# ΣΥΝΟΨΗ ΧΑΜΙΛΤΟΝΙΑΝΩΝ

προ (Θ)

$$\hat{H}_{HM,m} = \hbar\omega_m\left(\hat{a}_m^\dagger \hat{a}_m + \frac{1}{2}\right) = \hbar\omega\left(\hat{N}_m + \frac{1}{2}\right)$$

m τρόπος ΗΜ πεδίου στην κοιλότητα

και αγνοώντας τον όρο $\frac{\hbar\omega_m}{2}$

$$\hat{H}_{HM,m} = \hbar\omega_m \hat{a}_m^\dagger \hat{a}_m = \hbar\omega_m \hat{N}_m$$

$$\hat{H}_{\Delta\Sigma} = E_2 \hat{S}_+ \hat{S}_- + E_1 \hat{S}_- \hat{S}_+ \qquad E_2 - E_1 = \hbar\Omega$$

διεσταδιακό συστημα

και θέτοντας $E_1 \equiv 0$

$$\hat{H}_{\Delta\Sigma} = \hbar\Omega \hat{S}_+ \hat{S}_-$$

$$\hat{U}_{\Sigma m} = \hbar g_m\left(\hat{S}_+ + \hat{S}_-\right)\left(\hat{a}_m^\dagger + \hat{a}_m\right) = \hat{H}_{\Delta\Sigma - HMm}$$

αλληλεπίδραση

αλληλεπίδραση ΔΣ – m τρόπου ΗΜ πεδίου στην κοιλότητα

($\hat{H}_{AF}$ ατομική φυσική)

atom field

$$\hbar g_m := e x_{12}\left(\frac{\hbar\omega_m}{\varepsilon_0 V}\right)^{1/2} \sin\left(\frac{m\pi z}{L}\right)$$

$$\hbar|g_m| = \underbrace{|e x_{12}|}_{|\mathcal{P}|}\underbrace{\left(\frac{4\hbar\omega_m n_m}{\varepsilon_0 V}\right)^{1/2}\left|\sin\left(\frac{m\pi z}{L}\right)\right|}_{E_{0m}(z)}\frac{1}{2\sqrt{n_m}}$$

$$\Omega_{Rm} := \frac{|\mathcal{P}|\, E_{0m}}{\hbar}$$

$$\Omega_{Rm} := 2\sqrt{n_m}\, g_m$$

συχνότητα Rabi του m τρόπου

$|\uparrow n_m\rangle$, $|\downarrow n_m\rangle$ οι ίδιο καταστάσεις της $\hat{H}_{HMm} + \hat{H}_{\Delta\Sigma}$

Ενώ, η συνολική Χαμιλτονιανή γράφεται

$$\hat{H}_{Rm} = \hbar\omega_m \hat{a}_m^\dagger \hat{a}_m + \hbar\Omega \hat{S}_+ \hat{S}_- + \hbar g_m\left(\hat{S}_+ + \hat{S}_-\right)\left(\hat{a}_m^\dagger + \hat{a}_m\right)$$

ονομάζεται συχνά Χαμιλτονιανή Rabi

Αγνοώντας τους όρους $\hat{S}_+ \hat{a}_m^\dagger$ κ $\hat{S}_- \hat{a}_m$, οι οποίοι φαίνονται "παράλογοι" όταν υπάρχει ένας μόνο τρόπος!

$$\hat{H}_{JCm} = \hbar\omega_m \hat{a}_m^\dagger \hat{a}_m + \hbar\Omega \hat{S}_+ \hat{S}_- + \hbar g_m\left(\hat{S}_+ \hat{a}_m + \hat{S}_- \hat{a}_m^\dagger\right)$$

ονομάζεται συχνά Χαμιλτονιανή Jaynes–Cummings

**ΑΣΚΗΣΗ** (Παραλείπουμε για ευκολότητα τον δείκτη του ΗΜ τρόπου m)

**(A)**

Βρείτε τι κάνουν οι όροι

$\hat{a}^\dagger\hat{a}$, $\hat{a}\hat{a}^\dagger$, $\hat{S}_+\hat{S}_-$, $\hat{S}_-\hat{S}_+$, $\hat{S}_+\hat{a}^\dagger$, $\hat{S}_+\hat{a}$, $\hat{S}_-\hat{a}^\dagger$, $\hat{S}_-\hat{a}$ στις καταστάσεις

$|\uparrow, n_m\rangle$ ή $|\downarrow, n_m\rangle$

**(B)** Υπολογίστε τα

$\langle\hat{a}^\dagger\hat{a}\rangle$, $\langle\hat{a}\hat{a}^\dagger\rangle$, $\langle\hat{S}_+\hat{S}_-\rangle$, $\langle\hat{S}_-\hat{S}_+\rangle$, $\langle\hat{S}_+\hat{a}^\dagger\rangle$, $\langle\hat{S}_+\hat{a}\rangle$, $\langle\hat{S}_-\hat{a}^\dagger\rangle$, $\langle\hat{S}_-\hat{a}\rangle$

