases}$$

⑤ Προσέγγιση του $\sin 1$. $f(x)=\sin x$ $\sin 1 = f(1)$ $x=1$

Γνωρίζουμε ότι

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \Big|\ f(x)=\sin x$$

και

$$T_{2n+1}(f,0;x) = \sum_{k=0}^{n} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$\sin 1 \simeq T_{2n+1}(f,0;1)$ για n αρκετά μεγάλο

$$R_{2n+1}(f,0;x) = \sin 1 - T_{2n+1}(f,0;1)$$

Θ. Taylor

$$\left|R_{2n+1}(f,0;x)\right| \le \frac{\left|f^{(2n+2)}(\xi)\right|}{(2n+2)!}|x|^{2n+2} \le \frac{|x|^{2n+2}}{(2n+2)!}$$

για κάποιο $\xi \in [0,x]$ ή $[x,0]$

$$f^{(0)}(x) = \sin x \qquad f'(x) = \cos x$$

$$f''(x) = -\sin x \qquad f'''(x) = -\cos x$$

$$f^{(2k)}(x) = (-1)^k \sin x \qquad \left|f^{(2n+2)}(x)\right| \le |\sin x| \le 1$$

$\Rightarrow \quad x = 1$

$$|R_{2n+1}(f, 0; 1)| \le \frac{1}{(2n+2)!} \quad \text{αν} \le 10^{-7} \quad ✓$$

$x = 1$

Αρκεί να βρούμε το μικρότερο $n \ge 0$ ώστε

$$\frac{1}{(2n+2)!} \le 10^{-7}$$

$$\Leftrightarrow \quad (2n+2)! \ge 10^{7}$$

$10! = 3.628.800 < 10^7$

$10 = 2\cdot 4 + 2$

$11!$

$11! = 39.916.800 > 10^7.$

$\Rightarrow 12! > 10^7$

$12 = 2\cdot \underset{\substack{\shortparallel\\ 4}}{5} + 2$

Θςdoute $T_{2\cdot 5+2}(f,0;1)$

$= T_{12}(f,0;1) = T_{11}(f;0;1)$

$$T_{11}(f, 0; 8) =$$

$$= \sum_{k=0}^{5} (-1)^k \frac{1^{2k+1}}{(2k+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \frac{1}{11!}$$

$$11 = 2 \cdot 5 + 1$$

$$2n+1, \quad n = 5$$

Όμοια, αν αρκεί σφάλμα $\leq 10^{-3}$

αρκεί $n=3$

$$T_7(f,0,1) = 1 - \frac{1}{6} + \frac{1}{120} - \frac{1}{5040}$$

$$7 = 2\cdot 3+1$$

3 (ii)

$$g(x) = \frac{1}{1+x^2}$$

Πολυώνυμο Taylor;

(i) $$f(x) = \frac{1}{1+x}$$

$$T_n(f, 0; x) = 1 - x + x^2 - x^3 \cdots + (-1)^n x^n$$

Αν $$f(x) = \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}$$

$$g(x) = \frac{1}{1+x^2} = \frac{1}{1+(x^2)} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \rightarrow \frac{g^{(2k)}(0)}{(2k)!} = (-1)^k$$

Διων. Θεώρημα

$$(x+y)^n = \sum_{k=0}^{n} \underbrace{\binom{n}{k}}_{\frac{n!}{k!(n-k)!}} x^{n-k} y^k$$

$n=2$
$k=1$ | $2! = 2$

όμοια αν

f, g n φορές παραγωγίσιμες

$$(f\cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$$

$$\boxed{\left(f(x)g(x)\right)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)\, g^{(k)}(x)}$$

$$f(x) = \frac{1}{1+x^2} \qquad g(x) = 1+x^2$$

τότε $f(x)\cdot g(x) = \dfrac{1}{1+x^2}\,(1+x^2) = 1$ σταθερά

$$\hookrightarrow \left(f(x)g(x)\right)^{(n)} = (1)^{(n)} = 0$$

$(\text{αν } n \geq 1)$

$$= \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)\, g^{(k)}(x)$$

Άρα

$$0 = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x)\, g^{(k)}(x)$$

Επίσης για $\kappa \geq 3$

$$g^{(\kappa)}(x) = 0$$

(g πολυώνυμο βαθμού 2)

$$= \underbrace{\binom{n}{0}}_{\frac{n!}{0!\,n!} = 1} f^{(n)}(x)\, \underbrace{g(x)}_{= 1+x^2} + \underbrace{\binom{n}{1}}_{\frac{n!}{1!\,(n-1)!} = n} f^{(n-1)}(x)\, \underbrace{g'(x)}_{= 2x} + \underbrace{\binom{n}{2}}_{\frac{n!}{2!\,(n-2)!}} f^{(n-2)}(x)\, \underbrace{g''(x)}_{= 2}$$

$$= f^{(n)}(x) \cdot (1+x^2) + n\, f^{(n-1)}(x)\, 2x + \underbrace{\frac{n!}{\cancel{2}\,(n-2)!}}_{n(n-1)} f^{(n-2)}(x)\, \cancel{2}$$

