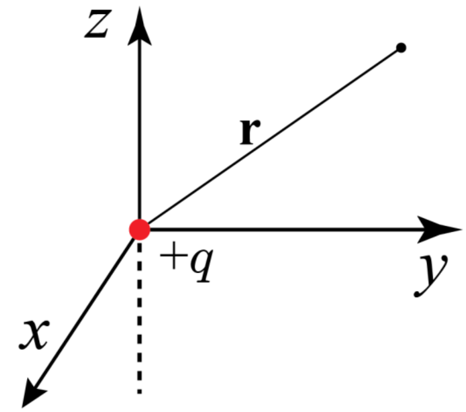
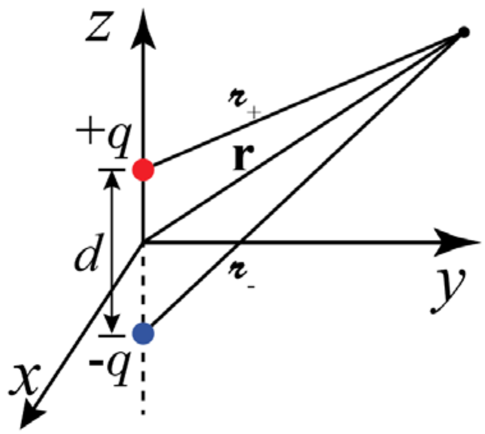
Multipole Expansion (Potentials at large distances)



- What is the potential due to a point charge (monopole)?

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$ **(goes like $1/r$)**



- What is the potential due to a dipole at large distance?

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}_+} - \frac{q}{\mathscr{r}_-}\right)$$

$$\mathscr{r}_\pm^2 = r^2 + \left(\frac{d}{2}\right)^2 \mp rd\cos\theta = r^2\left(1 \mp \frac{d}{r}\cos\theta + \frac{d^2}{4r^2}\right)$$

$$\approx r^2\left(1 \mp \frac{d}{r}\cos\theta\right) \quad (\text{for } r \gg d)$$

for $r \gg d$, and using binomial expansion

$$\frac{1}{\mathscr{r}_\pm} = \frac{1}{r}\left(1 \mp \frac{d}{r}\cos\theta\right)^{-\frac{1}{2}} \approx \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos\theta\right)$$

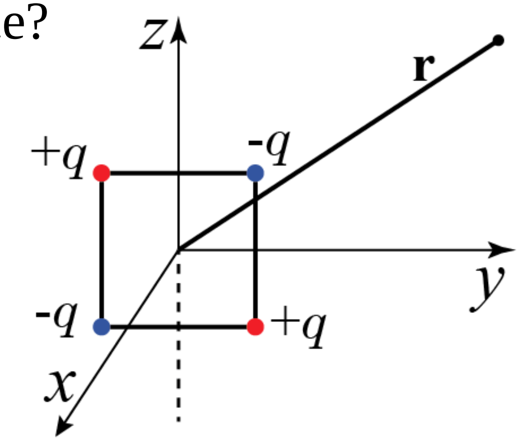
So, $\frac{1}{\mathscr{r}_+} - \frac{1}{\mathscr{r}_-} = \frac{d}{r^2}\cos\theta$

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{qd\cos\theta}{r^2}$$ **(goes like $1/r^2$ at large r)**

Multipole Expansion (Potentials at large distances)

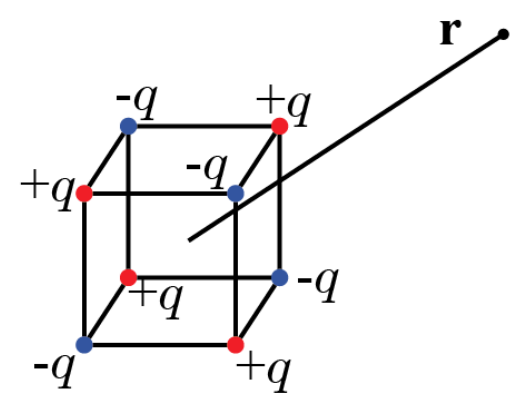
- What is the potential due to a quadrupole at large distance?