στις $|\uparrow n_m\rangle$, $|\downarrow n_m\rangle$

**(A)**

$\hat{a}^\dagger\hat{a}|\uparrow, n\rangle = n|\uparrow, n\rangle$ $\quad \hat{a}^\dagger\hat{a}|\downarrow, n\rangle = n|\downarrow, n\rangle$

$[\hat{a}, \hat{a}^\dagger] = 1 \Leftrightarrow \hat{a}\hat{a}^\dagger = 1 + \hat{a}^\dagger\hat{a}$

$\hat{a}\hat{a}^\dagger|\uparrow, n\rangle = (1 + \hat{a}^\dagger\hat{a})|\uparrow, n\rangle = (n+1)|\uparrow, n\rangle$

$\hat{a}\hat{a}^\dagger|\downarrow, n\rangle = (1 + \hat{a}^\dagger\hat{a})|\downarrow, n\rangle = (n+1)|\downarrow, n\rangle$

$\hat{S}_+\hat{S}_-|\uparrow, n\rangle = |\uparrow, n\rangle$ $\quad \hat{S}_+\hat{S}_-|\downarrow, n\rangle = |\emptyset, n\rangle$

$\hat{S}_-\hat{S}_+|\uparrow, n\rangle = |\emptyset, n\rangle$ $\quad \hat{S}_-\hat{S}_+|\downarrow, n\rangle = |\downarrow, n\rangle$

$\hat{S}_+\hat{a}^\dagger|\uparrow, n\rangle = \sqrt{n+1}\,|\emptyset, n+1\rangle$ $\quad \hat{S}_+\hat{a}^\dagger|\downarrow n\rangle = \sqrt{n+1}\,|\uparrow, n+1\rangle$

$\hat{S}_+\hat{a}|\uparrow n\rangle = \sqrt{n}\,|\emptyset, n-1\rangle$ $\quad \hat{S}_+\hat{a}|\downarrow n\rangle = \sqrt{n}\,|\uparrow, n-1\rangle$

$\hat{S}_-\hat{a}^\dagger|\uparrow n\rangle = \sqrt{n+1}\,|\downarrow, n+1\rangle$ $\quad \hat{S}_-\hat{a}^\dagger|\downarrow n\rangle = \sqrt{n+1}\,|\emptyset, n+1\rangle$

$\hat{S}_-\hat{a}|\uparrow, n\rangle = \sqrt{n}\,|\downarrow, n-1\rangle$ $\quad \hat{S}_-\hat{a}|\downarrow, n\rangle = \sqrt{n}\,|\emptyset, n-1\rangle$

**(B)**

$\langle\uparrow n|\hat{a}^\dagger\hat{a}|\uparrow n\rangle = n$

$\langle\downarrow n|\hat{a}^\dagger\hat{a}|\downarrow n\rangle = n$

$\langle\uparrow n|\hat{a}\hat{a}^\dagger|\uparrow n\rangle = n+1$

$\langle\downarrow n|\hat{a}\hat{a}^\dagger|\downarrow n\rangle = n+1$

$\langle\uparrow n|\hat{S}_+\hat{S}_-|\uparrow n\rangle = 1$

$\langle\downarrow n|\hat{S}_+\hat{S}_-|\downarrow n\rangle = \langle\downarrow n|\emptyset\, n\rangle = 0$

$\langle\uparrow n|\hat{S}_-\hat{S}_+|\uparrow n\rangle = \langle\uparrow n|\emptyset\, n\rangle = 0$

$\langle\downarrow n|\hat{S}_-\hat{S}_+|\downarrow n\rangle = \langle\downarrow n|\downarrow n\rangle = 1$

συνεχίζεται ...

| | |
|---|---|
| $\langle \hat{a}^\dagger \hat{a} \rangle_{\uparrow n\rangle} = n$ | $\langle \hat{a}^\dagger \hat{a} \rangle_{\downarrow n\rangle} = n$ |
| $\langle \hat{a} \hat{a}^\dagger \rangle_{\uparrow n\rangle} = n+1$ | $\langle \hat{a} \hat{a}^\dagger \rangle_{\downarrow n\rangle} = n+1$ |
| $\langle \hat{S}_+ \hat{S}_- \rangle_{\uparrow n\rangle} = 1$ | $\langle \hat{S}_+ \hat{S}_- \rangle_{\downarrow n\rangle} = 0$ |
| $\langle \hat{S}_- \hat{S}_+ \rangle_{\uparrow n\rangle} = 0$ | $\langle \hat{S}_- \hat{S}_+ \rangle_{\downarrow n\rangle} = 1$ |

$$\langle \hat{S}_+ \hat{a}^\dagger \rangle_{\uparrow n\rangle} = 0 \qquad \langle \hat{S}_+ \hat{a}^\dagger \rangle_{\downarrow n\rangle} = 0$$

$$\langle \hat{S}_+ \hat{a} \rangle_{\uparrow n\rangle} = 0 \qquad \langle \hat{S}_+ \hat{a} \rangle_{\downarrow n\rangle} = 0$$

$$\langle \hat{S}_- \hat{a}_m^\dagger \rangle_{\uparrow n\rangle} = 0 \qquad \langle \hat{S}_- \hat{a}_m^\dagger \rangle_{\downarrow n\rangle} = 0$$

$$\langle \hat{S}_- \hat{a} \rangle_{\uparrow n\rangle} = 0 \qquad \langle \hat{S}_- \hat{a} \rangle_{\downarrow n\rangle} = 0$$

