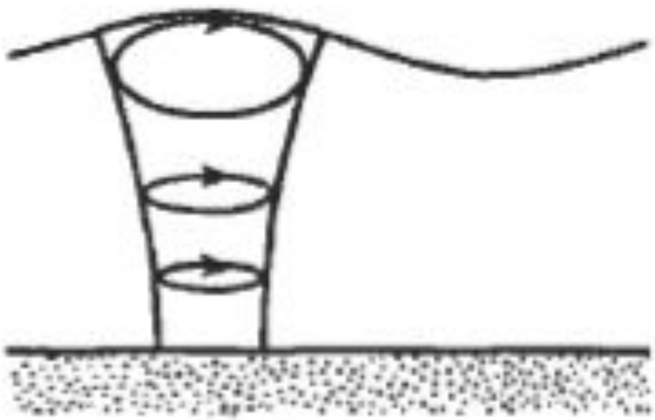
Κύματα παρουσία βαρύτητας

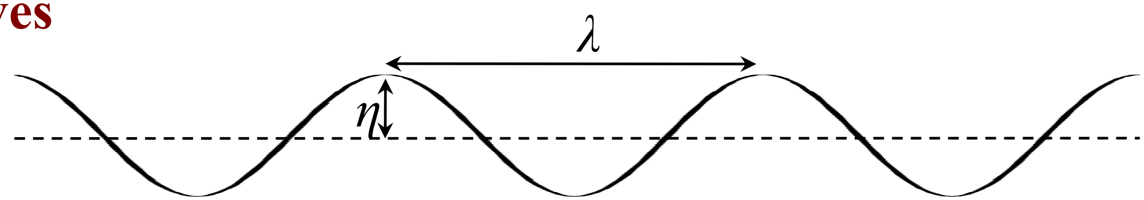


8. Gravity waves in the ocean

Sarantis Sofianos
Dept. of Physics, University of Athens

- **Waves in the ocean**
- **Surface gravity waves**
- **Short and long limit in gravity waves**
- **Wave characteristics**
- **Internal waves**

Characteristic properties of waves



• **Wavelength** (λ): The distance between two consecutive peaks and **Wavenumber** (K)

$$K = \frac{2\pi}{\lambda}$$

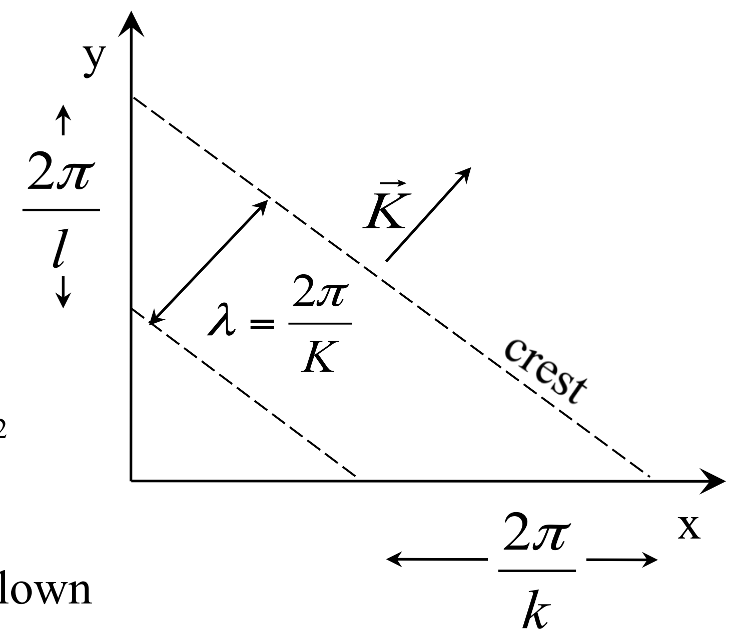
• **Period** (T): The time it takes for two consecutive peaks to pass from a point in space and **Frequency** (ω)

$$\omega = \frac{2\pi}{T}$$

•**Phase Speed** (C): The speed of a monochromatic wave

$$C \equiv \frac{\omega}{K} = \frac{\lambda}{T}$$

• **Group Speed** (C_g): The speed of a wave packet $C_g \equiv \frac{\partial \omega}{\partial K}$



