Τομέας Φυσικής Στερεάς Κατάστασης, Τμήμα Φυσικής, Εθνικό και Καποδιστριακό Πανεπιστήμιο Αθηνών.

Κβαντική Οπτική και Lasers.

Εξέταση της 28ης Ιουλίου 2020. Διδάσκων Κ. Σιμσερίδης

Θέμα 1.

Δίνεται η κατάσταση $|\psi_A(t)> = c_1(t)\,|\downarrow 1> \;+\; c_2(t)|\uparrow 0>$.

Να λυθεί το χρονοεξαρτημένο πρόβλημα $i\,\hbar \frac{\partial|\psi_A(t)>}{\partial t} = \hat{H}\;|\psi_A(t)>$,

για τη Χαμιλτονιανή Jaynes-Cummings, $\hat{H} = \hat{H}_{JC} = \hbar\omega\hat{a}^\dagger\hat{a} \;+\; \hbar\Omega\hat{S}_+\hat{S}_- \;+\; \hbar g\left(\hat{S}_+\,\hat{a} \;+\; \hat{S}_-\,\hat{a}^\dagger\right)$,

με αρχικές συνθήκες $c_1(0) = 1,\; c_2(0) = 0$.

Δηλαδή βρείτε τις ταλαντώσεις που κάνουν τα

$|c_1(t)|^2, |c_2(t)|^2, \;< \hat{a}^\dagger\hat{a} >, <\hat{S}_+\hat{S}_- >$,

ζωγραφίστε τες,

βρείτε την αντίστοιχη περίοδο T_R,

μέγιστο ποσοστό μεταβιβάσεως A_R, και

τις μέσες τιμές των $< \hat{a}^\dagger\hat{a} >, <\hat{S}_+\hat{S}_- >$, και

το μέσο ρυθμό μεταβιβάσεως k από την κάτω στάθμη στην άνω στάθμη.

ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ	ΕΠΩΝΥΜΟ..
ΤΟΜΕΑΣ ΦΥΣΙΚΗΣ	ΟΝΟΜΑ....................................ΑΜ..............
ΣΤΕΡΕΑΣ ΚΑΤΑΣΤΑΣΗΣ	ΜΑΘΗΜΑ..

$|\psi_A(t)\rangle = c_1(t)\,|\downarrow 1\rangle + c_2(t)\,|\uparrow 0\rangle \qquad c_1(0)=1 \qquad c_2(0)=0$

$i\hbar \frac{\partial |\psi_A(t)\rangle}{\partial t} = \hat{H}\,|\psi_A(t)\rangle \qquad \hat{H} = H_{JC} = \hbar\omega \hat{a}^{+}\hat{a} + \hbar\Omega \hat{S}_{+}\hat{S}_{-} + \hbar g(\hat{S}_{+}\hat{a} + \hat{S}_{-}\hat{a}^{+})$

$A' \quad i\hbar \frac{\partial |\psi_A(t)\rangle}{\partial t} = i\hbar\,\dot{c}_1(t)\,|\downarrow 1\rangle + i\hbar\,\dot{c}_2(t)\,|\uparrow 0\rangle$

$$\begin{aligned}
\Delta' = \hat{H}|\psi_A(t)\rangle = {} & \hbar\omega \hat{a}^{+}\hat{a}\,c_1(t)\,|\downarrow 1\rangle + \hbar\omega \hat{a}^{+}\hat{a}\,c_2(t)\,|\uparrow 0\rangle + \\
& \hbar\Omega \hat{S}_{+}\hat{S}_{-}\,c_1(t)\,|\downarrow 1\rangle + \hbar\Omega \hat{S}_{+}\hat{S}_{-}\,c_2(t)\,|\uparrow 0\rangle + \\
& \hbar g\,\hat{S}_{+}\hat{a}\,c_1(t)\,|\downarrow 1\rangle + \hbar g\,\hat{S}_{+}\hat{a}\,c_2(t)\,|\uparrow 0\rangle + \\
& \hbar g\,\hat{S}_{-}\hat{a}^{+}\,c_1(t)\,|\downarrow 1\rangle + \hbar g\,\hat{S}_{-}\hat{a}^{+}\,c_2(t)\,|\uparrow 0\rangle = \\
= {} & \hbar\omega \cdot 1\,c_1(t)\,|\downarrow 1\rangle + \hbar\omega \cdot 0 \cdot c_2(t)\,|\uparrow 0\rangle + \\
& \hbar\Omega\,c_1(t)\,|0\,1\rangle + \hbar\Omega\,c_2(t)\,|\uparrow 0\rangle + \\
& \hbar g \cdot c_1(t)\,1\,|\uparrow 0\rangle + \hbar g\,c_2(t) \cdot 0 + \\
& \hbar g\,c_1(t)\,\sqrt{2}\,|0\,2\rangle + \hbar g\,c_2(t)\,1\,|\downarrow 1\rangle
\end{aligned}$$