... sowie

$$\langle \uparrow n | \hat{S}_+ \hat{a}^\dagger | \uparrow n \rangle = \sqrt{n+1} \langle \uparrow n | \emptyset, n+1 \rangle = 0$$
$$\langle \downarrow n | \hat{S}_+ \hat{a}^\dagger | \downarrow n \rangle = \sqrt{n+1} \langle \downarrow n | \uparrow n+1 \rangle = 0$$
$$\langle \uparrow n | \hat{S}_+ \hat{a} | \uparrow n \rangle = \sqrt{n} \langle \uparrow n | \emptyset, n-1 \rangle = 0$$
$$\langle \downarrow n | \hat{S}_+ \hat{a} | \downarrow n \rangle = \sqrt{n} \langle \downarrow n | \uparrow n-1 \rangle = 0$$
$$\langle \uparrow n | \hat{S}_- \hat{a}^\dagger | \uparrow n \rangle = \sqrt{n+1} \langle \uparrow n | \downarrow n+1 \rangle = 0$$
$$\langle \downarrow n | \hat{S}_- \hat{a}^\dagger | \downarrow n \rangle = \sqrt{n+1} \langle \downarrow n | \emptyset\, n+1 \rangle = 0$$
$$\langle \uparrow n | \hat{S}_- \hat{a} | \uparrow n \rangle = \sqrt{n} \langle \uparrow n | \downarrow n-1 \rangle = 0$$
$$\langle \downarrow n | \hat{S}_- \hat{a} | \downarrow n \rangle = \sqrt{n} \langle \downarrow n | 0, n-1 \rangle = 0$$

# ΜΕΣΕΣ (ΑΝΑΜΕΝΟΜΕΝΕΣ) ΤΙΜΕΣ ΜΕΓΕΘΩΝ για τη Χαμιλτονιανή Jaynes – Cummings



Για την κατάσταση Α $|\psi_A(t)\rangle = c_1(t)\,|\downarrow, n\rangle + c_2(t)\,|\uparrow, n-1\rangle = |A\rangle$

$$\langle \hat{a}^\dagger \hat{a} \rangle_{|A\rangle} = \left( c_1(t)^* \langle \downarrow n| + c_2^*(t) \langle \uparrow n-1| \right) \hat{a}^\dagger \hat{a} \left( c_1(t)|\downarrow n\rangle + c_2(t)|\uparrow n-1\rangle \right)$$

$$= |c_1(t)|^2 \langle \downarrow n|\hat{a}^\dagger\hat{a}|\downarrow n\rangle + c_1^*(t)c_2(t)\langle \downarrow n|\hat{a}^\dagger\hat{a}|\uparrow n-1\rangle$$

$$+ c_2^*(t)c_1(t)\langle \uparrow n-1|\hat{a}^\dagger\hat{a}|\downarrow n\rangle + |c_2(t)|^2\langle \uparrow n-1|\hat{a}^\dagger\hat{a}|\uparrow n-1\rangle$$

$$= |c_1(t)|^2 n \langle \downarrow n|\downarrow n\rangle + c_1^*(t)c_2(t)(n-1)\langle \downarrow n|\uparrow n-1\rangle$$

$$+ c_2^*(t)c_1(t)\, n \langle \uparrow n-1|\downarrow n\rangle + |c_2(t)|^2 (n-1)\langle \uparrow n-1|\uparrow n-1\rangle$$

$$= |c_1(t)|^2 n + |c_2(t)|^2 n - |c_2(t)|^2 \Rightarrow \boxed{\langle \hat{a}^\dagger\hat{a} \rangle_{|A\rangle} = n - |c_2(t)|^2}$$

$$\langle \hat{a}\hat{a}^\dagger \rangle_{|A\rangle} = \left( c_1(t)^* \langle \downarrow n| + c_2^*(t)\langle \uparrow n-1| \right) \hat{a}\hat{a}^\dagger \left( c_1(t)|\downarrow n\rangle + c_2(t)|\uparrow n-1\rangle \right)$$

$$= |c_1(t)|^2 \langle \downarrow n|\hat{a}\hat{a}^\dagger|\downarrow n\rangle + c_1^*(t)c_2(t)\langle \downarrow n|\hat{a}\hat{a}^\dagger|\uparrow n-1\rangle$$