• **Wave Energy** (E_w):

$$E_w = \rho_w g \langle \eta^2 \rangle$$

• **Significant Wave Height** ($H_{1/3}$): The average height (double amplitude) of the 1/3 largest waves

$$H_{1/3} = 4\langle \eta^2 \rangle^{1/2}$$

• <u>**Fetch:**</u> The length of water over which a given wind has blown

Working equations

$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \nabla\right)\vec{u} = -\frac{1}{\rho}\nabla P - 2\vec{\Omega} \times \vec{u} - g + A_H \nabla_H^2 \vec{u} + A_V \frac{\partial^2 \vec{u}}{\partial z^2}$$ Momentum Conservation

$$\frac{1}{\rho}\frac{d\rho}{dt} + \left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] = 0$$ Mass Conservation

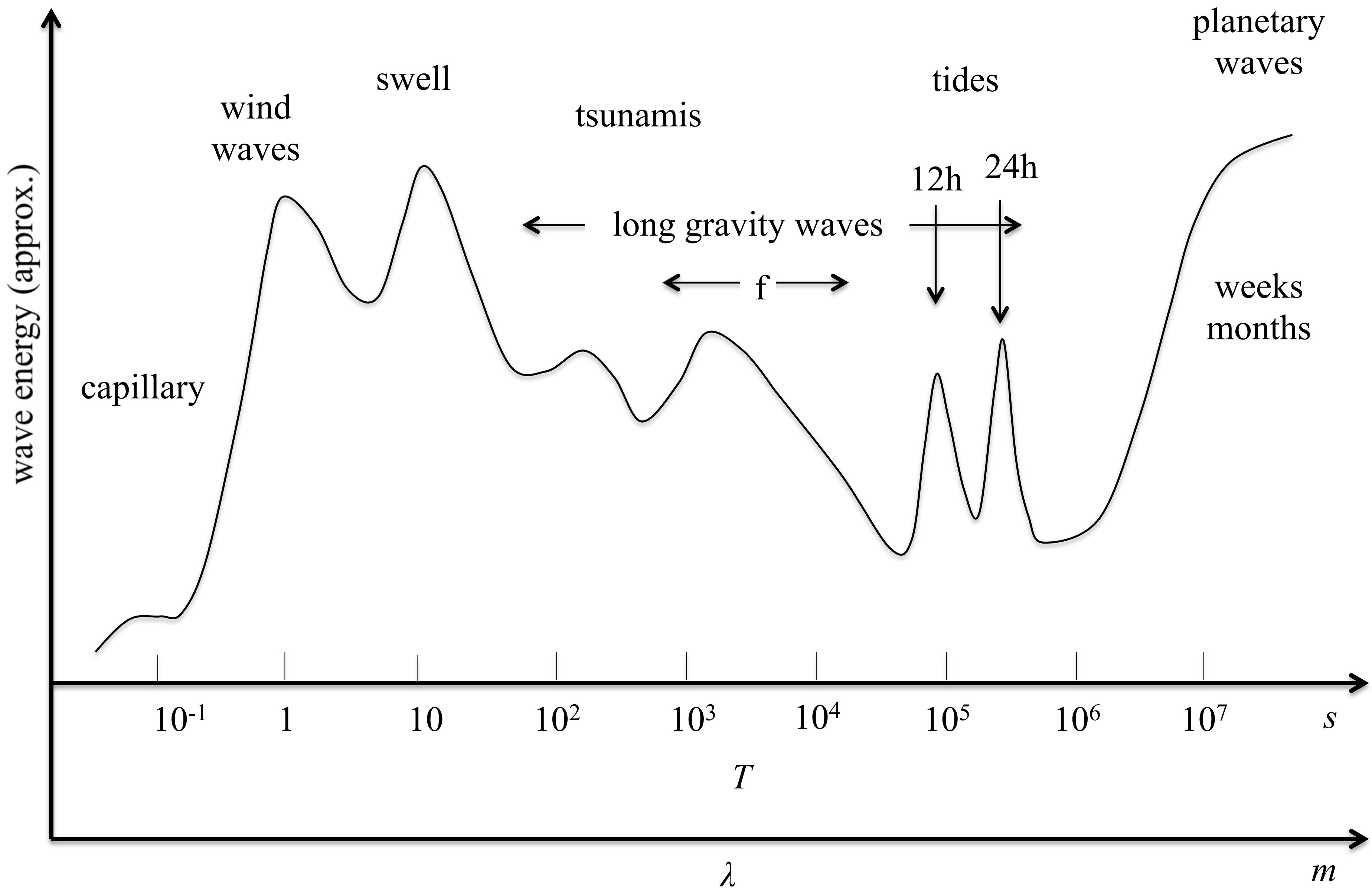
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + fv + A_H\frac{\partial^2 u}{\partial x^2} + A_H\frac{\partial^2 u}{\partial y^2} + A_V\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y} - fu + A_H\frac{\partial^2 v}{\partial x^2} + A_H\frac{\partial^2 v}{\partial y^2} + A_V\frac{\partial^2 v}{\partial z^2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial z} - g + A_H\frac{\partial^2 w}{\partial x^2} + A_H\frac{\partial^2 w}{\partial y^2} + A_V\frac{\partial^2 w}{\partial z^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$ (Incompressible flow)