επί $\langle\downarrow 1|$ $\quad \left.\begin{aligned} A' &= i\hbar\,\dot{c}_1(t) \\ \Delta' &= \hbar\omega\,c_1(t) + \hbar g\,c_2(t) \end{aligned}\right\} \quad i\,\dot{c}_1(t) = \omega\,c_1(t) + g\,c_2(t)$

επί $\langle\uparrow 0|$ $\quad \left.\begin{aligned} A' &= i\hbar\,\dot{c}_2(t) \\ \Delta' &= \hbar\Omega\,c_2(t) + \hbar g\,c_1(t) \end{aligned}\right\} \quad i\,\dot{c}_2(t) = \Omega\,c_2(t) + g\,c_1(t)$

$$i \begin{bmatrix} \dot{c}_1(t) \\ \dot{c}_2(t) \end{bmatrix} = \begin{bmatrix} \omega & g \\ g & \Omega \end{bmatrix} \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}$$

$i\,\dot{\vec{x}}(t) = A\,\vec{x}(t)$

$\dot{\vec{x}}(t) = -iA\,\vec{x}(t)$

ΔΛΜ $\vec{x}(t) = \vec{u}\,e^{-i\lambda t}$

$i\,\dot{\vec{x}}(t) = A\,\vec{x}(t)$

$\vec{u}\,(-i\lambda)\,e^{-i\lambda t} = -iA\vec{u}\,e^{-i\lambda t}$

$A\vec{u} = \lambda\vec{u}$

$A\vec{u} = \lambda\vec{u} \Leftrightarrow (A - \lambda I)\vec{u} = \vec{0}$ πρέπει $\det(A - \lambda I) = 0$

$$A-\lambda I = \begin{bmatrix} \omega-\lambda & g \\ g & \Omega-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (\omega-\lambda)(\Omega-\lambda) - g^2 = 0 \Longrightarrow$$

$$\lambda^2 - (\omega+\Omega)\lambda + \omega\Omega - g^2 = 0$$

$$\Delta' = (\omega+\Omega)^2 - 4(\omega\Omega - g^2) = \omega^2 + \Omega^2 + 2\omega\Omega - 4\omega\Omega + 4g^2$$

διακρίνουσα $\Delta' = (\omega-\Omega)^2 + 4g^2 \Longrightarrow \Delta' = \Delta^2 + 4g^2$

$$\lambda_{2,1} = \frac{(\omega+\Omega) \pm \sqrt{\Delta^2+4g^2}}{2}$$

$$\lambda_{2,1} = \frac{\omega+\Omega}{2} \pm \sqrt{\left(\frac{\Delta}{2}\right)^2 + g^2}$$

$$\lambda_{2,1} = H_1 \pm \Omega_1 \qquad H_1 = \frac{\omega+\Omega}{2}$$

ιδιοτιμές

$$\Omega_1 = \sqrt{\left(\frac{\Delta}{2}\right)^2 + g^2}$$

ιδιοανύσματα

• για $\lambda_1 = H_1 - \Omega_1$ $\quad A\vec{U}_1 = \lambda_1 \vec{U}_1 \Longrightarrow \begin{bmatrix} \omega & g \\ g & \Omega \end{bmatrix} \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix} = (H_1 - \Omega_1) \begin{bmatrix} U_{11} \\ U_{21} \end{bmatrix}$

$$\omega U_{11} + g U_{21} = (H_1 - \Omega_1) U_{11} \qquad g U_{21} = (H_1 - \Omega_1 - \omega) U_{11}$$

$$g U_{11} + \Omega U_{21} = (H_1 - \Omega_1) U_{21} \qquad g U_{11} = (H_1 - \Omega_1 - \Omega) U_{21}$$