$$+ c_2^*(t)c_1(t)\langle \uparrow n-1|\hat{a}\hat{a}^\dagger|\downarrow n\rangle + |c_2(t)|^2\langle \uparrow n-1|\hat{a}\hat{a}^\dagger|\uparrow n-1\rangle =$$

$$= |c_1(t)|^2 (n+1) + c_1^*(t)c_2(t)\, n \langle \downarrow n|\uparrow n-1\rangle$$

$$+ c_2^*(t)c_1(t)(n+1)\langle \uparrow n-1|\downarrow n\rangle + |c_2(t)|^2 n =$$

$$= |c_1(t)|^2 n + |c_1(t)|^2 + |c_2(t)|^2 n \Rightarrow \boxed{\langle \hat{a}\hat{a}^\dagger \rangle_{|A\rangle} = n + |c_1(t)|^2}$$

$$\langle \hat{a}\hat{a}^\dagger \rangle_{|A\rangle} - \langle \hat{a}^\dagger\hat{a} \rangle_{|A\rangle} = 1$$

$\langle \hat{S}_+ \hat{S}_- \rangle_{|A\rangle} = |c_2(t)|^2$

$\langle \hat{S}_- \hat{S}_+ \rangle_{|A\rangle} = |c_1(t)|^2$

Όμοιως αποδεικνύονται ...

$\langle \hat{S}_+ \hat{S}_- \rangle_{|A\rangle} + \langle \hat{S}_- \hat{S}_+ \rangle_{|A\rangle} = 1$

$\langle \hat{a}^\dagger \hat{a} \rangle_{|A\rangle} + \langle \hat{S}_+ \hat{S}_- \rangle_{|A\rangle} = n$

$\langle \hat{S}_+ \hat{a} \rangle_{|A\rangle} = c_2^*(t)\, c_1(t) \sqrt{n}$

$\langle \hat{S}_- \hat{a}^\dagger \rangle_{|A\rangle} = c_1^*(t)\, c_2(t) \sqrt{n}$

$\langle \hat{S}_+ \hat{a}^\dagger \rangle_{|A\rangle} = 0$

$\langle \hat{S}_- \hat{a} \rangle_{|A\rangle} = 0$

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Για την κατάσταση (E) $|\psi_E(t)\rangle = c_1(t)\,|\uparrow n\rangle + c_2(t)\,|\downarrow n+1\rangle = |E\rangle$ όμοιως αποδεικνύονται

$\langle \hat{a}^\dagger \hat{a} \rangle_{|E\rangle} =$

$\langle \hat{a} \hat{a}^\dagger \rangle_{|E\rangle} =$

$\langle \hat{a} \hat{a}^\dagger \rangle_{|E\rangle} - \langle \hat{a}^\dagger \hat{a} \rangle_{|E\rangle} =$

$\langle \hat{S}_+ \hat{S}_- \rangle_{|E\rangle} =$

$\langle \hat{S}_- \hat{S}_+ \rangle_{|E\rangle} =$

$\langle \hat{S}_+ \hat{S}_- \rangle_{|E\rangle} + \langle \hat{S}_- \hat{S}_+ \rangle_{|E\rangle} =$

$\langle \hat{a}^\dagger \hat{a} \rangle_{|E\rangle} + \langle \hat{S}_+ \hat{S}_- \rangle_{|E\rangle} =$

$\langle \hat{S}_+ \hat{a} \rangle_{|E\rangle} =$

$\langle \hat{S}_+ \hat{a}^\dagger \rangle_{|E\rangle} =$

$\langle \hat{S}_- \hat{a}^\dagger \rangle_{|E\rangle} =$

$\langle \hat{S}_- \hat{a} \rangle_{|E\rangle} =$

$$\langle \hat{S}_+ \hat{S}_- \rangle_{|A\rangle} = \left(c_1^*(t)\langle \downarrow n| + c_2^*(t)\langle \uparrow n-1|\right)\hat{S}_+\hat{S}_-\left(c_1(t)|\downarrow n\rangle + c_2(t)|\uparrow n-1\rangle\right)$$

$$= |c_1(t)|^2 \langle \downarrow n|\hat{S}_+\hat{S}_-|\downarrow n\rangle +$$

$$c_1^*(t)c_2(t)\langle \downarrow n|\hat{S}_+\hat{S}_-|\uparrow n-1\rangle +$$

$$c_2^*(t)c_1(t)\langle \uparrow n-1|\hat{S}_+\hat{S}_-|\downarrow n\rangle +$$

$$|c_2(t)|^2\langle \uparrow n-1|\hat{S}_+\hat{S}_-|\uparrow n-1\rangle =$$

$$= |c_1(t)|^2 \langle \downarrow n|\cancel{\emptyset n}\rangle +$$

$$c_1^*(t)c_2(t)\langle \downarrow n|\cancel{\uparrow n-1}\rangle +$$

$$c_2^*(t)c_1(t)\langle \uparrow n-1|\cancel{\emptyset n}\rangle +$$

$$|c_2(t)|^2\langle \uparrow n-1|\uparrow n-1\rangle = |c_2(t)|^2$$

$$\langle \hat{S}_-\hat{S}_+ \rangle_{|A\rangle} = \left(c_1^*(t)\langle \downarrow n| + c_2^*(t)\langle \uparrow n-1|\right)\hat{S}_-\hat{S}_+\left(c_1(t)|\downarrow n\rangle + c_2(t)|\uparrow n-1\rangle\right)$$

