

$$E_m = \frac{M_m \omega_m^2}{2} q_m^2 + \frac{M_m}{2} \dot{q}_m^2 = \frac{M_m \omega_m^2}{2} q_m^2 + \frac{p_m^2}{2M_m}$$

Ενέργεια  
m τρόπου

$q_m$  γεν. δέση  
 $\dot{q}_m$  γεν. ταχύτητα  
 $p_m$  γεν. όρμη

$$\hat{H}_{HM,m} = \frac{M_m \omega_m^2}{2} \hat{q}_m^2 + \frac{\hat{p}_m^2}{2M_m}$$

Χαμιλτονιανή m τρόπου

$$E_{m,n_m} = \hbar \omega_m \left( n_m + \frac{1}{2} \right)$$

Ιδιοενέργειες m τρόπου

$m \in \mathbb{N}^*$

$n_m \in \mathbb{N}$

$$\hat{q}_m = q_m$$

$$\hat{p}_m = -i\hbar \frac{\partial}{\partial q_m}$$

$$[\hat{q}_m, \hat{p}_m] = i\hbar$$

Εισάγουμε τους μετασχηματισμούς

$$\hat{a}_m = \frac{1}{\sqrt{2M_m \hbar \omega_m}} (M_m \omega_m \hat{q}_m + i \hat{p}_m)$$

"καταστροφής"

$$\hat{a}_m^\dagger = \frac{1}{\sqrt{2M_m \hbar \omega_m}} (M_m \omega_m \hat{q}_m - i \hat{p}_m)$$

"δημιουργίας"

Ποσότητα ή ιδιότητα

$$[\hat{a}_m, \hat{a}_m^\dagger] = \hat{a}_m \hat{a}_m^\dagger - \hat{a}_m^\dagger \hat{a}_m = 1$$

Από

$$[\hat{a}, \hat{a}^\dagger] = \hat{a} \hat{a}^\dagger - \hat{a}^\dagger \hat{a} = \frac{1}{2M\hbar\omega} (M\omega\hat{q} + i\hat{p})(M\omega\hat{q} - i\hat{p}) - \frac{1}{2M\hbar\omega} (M\omega\hat{q} - i\hat{p})(M\omega\hat{q} + i\hat{p}) =$$

$$= \frac{1}{2M\hbar\omega} \left( M^2 \omega^2 \hat{q}^2 + \hat{p}^2 - M\omega\hat{q}i\hat{p} + i\hat{p}M\omega\hat{q} - M^2 \omega^2 \hat{q}^2 - \hat{p}^2 - M\omega\hat{q}i\hat{p} + i\hat{p}M\omega\hat{q} \right)$$

$$= \frac{1}{2M\hbar\omega} (-2M\omega i \hat{q}\hat{p} + 2i\hat{p}M\omega\hat{q}) = \frac{1}{\hbar} (-i\hat{q}\hat{p} + i\hat{p}\hat{q})$$

$$= \frac{1}{\hbar} (-i)(\hat{q}\hat{p} - \hat{p}\hat{q}) = \frac{-i}{\hbar} [\hat{q}, \hat{p}] = \frac{-i}{\hbar} i\hbar = 1 \Rightarrow [\hat{a}, \hat{a}^\dagger] = 1$$

$$\hat{a}_m^\dagger + \hat{a}_m = \frac{1}{\sqrt{2M\omega\hbar}} 2M\omega\hat{q}_m = \sqrt{\frac{2M\omega\hbar}{\hbar}} \hat{q}_m \Rightarrow$$

$$\hat{q}_m = \sqrt{\frac{\hbar}{2M\omega}} (\hat{a}_m^\dagger + \hat{a}_m)$$

$$\hat{a}_m^\dagger - \hat{a}_m = \frac{1}{\sqrt{2M\omega\hbar}} (-2i)\hat{p}_m = (-i)\sqrt{\frac{2}{M\omega\hbar}} \hat{p}_m \Rightarrow$$

$$\hat{p}_m = i\sqrt{\frac{M\omega\hbar}{2}} (\hat{a}_m^\dagger - \hat{a}_m)$$

APA

$$\hat{H}_{HM,m} = \frac{M\omega^2}{2} \hat{q}_m^2 + \frac{\hat{p}_m^2}{2M} = \frac{M\omega^2}{2} \frac{\hbar}{2M\omega} (\hat{a}_m^\dagger + \hat{a}_m)^2 (\hat{a}_m^\dagger + \hat{a}_m)$$

$$+ \frac{1}{2M} (-1) \frac{M\omega\hbar}{2} (\hat{a}_m^\dagger - \hat{a}_m)^2 (\hat{a}_m^\dagger - \hat{a}_m)$$

$$= \frac{\hbar\omega}{4} (\hat{a}_m^\dagger \hat{a}_m + \hat{a}_m^\dagger \hat{a}_m + \hat{a}_m^\dagger \hat{a}_m + \hat{a}_m^\dagger \hat{a}_m - \hat{a}_m^\dagger \hat{a}_m + \hat{a}_m^\dagger \hat{a}_m + \hat{a}_m^\dagger \hat{a}_m - \hat{a}_m^\dagger \hat{a}_m)$$

$$\hat{H}_{HM,m} = \frac{\hbar\omega}{4} (2\hat{a}_m^\dagger \hat{a}_m + 2\hat{a}_m \hat{a}_m^\dagger) = \frac{\hbar\omega}{2} (\hat{a}_m^\dagger \hat{a}_m + \hat{a}_m \hat{a}_m^\dagger)$$