$$g^2 U_{21} = (H_1 - \Omega_1 - \omega)(H_1 - \Omega_1 - \Omega) U_{21} \Longrightarrow U_{21} = 0 \text{ ή}$$

$$g^2 = (H_1 - \Omega_1 - \omega)(H_1 - \Omega_1 - \Omega)$$

αλλά $U_{21} = 0 \Longrightarrow U_{11} = 0 \Longrightarrow \vec{U}_1 = \vec{0}$

επω $g^2 = (H_1 - \Omega_1 - \omega)(H_1 - \Omega_1 - \Omega) \Longrightarrow$

$$g^2 = \left(\frac{\omega+\Omega}{2} - \Omega_1 - \omega\right)\left(\frac{\omega+\Omega}{2} - \Omega_1 - \Omega\right)$$

$$g^2 = \left(\frac{\Omega-\omega}{2} - \Omega_1\right)\left(\frac{\omega-\Omega}{2} - \Omega_1\right) = -\left(\frac{\Delta}{2} + \Omega_1\right)\left(\frac{\Delta}{2} - \Omega_1\right)$$

$$g^2 = -\left[\frac{\Delta^2}{4} - \Omega_1^2\right] \Longrightarrow \Omega_1^2 = \left(\frac{\Delta}{2}\right)^2 + g^2$$ Το οποίο ισχύει, εξ ορισμού της Ω_1

δηλαδή το U_{21} μπορεί να είναι ότιδήποτε μη μηδενικό π.χ. ας θέσουμε $U_{21}=1$

$$\Rightarrow gU_{11} = (H_1 - \Omega_1 - \Omega) = \frac{\omega+\Omega}{2} - \Omega_1 - \Omega = \frac{\Delta}{2} - \Omega_1 = \frac{\Delta - 2\Omega_1}{2}$$

$$\Rightarrow U_{11} = \frac{\Delta - 2\Omega_1}{2g} \qquad \vec{U}_1 = \begin{bmatrix} \frac{\Delta - 2\Omega_1}{2g} \\ 1 \end{bmatrix}$$

• για $\lambda_2 = H_1 + \Omega_1$ $\quad A\vec{U}_2 = \lambda_2 \vec{U}_2 \Rightarrow \begin{bmatrix} \omega & g \\ g & \Omega \end{bmatrix} \begin{bmatrix} U_{12} \\ U_{22} \end{bmatrix} = (H_1+\Omega_1) \begin{bmatrix} U_{12} \\ U_{22} \end{bmatrix}$

$$\left.\begin{aligned} \omega U_{12} + gU_{22} &= (H_1+\Omega_1)U_{12} \\ gU_{12} + \Omega U_{22} &= (H_1+\Omega_1)U_{22} \end{aligned}\right\} \Rightarrow$$

$$\begin{aligned} gU_{22} &= (H_1+\Omega_1-\omega)U_{12} \\ gU_{12} &= (H_1+\Omega_1-\Omega)U_{22} \end{aligned} \Rightarrow g^2 U_{22} = (H_1+\Omega_1-\omega)(H_1+\Omega_1-\Omega)U_{22}$$

$$\Rightarrow U_{22} = 0 \;\; \text{ή} \;\; g^2 = (H_1+\Omega_1-\omega)(H_1+\Omega_1-\Omega)$$

αλλά $U_{22} = 0 \Rightarrow U_{12} = 0 \Rightarrow \vec{U}_2 = \vec{0}$

ενώ $g^2 = (H_1+\Omega_1-\omega)(H_1+\Omega_1-\Omega) \Rightarrow g^2 = \left(\frac{\Omega-\omega}{2} + \Omega_1\right)\left(\frac{\omega-\Omega}{2} + \Omega_1\right)$

$$\Rightarrow g^2 = \left(\Omega_1 - \frac{\Delta}{2}\right)\left(\Omega_1 + \frac{\Delta}{2}\right) \Rightarrow g^2 = \Omega_1^2 - \frac{\Delta^2}{4} \Rightarrow$$

$$\Rightarrow \Omega_1^2 = \left(\frac{\Delta}{2}\right)^2 + g^2$$ το οποίο ισχύει, εξ ορισμού της Ω_1