$$= |c_1(t)|^2 \langle \downarrow n|\hat{S}_-\hat{S}_+|\downarrow n\rangle +$$

$$c_1^*(t)c_2(t)\langle \downarrow n|\hat{S}_-\hat{S}_+|\uparrow n-1\rangle +$$

$$c_2^*(t)c_1(t)\langle \uparrow n-1|\hat{S}_-\hat{S}_+|\downarrow n\rangle +$$

$$|c_2(t)|^2\langle \uparrow n-1|\hat{S}_-\hat{S}_+|\uparrow n-1\rangle =$$

$$= |c_1(t)|^2 \langle \downarrow n|\downarrow n\rangle +$$

$$c_1^*(t)c_2(t)\langle \downarrow n|\cancel{\emptyset n-1}\rangle +$$

$$c_2^*(t)c_1(t)\langle \uparrow n-1|\cancel{\downarrow n}\rangle +$$

$$|c_2(t)|^2\langle \uparrow n-1|\cancel{\emptyset n-1}\rangle = |c_1(t)|^2$$

$$\langle \hat{S}_+ \hat{a} \rangle_{|A\rangle} = \left(c_1^*(t)\langle \downarrow n| + c_2^*(t)|\uparrow n-1\rangle\right) \hat{S}_+ \hat{a} \left(c_1(t)|\downarrow n\rangle + c_2(t)|\uparrow n-1\rangle\right)$$ (IB)

$$= |c_1(t)|^2 \langle \downarrow n| \hat{S}_+ \hat{a} |\downarrow n\rangle +$$
$$c_1^*(t) c_2(t) \langle \downarrow n| \hat{S}_+ \hat{a} |\uparrow n-1\rangle +$$
$$c_2^*(t) c_1(t) \langle \uparrow n-1| \hat{S}_+ \hat{a} |\downarrow n\rangle +$$
$$|c_2(t)|^2 \langle \uparrow n-1| \hat{S}_+ \hat{a} |\uparrow n-1\rangle =$$

$$= |c_1(t)|^2 \sqrt{n} \langle \downarrow n|\uparrow n-1\rangle +$$
$$c_1^*(t) c_2(t) \sqrt{n-1} \langle \downarrow n|\emptyset\, n-2\rangle +$$
$$c_2^*(t) c_1(t) \sqrt{n} \langle \uparrow n-1|\uparrow n-1\rangle +$$
$$|c_2(t)|^2 \sqrt{n-1} \langle \uparrow n-1|\emptyset\, n-2\rangle = c_2^*(t)\, c_1(t) \sqrt{n}$$

$$\langle \hat{S}_- \hat{a}^\dagger \rangle_{|A\rangle} = \left(c_1^*(t)\langle \downarrow n| + c_2^*(t)\langle \uparrow n-1|\right) \hat{S}_- \hat{a}^\dagger \left(c_1(t)|\downarrow n\rangle + c_2(t)|\uparrow n-1\rangle\right)$$

$$= |c_1(t)|^2 \langle \downarrow n| \hat{S}_- \hat{a}^\dagger |\downarrow n\rangle +$$
$$c_1^*(t) c_2(t) \langle \downarrow n| \hat{S}_- \hat{a}^\dagger |\uparrow n-1\rangle +$$
$$c_2^*(t) c_1(t) \langle \uparrow n-1| \hat{S}_- \hat{a}^\dagger |\downarrow n\rangle +$$
$$|c_2(t)|^2 \langle \uparrow n-1| \hat{S}_- \hat{a}^\dagger |\uparrow n-1\rangle =$$

$$= |c_1(t)|^2 \sqrt{n+1} \langle \downarrow n|\emptyset\, n+1\rangle$$
$$c_1^*(t) c_2(t) \sqrt{n} \langle \downarrow n|\downarrow n\rangle$$
$$c_2^*(t) c_1(t) \sqrt{n+1} \langle \uparrow n-1|\emptyset\, n+1\rangle$$
$$|c_2(t)|^2 \sqrt{n} \langle \uparrow n-1|\downarrow n\rangle = c_1^*(t)\, c_2(t) \sqrt{n}$$

$$\hat{H}_R^m = \hbar\omega_m \hat{a}_m^\dagger \hat{a}_m + \hbar\Omega \hat{S}_+\hat{S}_- + \hbar g^m (\hat{S}_+ + \hat{S}_-)(\hat{a}_m^\dagger + \hat{a}_m)$$

Rabi Hamiltonian describing the interaction of a two level atom with a single-mode harmonic field