The oceanic wave spectrum



Linear gravity waves in barotropic fluid

$$R_{oT} = \frac{\text{Temporal Changes}}{\text{Coriolis Term}} = \frac{U}{T}\frac{1}{fU} = \frac{1}{fT} >> 1$$

$$E_k = \frac{\text{Viscosity Terms}}{\text{Coriolis Term}} = \frac{A_H}{fL^2} \left(\frac{A_V}{fH^2}\right) << 1$$

$$F_R = \frac{U}{NH} >> 1$$

Scaling parameters

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv + A_H\frac{\partial^2 u}{\partial x^2} + A_H\frac{\partial^2 u}{\partial y^2} + A_V\frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu + A_H\frac{\partial^2 v}{\partial x^2} + A_H\frac{\partial^2 v}{\partial y^2} + A_V\frac{\partial^2 v}{\partial z^2}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g + A_H\frac{\partial^2 w}{\partial x^2} + A_H\frac{\partial^2 w}{\partial y^2} + A_V\frac{\partial^2 w}{\partial z^2}$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} \quad (1)$$

$$\frac{\partial \upsilon}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial y} \quad (2)$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial z} \quad (3)$$

where $P = p - g\rho z$

$$\frac{\partial}{\partial x}(1) + \frac{\partial}{\partial y}(2) + \frac{\partial}{\partial z}(3)$$

$$\Rightarrow \rho \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = -\left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2}\right)$$

$= 0$ (Continuity)

$$\Rightarrow \boxed{\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = 0}$$

Εξίσωση LAPLACE

Boundary Conditions:

at $z = 0$

$$P = \rho g \eta$$

$$w = \frac{\partial \eta}{\partial t}$$

at $z = -H$

$$w = 0 \Rightarrow \frac{\partial P}{\partial z} = 0$$

Looking for wave-like solutions:

$$\eta = \eta_0 \cos(kx + ly - \omega t) \quad \text{2-D solutions}$$

for simplicity we will investigate a 2-D (x-z) wave

Wave solution for Laplace equation:

$$\boxed{P = \left(Ae^{kz} + Be^{-kz}\right)\cos(kx - \omega t)}$$

Using the surface boundary condition:

$$A + B = \rho g \eta_0$$

Using the ocean bottom boundary condition:

$$Ae^{-kH} + Be^{kH} = 0$$

$$A = \rho g \eta_0 \frac{e^{kH}}{e^{kH} + e^{-kH}}$$

$$B = \rho g \eta_0 \frac{e^{-kH}}{e^{kH} + e^{-kH}}$$

$$P = \rho g \eta_0 \frac{e^{k(z+H)} + e^{-k(z+H)}}{e^{kH} + e^{-kH}} cos(kx - \omega t) = \frac{\rho g \eta_0 cosh[k(z+H)]}{cosh(kH)} cos(kx - \omega t)$$

From equation (3):

$$\frac{\partial w}{\partial t} = \frac{k g \eta_0 sinh[k(z+H)]}{cosh(kH)} cos(kx - \omega t)$$