(goes like $1/r^3$ at large r)



- What is the potential due to a octopole at large distance?

(goes like $1/r^4$ at large r)

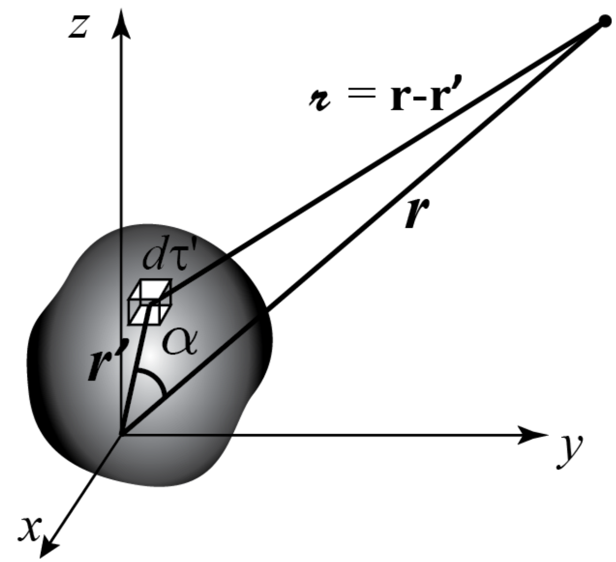


Note1: Multipole terms are defined in terms of their $\boldsymbol{r}$ dependence, not in terms of the number of charges.

Note 2: The dipole potential need not be produced by a two-charge system only . A general n-charge system can have any multipole contribution.

Multipole Expansion (Potentials at large distances)

- What is the potential due to a localized charge distribution?



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}} \quad = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'$$

Using the cosine rule,

$$\mathscr{r}^2 = r^2 + r'^2 - 2rr'\cos\alpha$$

$$\mathscr{r}^2 = r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]$$

$$\mathscr{r} = r\sqrt{1 + \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)}$$

$$\mathscr{r} = r\sqrt{1+\epsilon}$$

Source coordinates: (r', θ', ϕ')

Observation point coordinates: (r, θ, ϕ)

Angle between $\mathbf{r}$ and $\mathbf{r}'$: α

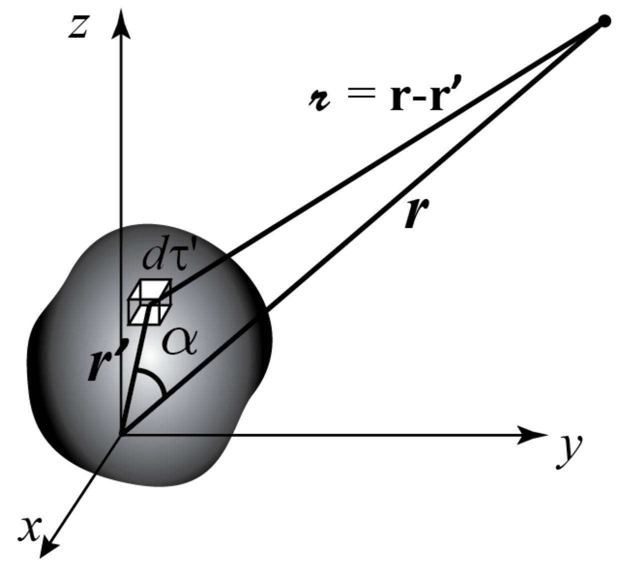
Define: $\epsilon \equiv \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right)$

So, $\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\epsilon)^{-1/2}$

Or, $\frac{1}{\mathscr{r}} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots\right)$ (using binomial expansion)

Multipole Expansion (Potentials at large distances)

- What is the potential due to a localized charge distribution?