$\hbar g^m (\hat{S}_+ \hat{a}_m^\dagger + \hat{S}_- \hat{a}_m)$ the so-called counter-rotating terms (αγνοούμε)

$$\hat{H}_{JC}^m = \hbar\omega_m \hat{a}_m^\dagger \hat{a}_m + \hbar\Omega \hat{S}_+\hat{S}_- + \hbar g^m (\hat{S}_+ \hat{a}_m + \hat{S}_- \hat{a}_m^\dagger)$$

Jaynes-Cummings Hamiltonian

Να υπολογιστούν τα $\langle \hat{a}_m^\dagger \hat{a}_m \rangle$, $\langle \hat{S}_+ \hat{S}_- \rangle$, $\langle \hat{S}_+ \hat{a}_m \rangle$, $\langle \hat{S}_- \hat{a}_m^\dagger \rangle$ για τις καταστάσεις

Ⓐ $|\psi_A(t)\rangle = c_1(t)|\downarrow, n\rangle + c_2(t)|\uparrow, n-1\rangle$

Ⓑ $|\psi_B(t)\rangle = c_1(t)|\downarrow, n+1\rangle + c_2(t)|\uparrow, n\rangle$

θα χρησιμοποιηθεί με αρχικές συνθήκες $c_1(0)=1, c_2(0)=0$ και $c_1(0)=0, c_2(0)=1$

Ⓐ

$$\langle \hat{a}_m^\dagger \hat{a}_m \rangle_Ⓐ = \langle \psi_A(t)| \hat{a}_m^\dagger \hat{a}_m |\psi_A(t)\rangle = \{c_1^* \langle \downarrow, n| + c_2^* \langle \uparrow, n-1|\}\{\hat{a}_m^\dagger \hat{a}_m \{c_1|\downarrow, n\rangle + c_2|\uparrow, n-1\rangle\}$$

$$= |c_1|^2 \langle \downarrow, n|\hat{a}_m^\dagger \hat{a}_m|\downarrow, n\rangle + c_1^* c_2 \langle \downarrow, n|\hat{a}_m^\dagger \hat{a}_m|\uparrow, n-1\rangle$$

$$+ c_2^* c_1 \langle \uparrow, n-1|\hat{a}_m^\dagger \hat{a}_m|\downarrow, n\rangle + |c_2|^2 \langle \uparrow, n-1|\hat{a}_m^\dagger \hat{a}_m|\uparrow, n-1\rangle$$

$$= |c_1|^2 \sqrt{n}\sqrt{n} \langle \downarrow, n|\downarrow, n\rangle + c_1^* c_2 \sqrt{n-1}\sqrt{n-1} \langle \downarrow, n|\uparrow, n-1\rangle$$

$$+ c_2^* c_1 \sqrt{n}\sqrt{n} \langle \uparrow, n-1|\downarrow, n\rangle + |c_2|^2 \sqrt{n-1}\sqrt{n-1} \langle \uparrow, n-1|\uparrow, n-1\rangle =$$

$$= n \cdot |c_1|^2 \cdot 1 + c_1^* c_2 (n-1) \cdot 0 + c_2^* c_1 \, n \cdot 0 + (n-1)|c_2|^2 \cdot 1 =$$

$$= n|c_1|^2 + n|c_2|^2 - |c_2|^2 = n(|c_1|^2 + |c_2|^2) - |c_2|^2 = n - |c_2|^2 \Rightarrow$$

$$\boxed{\langle \hat{a}_m^\dagger \hat{a}_m \rangle_Ⓐ = n_m - |c_2(t)|^2} \checkmark$$

$$\langle \hat{S}_+ \hat{S}_- \rangle_Ⓐ = \langle \psi_A(t)| \hat{S}_+ \hat{S}_- |\psi_A(t)\rangle = \{c_1^* \langle \downarrow, n| + c_2^* \langle \uparrow, n-1|\}\hat{S}_+\hat{S}_-\{c_1|\downarrow, n\rangle + c_2|\uparrow, n-1\rangle\}$$

$$= |c_1|^2 \langle \downarrow, n|\hat{S}_+\hat{S}_-|\downarrow, n\rangle + c_1^* c_2 \langle \downarrow, n|\hat{S}_+\hat{S}_-|\uparrow, n-1\rangle$$

$$+ c_2^* c_1 \langle \uparrow, n-1|\hat{S}_+\hat{S}_-|\downarrow, n\rangle + |c_2|^2 \langle \uparrow, n-1|\hat{S}_+\hat{S}_-|\uparrow, n-1\rangle =$$

$$= |c_1|^2 \cdot 0 + c_1^* c_2 \langle \downarrow, n|\uparrow, n-1\rangle + c_2^* c_1 \cdot 0 + |c_2|^2 \langle \uparrow, n-1|\uparrow, n-1\rangle \Rightarrow$$