δηλαδή το U_{22} μπορεί να είναι ότιδήποτε μη μηδενικό, για ευκολία ας θέσουμε κι εδώ $U_{22} = 1 \Rightarrow$

$$U_{12} = \frac{H_1+\Omega_1-\Omega}{g} = \frac{\frac{\omega-\Omega}{2}+\Omega_1}{g} = \frac{\Delta+2\Omega_1}{2g}$$

$$\vec{U}_2 = \begin{bmatrix} \frac{\Delta+2\Omega_1}{2g} \\ 1 \end{bmatrix}$$

Η γενική λύση είναι $\vec{x}(t) = \sigma_1 \vec{u}_1 e^{-i\lambda_1 t} + \sigma_2 \vec{u}_2 e^{-i\lambda_2 t}$

$$\vec{x}(t) = \begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix} = \begin{bmatrix} \sigma_1 \cdot \frac{\Delta - 2\Omega_1}{2g} e^{-i(H_1-\Omega_1)t} + \sigma_2 \cdot \frac{\Delta + 2\Omega_1}{2g} e^{-i(H_1+\Omega_1)t} \\ \sigma_1 \cdot 1 \cdot e^{-i(H_1-\Omega_1)t} + \sigma_2 \cdot 1 \cdot e^{-i(H_1+\Omega_1)t} \end{bmatrix}$$

Αρχικές Συνθήκες $c_1(0)=1$, $c_2(0)=0 \Rightarrow$

$$\left.\begin{aligned} 1 &= \sigma_1 \frac{\Delta - 2\Omega_1}{2g} + \sigma_2 \frac{\Delta + 2\Omega_1}{2g} \\ 0 &= \sigma_1 + \sigma_2 \Rightarrow \sigma_2 = -\sigma_1 = -\sigma \end{aligned}\right\} \quad 2g = \cancel{\sigma\Delta} - \sigma 2\Omega_1 - \cancel{\sigma\Delta} - \sigma 2\Omega_1$$

$$\Rightarrow g = -2\sigma\Omega_1 \Rightarrow$$

$$\sigma_1 = \sigma = -\frac{g}{2\Omega_1} = -\sigma_2$$

Άρα

$$c_2(t) = -\frac{g}{2\Omega_1} e^{-i(H_1-\Omega_1)t} + \frac{g}{2\Omega_1} e^{-i(H_1+\Omega_1)t}$$

$$c_2(t) = -\frac{g}{2\Omega_1} e^{-iH_1 t} e^{i\Omega_1 t} + \frac{g}{2\Omega_1} e^{-iH_1 t} e^{-i\Omega_1 t}$$

$$c_2(t) = e^{-iH_1 t} \frac{g}{2\Omega_1}\left(e^{-i\Omega_1 t} - e^{i\Omega_1 t}\right)$$

$$c_2(t) = e^{-iH_1 t} \frac{g}{2\Omega_1}(-2i)\sin(\Omega_1 t) \Rightarrow c_2(t) = \frac{(-i)g}{\Omega_1} e^{-iH_1 t} \sin(\Omega_1 t)$$

$$|c_2(t)|^2 = \frac{g^2}{\Omega_1^2}\sin^2(\Omega_1 t) \qquad \frac{g^2}{\Omega_1^2} = \frac{g^2}{g^2 + \frac{\Delta^2}{4}} = \frac{4g^2}{\Delta^2 + 4g^2}$$

$$|c_1(t)|^2 = 1 - |c_2(t)|^2 = 1 - \frac{g^2}{\Omega_1^2}\left(1 - \cos^2(\Omega_1 t)\right) = \frac{\Omega_1^2 - g^2}{\Omega_1^2} + \frac{g^2\cos^2(\Omega_1 t)}{\Omega_1^2}$$

$$|c_1(t)|^2 = \frac{\Omega_1^2 - g^2}{\Omega_1^2} + \frac{g^2}{\Omega_1^2}\cos^2(\Omega_1 t) \qquad \Omega_1^2 - g^2 = \frac{\Delta^2}{4}$$

$$\frac{\Omega_1^2 - g^2}{\Omega_1^2} = \frac{\frac{\Delta^2}{4}}{g^2 + \frac{\Delta^2}{4}} = \frac{\Delta^2}{\Delta^2 + 4g^2}$$

ΠΑΝΕΠΙΣΤΗΜΙΟ ΑΘΗΝΩΝ	ΕΠΩΝΥΜΟ..
ΤΟΜΕΑΣ ΦΥΣΙΚΗΣ	ΟΝΟΜΑ...ΑΜ.............
ΣΤΕΡΕΑΣ ΚΑΤΑΣΤΑΣΗΣ	ΜΑΘΗΜΑ..