Alle  $[\hat{a}_m, \hat{a}_m^\dagger] = \hat{a}_m \hat{a}_m^\dagger - \hat{a}_m^\dagger \hat{a}_m = 1 \Rightarrow \hat{a}_m \hat{a}_m^\dagger = 1 + \hat{a}_m^\dagger \hat{a}_m$

$$\hat{H}_{HM,m} = \frac{\hbar\omega}{2} (2\hat{a}_m^\dagger \hat{a}_m + 1) \Rightarrow$$

$$\hat{H}_{HM,m} = \hbar\omega \left( \hat{a}_m^\dagger \hat{a}_m + \frac{1}{2} \right)$$

ergibt

$$E_{n,m} = \hbar\omega \left( n_m + \frac{1}{2} \right)$$



Αν θεωρήσουμε  $|n_m\rangle$  την κατάσταση του ΗΜ πεδίου

με  $n_m$  αριθμό φωτονίων στον ΗΜ τρόπο  $m$



$$\hat{H}_{HM,m} |n_m\rangle = E_{m,n_m} |n_m\rangle$$

$$\hbar\omega_m \left( \hat{a}_m^\dagger \hat{a}_m + \frac{1}{2} \right) |n_m\rangle = \hbar\omega_m \left( n_m + \frac{1}{2} \right) |n_m\rangle$$

$$\hat{a}_m^\dagger \hat{a}_m |n_m\rangle = n_m |n_m\rangle$$

Άρα ο τελεστής  $\hat{N}_m = \hat{a}_m^\dagger \hat{a}_m$  μετρά τον αριθμό των φωτονίων στον ΗΜ τρόπο  $m$ .

$$[A+B, C] = [A, C] + [B, C]$$

$$[AB, C] = A[B, C] + [A, C]B$$

$$\hat{H} = \kappa \hat{q}^2 + \lambda \hat{p}^2$$

$$\hat{a} = \mu \hat{q} + \nu \hat{p}$$

$$\hat{a}^\dagger = \gamma \hat{q} - \nu \hat{p}$$

Με τη βοήθεια των παραπάνω σχέσεων μπορεί να αποδειχτεί ότι

$$[\hat{H}, \hat{a}] = -\hbar\omega \hat{a}$$

$$[\hat{H}, \hat{a}^\dagger] = \hbar\omega \hat{a}^\dagger$$

$$\begin{aligned} \hat{H} \hat{a} |n\rangle - \hat{a} \hat{H} |n\rangle &= -\hbar\omega \hat{a} |n\rangle \\ \hat{H} \hat{a} |n\rangle - E_n \hat{a} |n\rangle &= -\hbar\omega \hat{a} |n\rangle \end{aligned}$$

$$\hat{H} \hat{a} |n\rangle = (E_n - \hbar\omega) \hat{a} |n\rangle$$

Ιδιοκατάσταση με ενέργεια κατεβαγμένη κατά  $\hbar\omega$  (ένα φωτόνιο λιγότερο)

$$\Rightarrow \hat{a} |n\rangle = \xi |n-1\rangle$$

$$\begin{aligned} \hat{H} \hat{a}^\dagger |n\rangle - \hat{a}^\dagger \hat{H} |n\rangle &= \hbar\omega \hat{a}^\dagger |n\rangle \\ \hat{H} \hat{a}^\dagger |n\rangle - E_n \hat{a}^\dagger |n\rangle &= \hbar\omega \hat{a}^\dagger |n\rangle \end{aligned}$$

$$\hat{H} \hat{a}^\dagger |n\rangle = (E_n + \hbar\omega) \hat{a}^\dagger |n\rangle$$

Ιδιοκατάσταση με ενέργεια ανεβασμένη κατά  $\hbar\omega$  (ένα φωτόνιο περισσότερο)

$$\Rightarrow \hat{a}^\dagger |n\rangle = \rho |n+1\rangle$$

... πράξεις

$$[\hat{H}, \hat{a}] = -\hbar\omega \hat{a}$$

$$[\hat{H}, \hat{a}] |n\rangle = -\hbar\omega \hat{a} |n\rangle$$

$$\hat{H} \hat{a} |n\rangle - \hat{a} \hat{H} |n\rangle = -\hbar\omega \hat{a} |n\rangle$$

$$\hat{H} \hat{a} |n\rangle - \hat{a} E_n |n\rangle = -\hbar\omega \hat{a} |n\rangle$$

$$\hat{H} \hat{a} |n\rangle = (E_n - \hbar\omega) \hat{a} |n\rangle$$



? ιδιοκατάσταση με ενέργεια  
κατεβαμένη κατά  $\hbar\omega$   
(ένα φωτόνιο λιγότερο)