Για $x=0$

$$\boxed{f^{(n)}(0) = -n(n-1)\, f^{(n-2)}(0)}$$

$$f^{(n+2)}(0) = -(n+2)(n+1)\, f^{(n)}(0)$$

$$f'(0) = 0 \rightsquigarrow f^{(3)}(0) = 0 \rightsquigarrow f^{(5)}(0) = 0$$

Γενικά $f^{(2k+1)}(0) = 0$ για κάθε $k \geq 0$

$$f(0) = 1 \rightsquigarrow f''(0) = -2\cdot 1\cdot f(0) = -2\cdot 1 = -2!$$

$(n=0)$

$$n=2 \quad f^{(4)}(0) = -4\cdot 3\cdot f''(0) = -4\cdot 3\cdot(-2!) = (-1)^2\, 4!$$

Γενικά

$$f^{(2k)}(0) = (-1)^k (2k)!$$

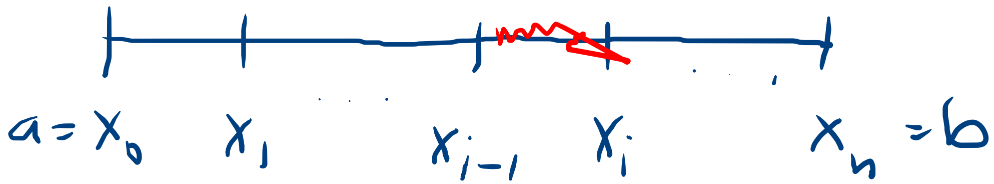
$$\Rightarrow \begin{cases} \dfrac{f^{(2k)}(0)}{(2k)!} = (-1)^k \\[2ex] \dfrac{f^{(2k+1)}(0)}{(2k+1)!} = 0 \end{cases}$$

$$T_{2n}(f,0;x) \overset{?}{=} \sum_{k=0}^{n} \frac{(-1)^k (2n)!}{(2n)!} x^{2n}$$

$$\|$$

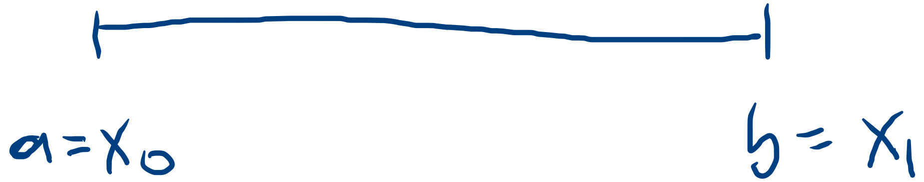
$$T_{2n+1}(f,0,x) = \sum_{k=0}^{n} (-1)^k x^{2k}.$$