$$|c_1(t)|^2 = \frac{\Delta^2}{\Delta^2+4g^2} + \frac{4g^2}{\Delta^2+4g^2}\cos^2(\Omega_1 t)$$

$$|c_2(t)|^2 = \frac{4g^2}{\Delta^2+4g^2}\sin^2(\Omega_1 t)$$

$$\cos^2(\Omega_1 t) = \frac{1+\cos(2\Omega_1 t)}{2}$$

$$T_R = \frac{2\pi}{2\Omega_1} = \frac{\pi}{\Omega_1}$$

$$\sin^2(\Omega_1 t) = \frac{1-\cos(2\Omega_1 t)}{2}$$

$$|c_1(t)|^2 = \frac{\Delta^2}{\Delta^2+4g^2} + \frac{4g^2}{2(\Delta^2+4g^2)} + \frac{4g^2}{2(\Delta^2+4g^2)}\cos(2\Omega_1 t)$$

$$|c_2(t)|^2 = \frac{4g^2}{2(\Delta^2+4g^2)} - \frac{4g^2}{2(\Delta^2+4g^2)}\cos(2\Omega_1 t)$$

$$|c_1(t)|^2 + |c_2(t)|^2 = 1$$

$$\langle \hat{a}^\dagger \hat{a}\rangle = \left(c_1^*(t)\langle\downarrow 1| + c_2^*(t)\langle\uparrow 0|\right)\hat{a}^\dagger\hat{a}\left(c_1(t)|\downarrow 1\rangle + c_2(t)|\uparrow 0\rangle\right)$$

$$\left(c_1^*(t)\langle\downarrow 1| + c_2^*(t)\langle\uparrow 0|\right)\left(c_1(t)\cdot 1|\downarrow 1\rangle + c_2(t)\,0\,|\uparrow 0\rangle\right)$$

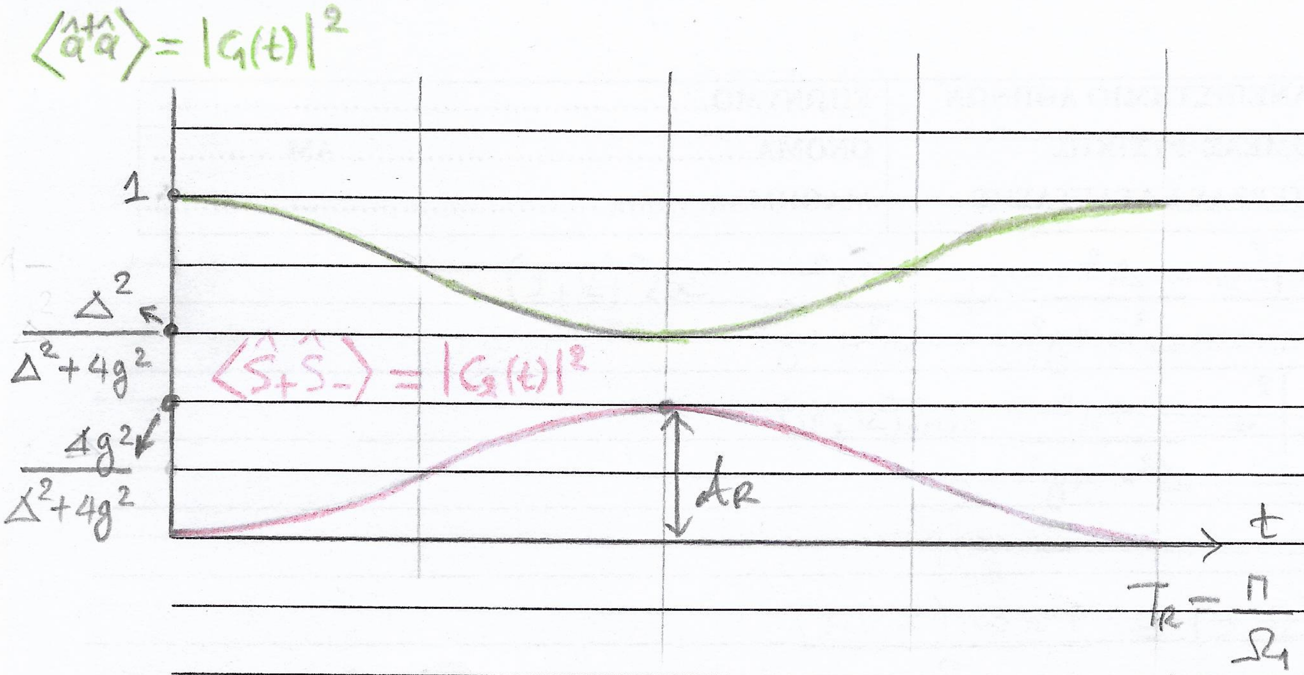
$$\langle \hat{a}^\dagger \hat{a}\rangle = |c_1(t)|^2 = 1 - |c_2(t)|^2$$