Using the surface boundary condition: $\left(w = \frac{\partial \eta}{\partial t} \Rightarrow \frac{\partial w}{\partial t} = \frac{\partial^2 \eta}{\partial t^2} \right)$

$$\frac{k g \eta_0 sinh(kH)}{cosh(kH)} cos(kx - \omega t) = \omega^2 \eta_0 cos(kx - \omega t)$$

$$\omega^2 = gk \tanh(kH)$$

Dispersion relation – Σχέση διασποράς

Short waves

$\lambda << H$ (short wavelength or deep ocean) => ***kH>>1***

$$\tanh(kH) \cong 1$$

$$\omega^2 = gk$$

$$\text{Phase Speed}: C = \frac{\omega}{k}$$

$$C = \sqrt{\frac{g}{k}}$$

Dispersive

$$\text{Group Velocity}: C_g = \frac{\partial \omega}{\partial k}$$

$$C_g = \frac{C}{2}$$

Long waves

$\lambda >> H$ (long wavelength or shallow ocean) => ***kH<<1***

$$\tanh(kH) \cong kH$$

$$\omega^2 = gk^2 H$$

$$\text{Phase Speed}: C = \frac{\omega}{k}$$

$$C = \sqrt{gH}$$

Non-dispersive

$$\text{Group Velocity}: C_g = \frac{\partial \omega}{\partial k}$$

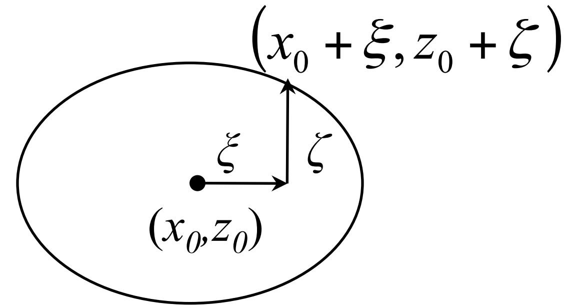
$$C_g = C$$

Water parcel orbits

$$u = \frac{\partial \xi}{\partial t}, \qquad w = \frac{\partial \zeta}{\partial t}$$

$$\frac{\partial \xi}{\partial t} = kg\eta_0 \frac{\cosh[k(z_0 + H)]}{\omega \cosh(kH)} \cos(kx_0 - \omega t)$$

$$\frac{\partial \zeta}{\partial t} = kg\eta_0 \frac{\sinh[k(z_0 + H)]}{\omega \cosh(kH)} \sin(kx_0 - \omega t)$$

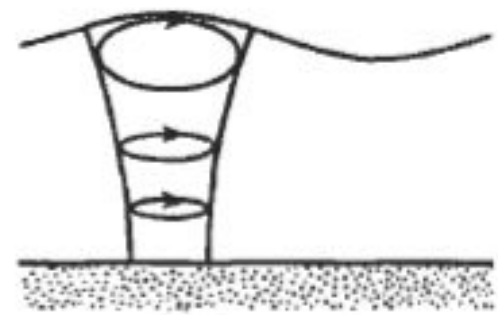


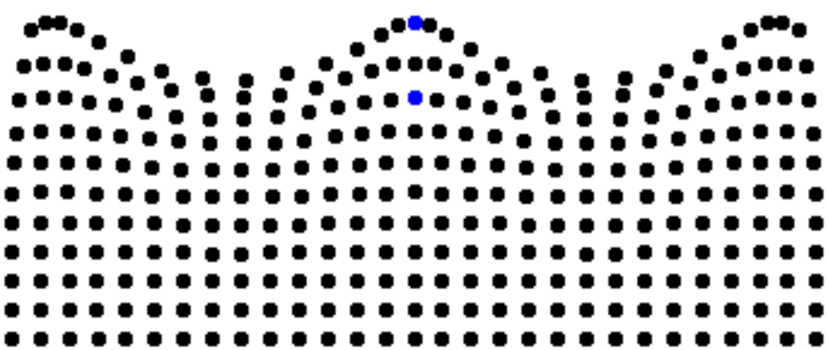
$$\xi = \eta_0 \frac{\cosh[k(z_0 + H)]}{\cosh(kH)} \sin(kx_0 - \omega t)$$

$$\zeta = -\eta_0 \frac{\sinh[k(z_0 + H)]}{\cosh(kH)} \cos(kx_0 - \omega t)$$

$$\frac{\xi^2}{\left[\dfrac{\eta_0 \cosh[k(z_0 + H)]}{\cosh(kH)}\right]^2} + \frac{\zeta^2}{\left[\dfrac{\eta_0 \sinh[k(z_0 + H)]}{\cosh(kH)}\right]^2} = 1$$

semimajor axis ***semiminor axis***





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- Both axes decrease with depth
- The seminor axis is zero at bottom (-*H*)

