9.5 Comments on $\frac{d}{dx}\left(f\frac{d\Phi}{dx}\right)+(M^2h+p)\Phi=0$

Consider the differential eq.

$$\frac{d}{dx}\left(f\frac{d\Phi}{dx}\right)+(M^2h+p)\Phi=0\,, \tag{9.52}$$

where f, h and p are functions of x and M^2 is constant. Define

$$F=fh\,,\qquad H=\frac{f}{h}\,. \tag{9.53}$$

If we change variables as

$$dx=H^{1/2}dz\,,\qquad \Phi=F^{-1/4}\Psi\,, \tag{9.54}$$

we obtain from (9.52) the Schrödinger equation

$$-\frac{d^2\Psi}{dz^2}+V\Psi=M^2\Psi\,, \tag{9.55}$$

with potential

$$V=-\frac{p}{h}+F^{-1/4}\frac{d^2F^{1/4}}{dz^2}=-\frac{p}{h}+\frac{H^{1/2}}{F^{1/4}}\frac{d}{dx}\left(H^{1/2}\frac{d}{dx}F^{1/4}\right)\,. \tag{9.56}$$

The inner product is defined as

$$\langle\Psi_1|\Psi_2\rangle=\int dz\Psi_1^*\Psi_2=\int dxh\Phi_1^*\Phi_2=\langle\Phi_1|\Phi_2\rangle\,. \tag{9.57}$$

We also note that if $\Phi(x)$ solves (9.52) so does

$$\tilde{\Phi}(x)=\Phi(x)\int^x\frac{dx'}{f(x')\Phi^2(x')}\,, \tag{9.58}$$

or in terms of the Schrödinger eq. (9.55) if $\Psi(z)$ solves it so does

$$\tilde{\Psi}(z)=\Psi(z)\int^z\frac{dz'}{\Psi^2(z')}\,. \tag{9.59}$$

For two solutions $\Phi_{n,m}$ of (9.52) corresponding to eigenvalues $M^2_{n,m}$ we can easily prove that

$$\begin{aligned}(M_n^2 - M_m^2)\int_a^b dx\, h\Phi_n\Phi_m &= \int_a^b dx\left[\Phi_n(f\Phi_m')' - \Phi_m(f\Phi_n')'\right] \\ &= \left[f(\Phi_n\Phi_m' - \Phi_m\Phi_n')\right]\Big|_a^b . \end{aligned} \tag{9.60}$$

With appropriate boundary conditions the right hand side is zero and therefore

$$\int_a^b dx\, h\Phi_n(x)\Phi_m(x) = \mathcal{N}_n\delta_{n,m} . \tag{9.61}$$

The normalization constant $\mathcal{N}_n$ can be found if we take the derivative w.r.t. M_n^2 of both sides in (9.60) and then set $M_n = M_m$. We note also the completeness relation

$$\sum_n \frac{\Phi_n(x)\Phi_n(x')}{\mathcal{N}_n} = \frac{\delta(x-x')}{h(x)} . \tag{9.62}$$

In addition it can be easily shown that for the two independent solutions of (9.52) (corresponding to the same eigenvalue M^2), the Wroskian

$$W(\Phi_1,\Phi_2) \equiv \Phi_1\Phi_2' - \Phi_1'\Phi_2 = \frac{\text{const.}}{f(x)} , \tag{9.63}$$

that is, it is either zero or non-vanishing anywhere. Moreover, being essentially the inverse of $f(x)$, it depends on properties of the differential equation and is easily determined.

• We also note the relation of the potential (9.56) to the theory of supersymmetric quantum mechanics that we mention below. For $p = 0$ the potential (9.56) can be written in the form (9.106) with the superpotential determined from

$$W(z) = -\frac{1}{4}\frac{F'}{F} \qquad \Longleftrightarrow \qquad F(z) = e^{-4\int^z dz' W(z')} . \tag{9.64}$$

The zero eigenvalue wavefuction is then $\Psi_0 = F^{1/4}$ and is normalizable if $\int dz\, F^{1/4} = \int dx\, h^{5/4} f^{1/4} < \infty$. The partner potential is then given by

$$V_2 = F^{1/4}\frac{d^2F^{-1/4}}{dz^2} = -\frac{p}{h} + H^{1/2}F^{1/4}\frac{d}{dx}\left(H^{1/2}\frac{d}{dx}F^{-1/4}\right) , \tag{9.65}$$