$$\langle \hat{S}_+\hat{S}_-\rangle = \left(c_1^*(t)\langle\downarrow 1| + c_2^*(t)\langle\uparrow 0|\right)\hat{S}_+\hat{S}_-\left(c_1(t)|\downarrow 1\rangle + c_2(t)|\uparrow 0\rangle\right)$$

$$\left(c_1^*(t)\langle\downarrow 1| + c_2^*(t)\langle\uparrow 0|\right)\left(c_1(t)\,0\,|\downarrow 1\rangle + c_2(t)\cdot 1|\uparrow 0\rangle\right)$$

$$\langle \hat{S}_+\hat{S}_-\rangle = |c_2(t)|^2$$

$$\langle \hat{a}^\dagger \hat{a}\rangle + \langle \hat{S}_+\hat{S}_-\rangle = 1$$



$$\Omega_1 = \sqrt{\left(\frac{\Delta}{2}\right)^2 + g^2}$$

$$A_R = \frac{4g^2}{\Delta^2+4g^2}$$

$$\langle\langle \hat a^{\dagger}\hat a\rangle\rangle = \langle |c_1(t)|^2\rangle = \frac{\Delta^2}{\Delta^2+4g^2} + \frac{4g^2}{2(\Delta^2+4g^2)}$$

$$\langle\langle \hat S_+ \hat S_-\rangle\rangle = \langle |c_2(t)|^2\rangle = \frac{4g^2}{2(\Delta^2+4g^2)}$$

κ το άθροισμά τους, κάνει, φυσικά, μονάδα.

t_{2mean}: $\langle |c_2(t)|^2\rangle = |c_2(t)|^2$ για πρώτη φορά

$$\Rightarrow \frac{4g^2}{2(\Delta^2+4g^2)} = \frac{4g^2}{2(\Delta^2+4g^2)} - \frac{4g^2}{2(\Delta^2+4g^2)}\cos(2\Omega_1 t_{2mean})$$

$\Rightarrow \cos(2\Omega_1 t_{2mean}) = 0$ για πρώτη φορά $\Rightarrow 2\Omega_1 t_{2mean} = \frac{\pi}{2}$

$$t_{2mean} = \frac{\pi}{4\Omega_1}$$

$$k = \frac{\langle |c_2(t)|^2 \rangle}{t_{2mean}} = \frac{\dfrac{4g^2}{2(\Delta^2+4g^2)}}{\dfrac{\pi}{4\Omega_1}} = \frac{16g^2\Omega_1}{2\pi(\Delta^2+4g^2)} = \frac{4g^2\sqrt{\Delta^2+4g^2}}{\pi(\Delta^2+4g^2)} = \frac{4g^2}{\pi\sqrt{\Delta^2+4g^2}}$$

ΕΛΕΓΧΟΣ 1

για $\Delta = 0$ $\quad k = \frac{16g^2 g}{2\pi \cdot 4g^2} = \frac{2}{\pi} g$

κι επειδή $\Omega_R = 2\sqrt{n}\, g \Rightarrow \Omega_R = 2g$

όποτε $k = \frac{\Omega_R}{\pi}$

ΕΛΕΓΧΟΣ 2

$$k_{ΗΜΙΚΛΑΣΣΙΚΟ} = \frac{\Omega_R^2}{\pi\sqrt{\Omega_R^2+\Delta^2}} = \frac{4g^2}{\pi\sqrt{4g^2+\Delta^2}} = \frac{4g^2}{\pi\sqrt{\Delta^2+4g^2}}$$

$$\Omega_1 = \sqrt{\frac{\Delta^2}{4}+g^2} \Rightarrow 2\Omega_1 = \sqrt{\Delta^2+4g^2}$$