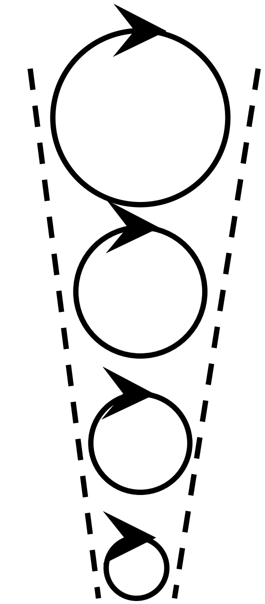
Short waves

$\lambda << H$ (short wavelength or deep ocean) => ***kH>>1***

$$\frac{\cosh[k(z_0+H)]}{\cosh(kH)} \approx \frac{\sinh[k(z_0+H)]}{\cosh(kH)} \approx e^{kz_0}$$

$$\xi = \eta_0 e^{kz_0}\sin(kx_0 - \omega t)$$
$$\zeta = -\eta_0 e^{kz_0}\cos(kx_0 - \omega t)$$

- Circles
- Both decrease exponentially with depth



Long waves

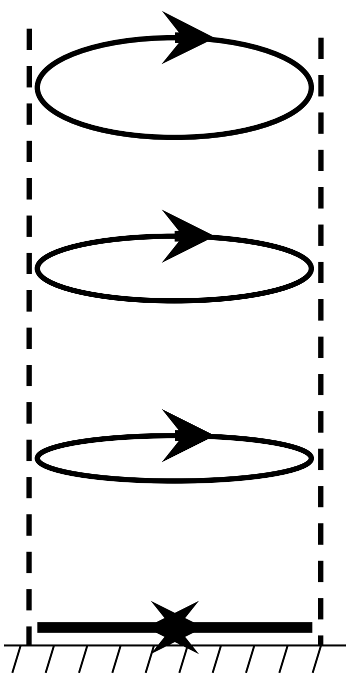
$\lambda >> H$ (long wavelength or shallow ocean) => ***kH<<1***

$$\cosh[k(z_0+H)] \approx 1$$
$$\sinh[k(z_0+H)] \approx k(z_0+H)$$
$$\cosh(kH) \approx 1$$

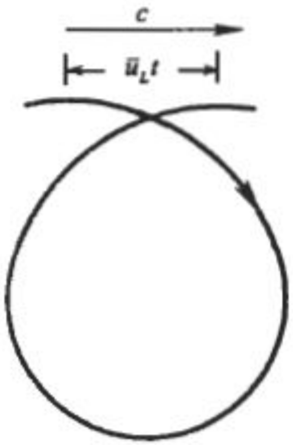
$$\xi = \eta_0 \sin(kx_0 - \omega t)$$
$$\zeta = -k\eta_0 (z_0+H)\cos(kx_0 - \omega t)$$

Constant

Decrease linearly with depth and becomes zero at $z_0=H$



Stokes Drift



If we do not use the approximation (i.e. the Lagrangian velocity at time t is equal with the Eulerian velocity at x_0, z_0 at time t)

The Lagrangian velocity $u_L(x_0,z_0,t)$ can be defined as the Taylor series expansion of the Eulerian velocity $u(x,z,t)$ around x_0, z_0

$$u_L(x_0,z_0,t) = u(x_0,z_0,t) + (x - x_0)\left(\frac{\partial u}{\partial x}\right)_0 + (z - z_0)\left(\frac{\partial u}{\partial z}\right)_0 + \ldots$$

For short gravity waves (deep water limit):