$$\boxed{\langle \hat{S}_+ \hat{S}_- \rangle_Ⓐ = |c_2(t)|^2} \checkmark$$

ΑΡΑ $\boxed{\langle \hat{a}_m^\dagger \hat{a}_m \rangle_Ⓐ + \langle \hat{S}_+ \hat{S}_- \rangle_Ⓐ = n_m}$ ✓

$$\langle \hat{S}_+ \hat{a}_m \rangle_A = \langle \psi_A(t) | \hat{S}_+ \hat{a}_m | \psi_A(t) \rangle = \{ c_1^* \langle \downarrow, n | + c_2^* \langle \uparrow, n-1 | \} \hat{S}_+ \hat{a}_m \{ c_1 | \downarrow, n \rangle + c_2 | \uparrow, n-1 \}$$

$$= |c_1|^2 \langle \downarrow, n | \hat{S}_+ \hat{a}_m | \downarrow, n \rangle + c_1^* c_2 \langle \downarrow, n | \hat{S}_+ \hat{a}_m | \uparrow, n-1 \rangle +$$

$$+ c_2^* c_1 \langle \uparrow, n-1 | \hat{S}_+ \hat{a}_m | \downarrow, n \rangle + |c_2|^2 \langle \uparrow, n-1 | \hat{S}_+ \hat{a}_m | \uparrow, n-1 \rangle =$$

$$= |c_1|^2 \sqrt{n} \langle \downarrow, n | \uparrow, n-1 \rangle + c_1^* c_2 \cdot \sqrt{n-1} \cdot \langle \downarrow, n | \hat{S}_+ | \uparrow, n-2 \rangle$$

$$+ c_2^* c_1 \sqrt{n} \langle \uparrow, n-1 | \uparrow, n-1 \rangle + |c_2|^2 \sqrt{n-1} \langle \uparrow, n-1 | \hat{S}_+ | \uparrow, n-2 \rangle \Rightarrow$$

$$\boxed{\langle \hat{S}_+ \hat{a}_m \rangle_A = c_2^*(t) c_1(t) \cdot \sqrt{n}}$$

$$\langle \hat{S}_- \hat{a}_m^+ \rangle_A = \langle \psi_A(t) | \hat{S}_- \hat{a}_m^+ | \psi_A(t) \rangle = \{ c_1^* \langle \downarrow, n | + c_2^* | \uparrow, n-1 \rangle \} \{ \hat{S}_- \hat{a}_m^+ \{ c_1 | \downarrow, n \rangle + c_2 | \uparrow, n-1 \rangle \}$$

$$= |c_1|^2 \langle \downarrow, n | \hat{S}_- \hat{a}_m^+ | \downarrow, n \rangle + c_1^* c_2 \langle \downarrow, n | \hat{S}_- \hat{a}_m^+ | \uparrow, n-1 \rangle +$$

$$c_2^* c_1 \langle \uparrow, n-1 | \hat{S}_- \hat{a}_m^+ | \downarrow, n \rangle + |c_2|^2 \langle \uparrow, n-1 | \hat{S}_- \hat{a}_m^+ | \uparrow, n-1 \rangle =$$

$$= |c_1|^2 \sqrt{n+1} \langle \downarrow, n | \hat{S}_- | \downarrow, n+1 \rangle + c_1^* c_2 \sqrt{n} \langle \downarrow, n | \downarrow, n \rangle$$

$$+ c_2^* c_1 \langle \uparrow, n-1 | \hat{S}_- | \downarrow, n+1 \rangle \sqrt{n+1} + |c_2|^2 \langle \uparrow, n-1 | \downarrow, n \rangle \sqrt{n} \Rightarrow$$

$$\boxed{\langle \hat{S}_- \hat{a}_m^+ \rangle_A = c_1^*(t) c_2(t) \sqrt{n}}$$

(E)

$$\langle \hat{a}_m^\dagger \hat{a}_m \rangle_{(E)} = \langle \psi_E(t) | \hat{a}_m^\dagger \hat{a}_m | \psi_E(t) \rangle = \{ c_1^* \langle \downarrow, n+1 | + c_2^* \langle \uparrow, n | \} | \hat{a}_m^\dagger \hat{a}_m | \{ c_1 | \downarrow, n+1 \rangle + c_2 | \uparrow, n \rangle \}$$

$$= |c_1|^2 \langle \downarrow, n+1 | \hat{a}_m^\dagger \hat{a}_m | \downarrow, n+1 \rangle + c_1^* c_2 \langle \downarrow, n+1 | \hat{a}_m^\dagger \hat{a}_m | \uparrow, n \rangle$$

$$+ c_2^* c_1 \langle \uparrow, n | \hat{a}_m^\dagger \hat{a}_m | \downarrow, n+1 \rangle + |c_2|^2 \langle \uparrow, n | a_m^\dagger \hat{a}_m | \uparrow, n \rangle$$

$$= |c_1|^2 \sqrt{n+1}\sqrt{n+1} \langle \downarrow, n+1 | \downarrow, n+1 \rangle + c_1^* c_2 \, n \langle \downarrow, n+1 | \uparrow, n \rangle$$