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}} \quad = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\left(1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 - \frac{5}{16}\epsilon^3 + \cdots\right)$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\alpha\right) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\alpha\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\alpha\right)^3 + \cdots\right]$$

$$= \frac{1}{r}\left[1 + \left(\frac{r'}{r}\right)(\cos\alpha) + \left(\frac{r'}{r}\right)^2 (3\cos^2\alpha - 1)/2 - \left(\frac{r'}{r}\right)^3 (5\cos^3\alpha - 3\cos\alpha)/2 + \cdots\right]$$

$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\alpha)$$

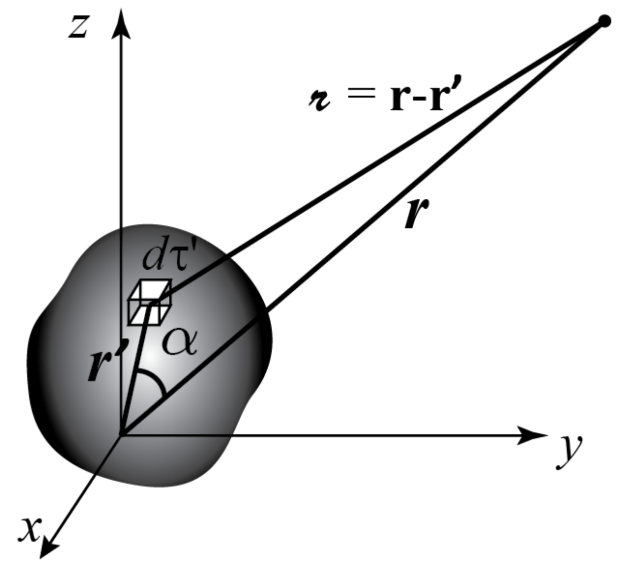
$P_n(\cos\alpha)$ are Legendre polynomials

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau' = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')d\tau'$$

Multipole Expansion (Potentials at large distances)

- What is the potential due to a localized charge distribution?

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{\mathscr{r}} \quad = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'$$



$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int (r')^n P_n(\cos\alpha)\rho(\mathbf{r}')d\tau'$$

$$= \frac{1}{4\pi\epsilon_0 r}\int \rho(\mathbf{r}')d\tau' + \frac{1}{4\pi\epsilon_0 r^2}\int r'(\cos\alpha)\rho(\mathbf{r}')d\tau' + \frac{1}{4\pi\epsilon_0 r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')d\tau' + \cdots$$

Monopole potential ($1/r$ dependence)

Dipole potential ($1/r^2$ dependence)

Quadrupole potential ($1/r^3$ dependence)

Multipole Expansion of V(r)

Multipole Expansion (Few comments)

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0 r}\int \rho(\mathbf{r}')d\tau' + \frac{1}{4\pi\epsilon_0 r^2}\int r'(\cos\alpha)\rho(\mathbf{r}')d\tau' + \frac{1}{4\pi\epsilon_0 r^3}\int (r')^2\left(\frac{3}{2}\cos^2\alpha - \frac{1}{2}\right)\rho(\mathbf{r}')d\tau' + \cdots$$

Monopole potential ($1/r$ dependence)

Dipole potential ($1/r^2$ dependence)

Quadrupole potential ($1/r^3$ dependence)

- **It is an exact expression, not an approximation.**
- **A particular term in the expansion is defined by its r dependence**
- **At large r, the potential can be approximated by the first non-zero term.**
- **More terms can be added if greater accuracy is required**

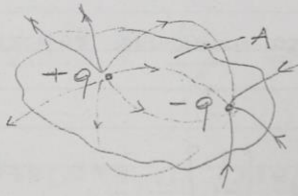
Νόμος του Gauss (για το ηλ. πεδίο)

$$\Phi_E = \oint_A \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\varepsilon_0}$$

A : οποιαδήποτε κλειστή επιφάνεια (επιφάνεια Gauss)

Q_{enc} : το συνολικό φορτίο που περικλείεται από την επιφάνεια.

Παράδειγμα:

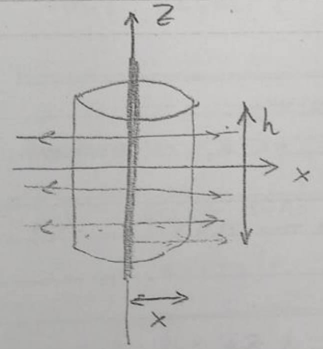


Εδώ: $Q_{enc} = +q - q = 0$

οπότε $\Phi_E = 0$

Παραδείγματα → [Από το Νόμο του Gauss → βρίσκουμε E]

① Πεδίο από μακρύ ευθύγραμμο τμήμα με πυκνότητα φορτίου ρ [C/m].