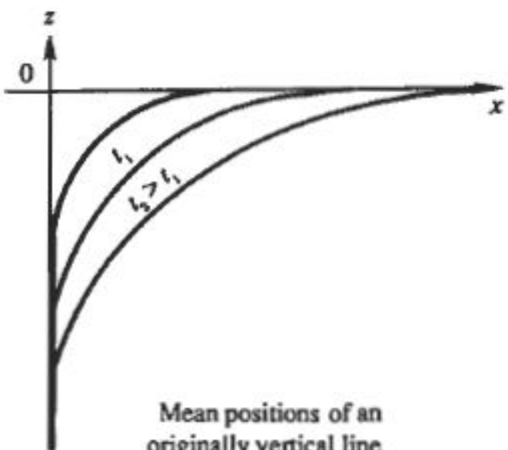
$$u(x_0,z_0,t) = \eta_0 \omega e^{kz_0} \cos(kx_0 - \omega t)$$

$$x - x_0 = \eta_0 e^{kz_0} \sin(kx_0 - \omega t)$$

$$z - z_0 = -\eta_0 e^{kz_0} \cos(kx_0 - \omega t)$$

Averaging over one period

$$\bar{u}_s = \bar{u}_L - \bar{u} = \eta_0^2 \omega \, k \, e^{2kz_0}$$



In general

$$\bar{u}_s = \eta_0^2 \omega k \frac{\cos\left[2k\left(z_0 + H\right)\right]}{2\sinh^2(kH)}$$

Stokes velocity

$$\eta = \eta_0 \cos(kx - \omega t) + \eta_0 \cos(kx + \omega t) = 2\eta_0 \cos(kx)\cos(\omega t)$$

Seiches

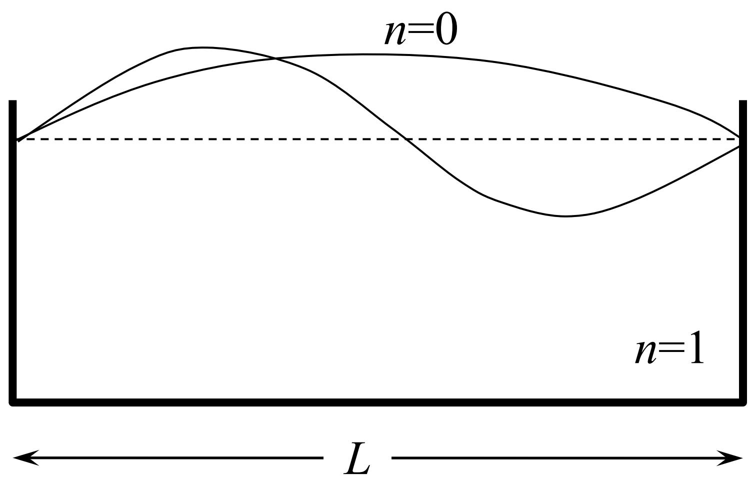
$$u = \frac{2kg\eta_0 \cosh[k(z+H)]}{\omega \cosh(kH)} \sin(kx)\sin(\omega t)$$

Boundary condition:
u=0 at x=0 και L

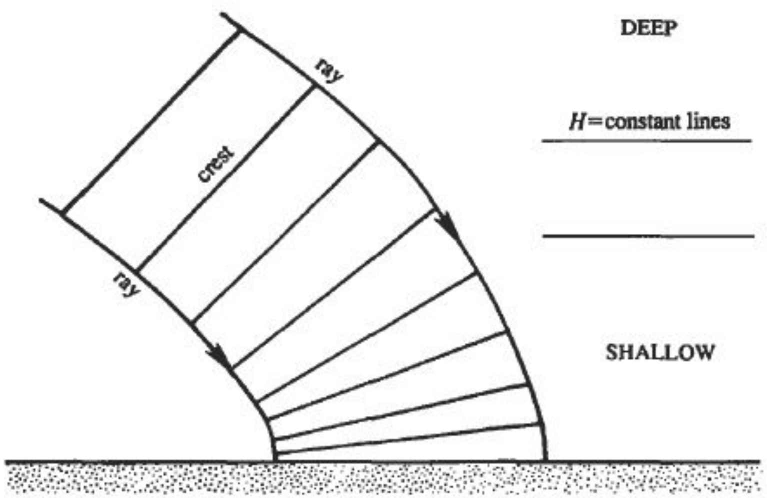
$$kL = (n+1)\pi \quad n = 0,\ 1,\ 2,\ \ldots$$