$$\frac{f^{(2n+1)}(0) = 0}{(2n+1)!}$$

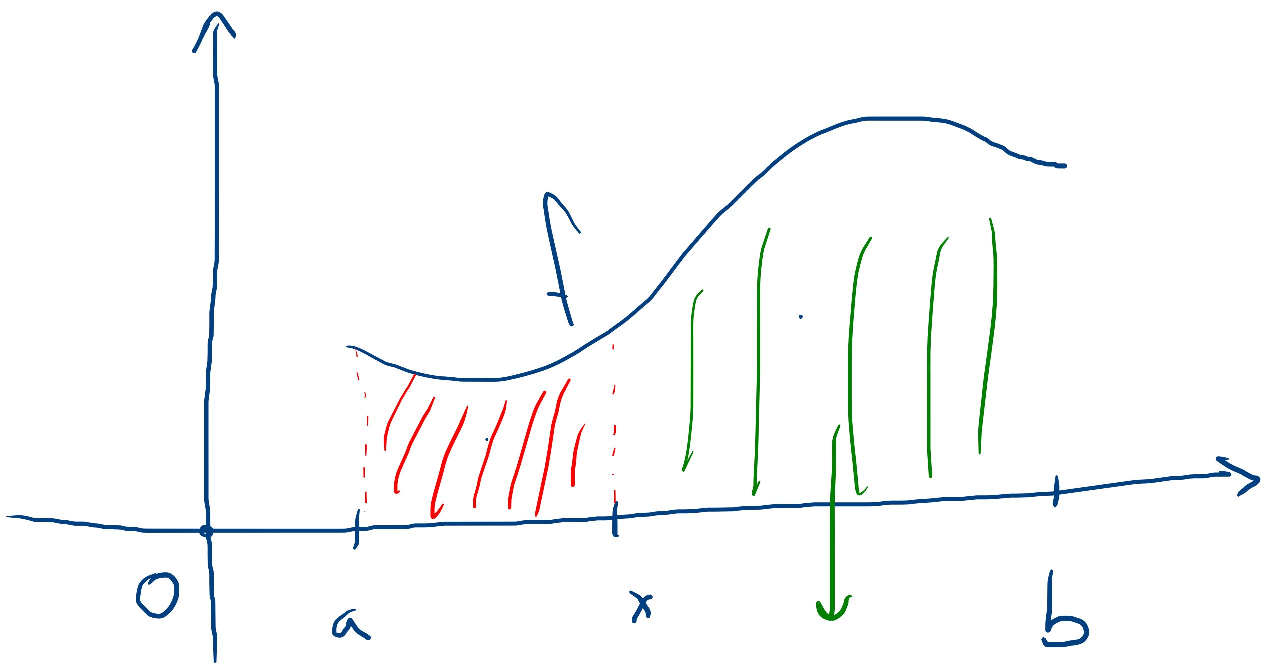


$$m_i = \inf\{ f(x) \,,\, x \in [x_{i-1}, x_i] \}$$

$$1 \le i \le n$$



$$m = m_1 = \inf\{ f(x) \mid x \in [x_0, x_1] \}$$
$$= \inf\{ f(x) \mid x \in [a, b] \}$$



$$\int_a^x f(t)\,dt = \int_x^b f(t)\,dt$$

$h(x) =$

$= \int_a^x - \int_x^b$

h συνεχής

αν f ολοκλ

$h(a) = -\int_a^b$

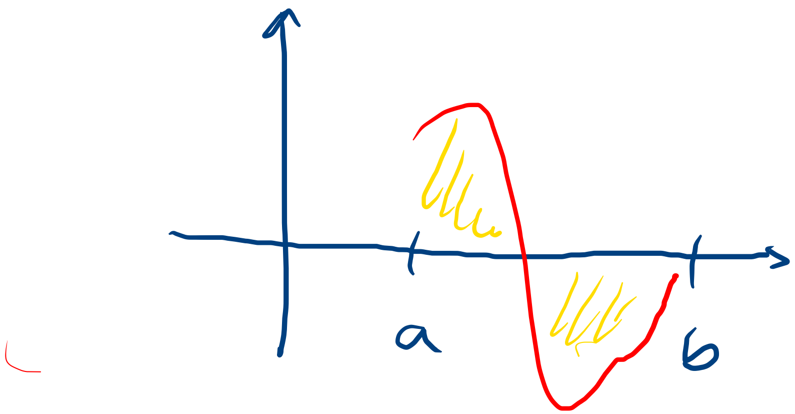
$h(b) = \int_a^b = -h(a)$

Bolzano

αν $h(a) \neq 0$

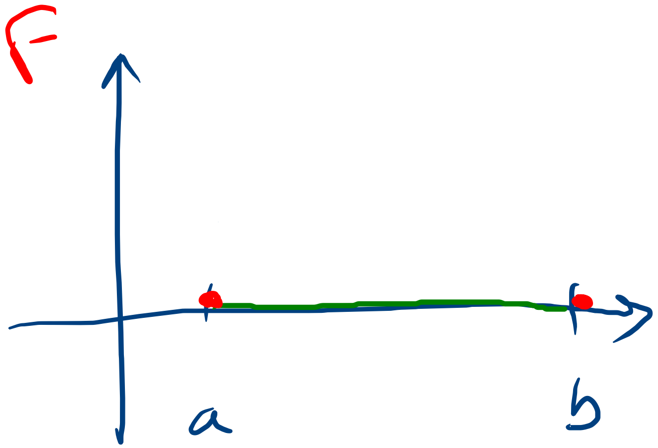
Ερώτηση: αν f (συνεχής) ν.δ.ο υπάρχει $x \in [a,b]$: $\int_a^x f(t)\,dt = \int_x^b f(t)\,dt$

με $\int_a^b \neq 0$, $f : [a,b] \to \mathbb{R}$



$$\int_a^b f(x)\,dx = 0$$

$$\int f(x)\,dx = 0$$



αν $f(x) \geq 0$

και $\int_a^b f(x)\,dx = 0$

$\Rightarrow f(x) = 0 \quad \forall x$

u

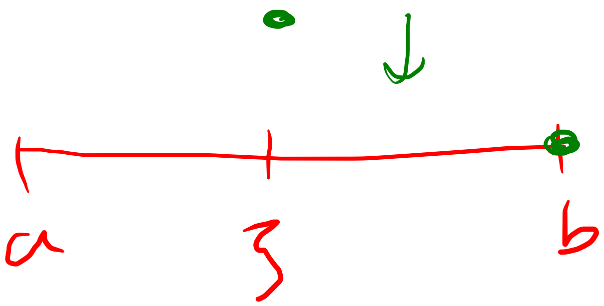
$$F(x) = \int_a^x f(t)\,dt \quad F'(x) = f(x) \geq 0$$

$$F(b) = F(a) = 0$$

$\Rightarrow \int f(x)\,dx = 0 \quad f(x) = 0$

$\Rightarrow F(x) = 0 \quad \forall x \quad F'(x) = 0$

Αν για $\xi \in [a,b]$ $F(\xi) > 0$



$$\Rightarrow \frac{F(b) - F(\xi)}{b - \xi} < 0$$

$\parallel$ ΘΜΤ $\exists\ x_0 \in [\xi, b]$

$F'(x_0) < 0$ άτοπο

$\parallel$

$0 \leq f(x_0)$