Επιφάνεια Gauss:

Κύλινδρος με κέντρο το ευθ. τμήμα με ακτίνα x και ύψος h.

▷ Στην καμπύλη επιφάνεια:

$\hat{n} \perp$ επιφάνεια $\Rightarrow \cos\theta = 1$

και $\int_A E\cos\theta \, dA = \int_A E \, dA =$

$= E \times (2\pi x h)$

και ο νόμος Gauss δίνει:

$$\int_{A \text{ (καμπύλη επιφάνεια)}} E\cos\theta\, dA = \int_A E\, dA = E\times(2\pi x h) = \frac{Q_{enc}}{\varepsilon_0}$$

όπου $Q_{enc} = \dfrac{\rho h}{\varepsilon_0}$ $\Rightarrow$

$$\Rightarrow E\times 2\pi x h = \frac{\rho h}{\varepsilon_0} \Rightarrow \boxed{E = \frac{1}{2\pi\varepsilon_0}\,\frac{\rho}{x}}$$

όπως είχαμε βρεί προηγουμένως!

▷ <u>Σημείωση</u>: στα "καπάκια" του κυλίνδρου:

$\hat{n} \perp$ πεδίο $\Rightarrow \cos\theta = \cos 90^\circ = 0$ άρα

στις βάσεις: $\displaystyle\int_{\text{βάσεις}} E\cos\theta\, dA = 0.$

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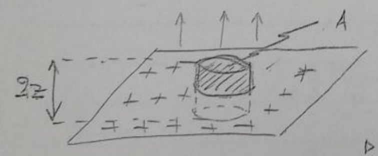
② Πεδίο από μεγάλο επίπεδο φύλλο με πυκνότητα φορτίου ρ $[C/m^2]$



Επιφάνεια Gauss:
κύλινδρος $\perp$ φύλλο με
▷▷ ύψος $= 2z$
▷▷ εμβαδόν βάσης $= A$.

$$\int_{\text{βάση}} E\cos\theta\, dA = \int E\, dA = E\int dA = E\times(2A)$$

($\cos\theta = 1$ γιατί $\theta = 0^\circ$) (2 βάσεις)

Στην καμπύλη επιφάνεια του κυλίνδρου:

$\theta = 90^\circ$ οπότε $\cos\theta = 0$ και

$$\int E \cos\theta \, dA = 0.$$

▷▷ Νόμος Gauss: $2\cancel{A}E = \dfrac{Q_{enc}}{\varepsilon_0} = \dfrac{\rho \cdot \cancel{A}}{\varepsilon_0} \Rightarrow$

$$\Rightarrow \boxed{E = \frac{\rho}{2\varepsilon_0}}$$

όπως είχαμε βρει προηγουμένως!

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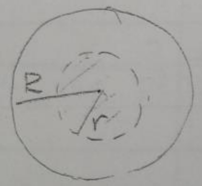
③ Πεδίο σε διηλεκτρική σφαίρα (*) [Μετά και σε αγώγιμη]

Σφαίρα ακτίνας R περιέχει ολικό φορτίο Q που είναι ομοιόμορφα κατανεμημένο σε όλο τον όγκο της.

1) Ποιο είναι το ηλ. πεδίο στο εσωτερικό ?

2) ——— " ——— εξωτερικό ?



1) Είναι: $\rho = \dfrac{Q}{\text{όγκος}} = \dfrac{Q}{\frac{4}{3}\pi R^3} = \dfrac{3Q}{4\pi R^3}.$

Στο εσωτερικό: Επιφάνεια Gauss σφαίρα ακτίνας $r < R$

Εδώ $\cos\theta = 1$ $(\theta = 0^\circ)$ και $\int_A E\cos\theta \, dA = \int_A E \, dA = E\int_A dA = 4\pi r^2 E$