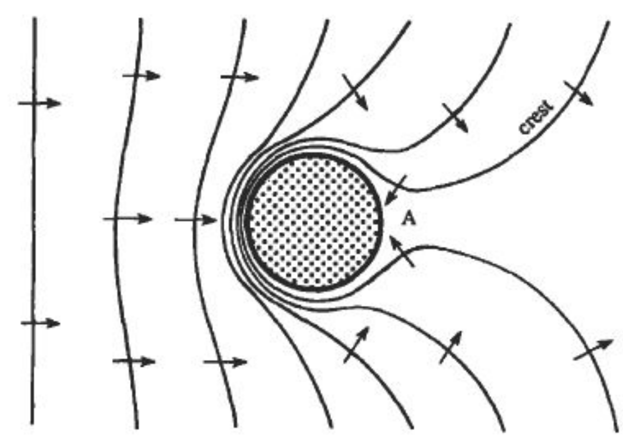
$$\lambda = \frac{2L}{n+1}$$

$$\omega = \sqrt{\frac{\pi g(n+1)}{L}\tanh\left[\frac{(n+1)\pi H}{L}\right]}$$



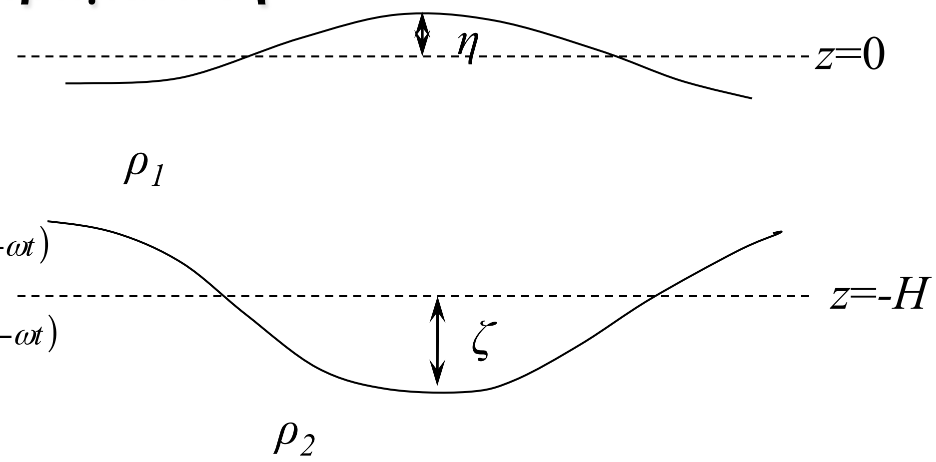
Wave Refraction and the island effect on waves





Internal gravity waves: Layered stratification
Εσωτερικά κύματα βαρύτητας: Απλή στρωμάτωση

Δύο στρώματα. Οι εξισώσεις Laplace για κάθε στρώμα:



$$\left.\begin{aligned}\frac{\partial^2 P_1}{\partial x^2}+\frac{\partial^2 P_1}{\partial z^2}=0\\ \frac{\partial^2 P_2}{\partial x^2}+\frac{\partial^2 P_2}{\partial z^2}=0\end{aligned}\right\}$$

$$P_1=\left(Ae^{kz}+Be^{-kz}\right)e^{i(kx-\omega t)}$$

$$P_2=\left(Ce^{kz}+De^{-kz}\right)e^{i(kx-\omega t)}$$

$$\eta=ae^{i(kx-\omega t)}$$

$$\zeta=be^{i(kx-\omega t)}$$

Οριακές συνθήκες:

at $z=0$ $\quad P_1=\rho_1 g\eta \quad w_1=\frac{\partial\eta}{\partial t}\Rightarrow\frac{\partial P_1}{\partial z}=-\rho\frac{\partial^2\eta}{\partial t^2}$