$$+ c_2^* c_1 (n+1) \langle \uparrow, n | \downarrow, n+1 \rangle + |c_2|^2 n \langle \uparrow, n | \uparrow, n \rangle =$$

$$= |c_1|^2 (n+1) + n |c_2|^2 = n(|c_1|^2 + |c_2|^2) + |c_1|^2 \Rightarrow$$

$$\boxed{\langle \hat{a}_m^\dagger \hat{a}_m \rangle_{(E)} = n + |c_1(t)|^2} \checkmark \quad (4.60)$$

$$\langle \hat{S}_+ \hat{S}_- \rangle_{(E)} = \langle \psi_E(t) | \hat{S}_+ \hat{S}_- | \psi_E(t) \rangle = \{ c_1^* \langle \downarrow, n+1 | + c_2^* \langle \uparrow, n | \} \{ \hat{S}_+ \hat{S}_- \{ c_1 | \downarrow, n+1 \rangle + c_2 | \uparrow, n \rangle \}$$

$$= |c_1|^2 \cdot 0 + c_1^* c_2 \langle \downarrow, n+1 | \uparrow, n \rangle + c_2^* c_1 \cdot 0 + |c_2|^2 \langle \uparrow, n | \uparrow, n \rangle \Rightarrow$$

$$\boxed{\langle \hat{S}_+ \hat{S}_- \rangle_{(E)} = |c_2(t)|^2} \checkmark \quad (4.60)$$

$$\langle \hat{S}_+ \hat{a}_m \rangle_{(E)} = \langle \psi_E(t) | \hat{S}_+ \hat{a}_m | \psi_E(t) \rangle = \{ c_1^* \langle \downarrow, n+1 | + c_2^* \langle \uparrow, n | \} \hat{S}_+ \hat{a}_m \{ c_1 | \downarrow, n+1 \rangle + c_2 | \uparrow, n \rangle \}$$

$$= |c_1|^2 \langle \downarrow, n+1 | \hat{S}_+ \hat{a}_m | \downarrow, n+1 \rangle + c_1^* c_2 \langle \downarrow, n+1 | \hat{S}_+ \hat{a}_m | \uparrow, n \rangle +$$

$$c_2^* c_1 \langle \uparrow, n | \hat{S}_+ \hat{a}_m | \downarrow, n+1 \rangle + |c_2|^2 \langle \uparrow, n | \hat{S}_+ \hat{a}_m | \uparrow, n \rangle$$

$$= |c_1|^2 \langle \downarrow, n+1 | \uparrow, n \rangle \sqrt{n+1} + c_1^* c_2 \cdot 0 + c_2^* c_1 \langle \uparrow, n | \uparrow, n \rangle \sqrt{n+1} + |c_2|^2 \cdot 0$$

$$\boxed{\langle \hat{S}_+ \hat{a}_m \rangle_{(E)} = c_2^*(t) \, c_1(t) \cdot \sqrt{n+1}}$$

$$\langle \hat{S}_- \hat{a}_m^\dagger \rangle_{(E)} = \langle \psi_E(t) | \hat{S}_- \hat{a}_m^\dagger | \psi_E(t) \rangle = \{ c_1^* \langle \downarrow, n+1 | + c_2^* \langle \uparrow, n | \} \hat{S}_- \hat{a}_m^\dagger \{ c_1 | \downarrow, n+1 \rangle + c_2 | \uparrow, n \rangle \}$$

$$= |c_1|^2 \langle \downarrow, n+1 | \hat{S}_- \hat{a}_m^\dagger | \downarrow, n+1 \rangle + c_1^* c_2 \langle \downarrow, n+1 | \hat{S}_- \hat{a}_m^\dagger | \uparrow, n \rangle$$

$$+ c_2^* c_1 \langle \uparrow, n | \hat{S}_- \hat{a}_m | \downarrow, n+1 \rangle + |c_2|^2 \langle \uparrow, n | \hat{S}_- \hat{a}_m^\dagger | \uparrow, n \rangle =$$

$$= |c_1|^2 \langle \downarrow, n+1 | \uparrow, n+2 \rangle \sqrt{n+2} + c_1^* c_2 \langle \downarrow, n+1 | \downarrow, n+1 \rangle \sqrt{n+1}$$

$$+ c_2^* c_1 \cdot 0 + |c_2|^2 \langle \uparrow, n | \downarrow, n+1 \rangle \sqrt{n+1} \Rightarrow$$

$$\boxed{\langle \hat{S}_- \hat{a}_m^\dagger \rangle_{(E)} = c_1^*(t) \cdot c_2(t) \cdot \sqrt{n+1}}$$

APA $\boxed{\langle \hat{a}_m^\dagger \hat{a}_m \rangle_{(E)} + \langle \hat{S}_+ \hat{S}_- \rangle_{(E)} = n+1}$ ✓ (4.61)