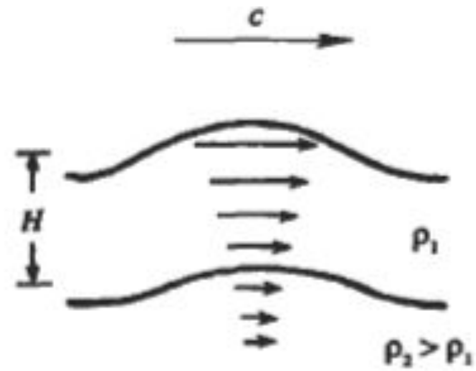
at $z=-H$ $\quad P_1=P_2=\rho_1 g\left(H+\eta-\zeta\right)+\rho_2\zeta \quad w_1=w_2=\frac{\partial\zeta}{\partial t}$

at $z=-\infty$ $\quad w_2=0 \quad \frac{\partial P_2}{\partial z}=0$

Χρησιμοποιώντας τις οριακές συνθήκες:

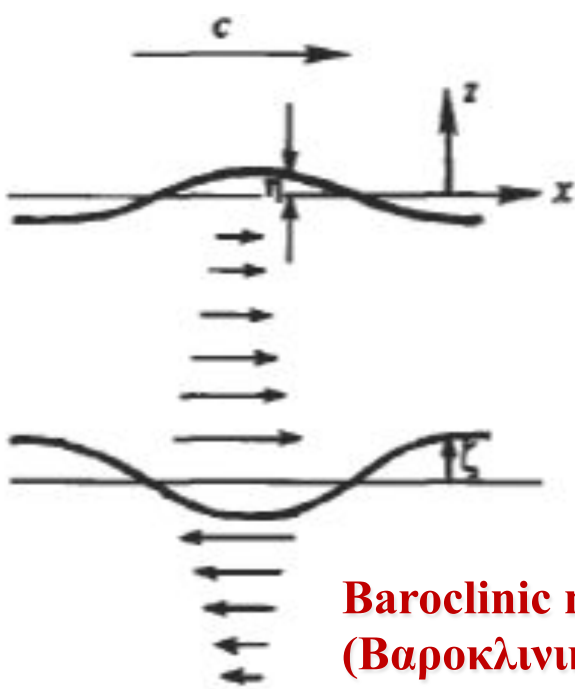
$$\left(\frac{\omega^2}{gk}-1\right)\left\{\frac{\omega^2}{gk}\left[\rho_1\sinh(kH)+\rho_2\cosh(kH)\right]-\left(\rho_2-\rho_1\right)\sinh(kH)\right\}=0$$

<u>First solution</u>: $\omega^2 = gk$



Barotropic mode (Βαροτροπικός τρόπος ταλάντωσης)

<u>Second solution</u>: $$\omega^2 = \frac{gk(\rho_2 - \rho_1)\sinh(kH)}{\rho_2 \cosh(kH) + \rho_1 \sinh(kH)}$$



Baroclinic mode (Βαροκλινικός τρόπος ταλάντηωσης)

In general:

For an ocean of *n* layers, there is one barotropic mode and *n*-1 baroclinic modes (a total of *n* modes of oscillation).

Internal waves: Baroclinic mode

Short waves

$$kH \to \infty$$

$$\coth(kH) \approx 1$$

$$\omega^2 = \frac{gk(\rho_2 - \rho_1)\sinh(kH)}{\rho_2 \cosh(kH) + \rho_1 \sinh(kH)} =$$

$$= \frac{gk(\rho_2 - \rho_1)}{\rho_2 \coth(kH) + \rho_1}$$

$$\omega = \sqrt{gk\left(\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}\right)}$$

$$\eta = -\zeta\left(\frac{\rho_2 - \rho_1}{\rho_1}\right)e^{-kH}$$

Long waves

$$kH << 1$$

$$\sinh(kH) \approx kH$$

$$\cosh(kH) \approx 1$$

$$\omega^2 = \frac{gk^2 H(\rho_2 - \rho_1)}{\rho_2}$$

$$c = \sqrt{g'H}, \quad g' = \frac{\rho_2 - \rho_1}{\rho_2} g$$

$$\eta = -\zeta\left(\frac{\rho_2 - \rho_1}{\rho_2}\right)$$

For a typical oceanic stratificatiob:

$\Delta\rho = O(1 kg/m^3)$, $\rho_2 = O(1000 kg/m^3)$

$g' = 10^{-3} g$