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle$$

$$[\hat{H}, \hat{a}^\dagger] = \hbar\omega \hat{a}^\dagger$$

$$[\hat{H}, \hat{a}^\dagger] |n\rangle = \hbar\omega \hat{a}^\dagger |n\rangle$$

$$\hat{H} \hat{a}^\dagger |n\rangle - \hat{a}^\dagger \hat{H} |n\rangle = \hbar\omega \hat{a}^\dagger |n\rangle$$

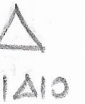
$$\hat{H} \hat{a}^\dagger |n\rangle - \hat{a}^\dagger E_n |n\rangle = \hbar\omega \hat{a}^\dagger |n\rangle$$

$$\hat{H} \hat{a}^\dagger |n\rangle = (E_n + \hbar\omega) \hat{a}^\dagger |n\rangle$$



ιδιοκατάσταση με ενέργεια  
άνεβαμένη κατά  $\hbar\omega$   
(ένα φωτόνιο περισσότερο)

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$



$$\begin{aligned}
 [\hat{H}, \hat{a}] &= [\hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}), \hat{a}] = \hbar\omega [\hat{a}^\dagger\hat{a} + \frac{1}{2}, \hat{a}] = \\
 &= \hbar\omega \left( [\hat{a}^\dagger\hat{a}, \hat{a}] + [\frac{1}{2}, \hat{a}] \right) = \hbar\omega [\hat{a}^\dagger\hat{a}, \hat{a}] = \\
 &= \hbar\omega \left( \underset{-1}{[\hat{a}^\dagger, \hat{a}]\hat{a}} + \underset{0}{\hat{a}^\dagger[\hat{a}, \hat{a}]} \right) = -\hbar\omega\hat{a}
 \end{aligned}$$

$$\begin{aligned}
 [\hat{H}, \hat{a}^\dagger] &= [\hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}), \hat{a}^\dagger] = \hbar\omega [\hat{a}^\dagger\hat{a} + \frac{1}{2}, \hat{a}^\dagger] = \\
 &= \hbar\omega \left( [\hat{a}^\dagger\hat{a}, \hat{a}^\dagger] + [\frac{1}{2}, \hat{a}^\dagger] \right) = \hbar\omega [\hat{a}^\dagger\hat{a}, \hat{a}^\dagger] \\
 &= \hbar\omega \left( \hat{a}^\dagger[\hat{a}, \hat{a}^\dagger] + [\hat{a}^\dagger, \hat{a}^\dagger]\hat{a} \right) = \hbar\omega\hat{a}^\dagger
 \end{aligned}$$



(1) The lowest energy state is the ground state. It is the state with the lowest energy. The energy of the ground state is  $\frac{1}{2}\hbar\omega$ . The energy of the first excited state is  $\frac{3}{2}\hbar\omega$ . The energy of the second excited state is  $\frac{5}{2}\hbar\omega$ . The energy of the third excited state is  $\frac{7}{2}\hbar\omega$ . The energy of the fourth excited state is  $\frac{9}{2}\hbar\omega$ . The energy of the fifth excited state is  $\frac{11}{2}\hbar\omega$ . The energy of the sixth excited state is  $\frac{13}{2}\hbar\omega$ . The energy of the seventh excited state is  $\frac{15}{2}\hbar\omega$ . The energy of the eighth excited state is  $\frac{17}{2}\hbar\omega$ . The energy of the ninth excited state is  $\frac{19}{2}\hbar\omega$ . The energy of the tenth excited state is  $\frac{21}{2}\hbar\omega$ .



$$\hat{a}|n\rangle = \xi|n-1\rangle \quad \left\{ \begin{array}{l} \Rightarrow \langle n|a^\dagger a|n\rangle = |\xi|^2 \langle n-1|n-1\rangle \Rightarrow \\ \langle n|\hat{a}^\dagger = \xi^* \langle n-1| \end{array} \right. \quad n \langle n|n\rangle = |\xi|^2 \langle n-1|n-1\rangle \Rightarrow |\xi|^2 = n \Rightarrow \overset{\text{p.x.}}{\xi = \sqrt{n}}$$

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$$

$$[a, a^\dagger] = 1 \Rightarrow aa^\dagger = 1 + a^\dagger a$$

$$\hat{a}^\dagger|n\rangle = \rho|n+1\rangle \quad \left\{ \begin{array}{l} \Rightarrow \langle n|aa^\dagger|n\rangle = |\rho|^2 \langle n+1|n+1\rangle \\ \langle n|a = \rho^* \langle n+1| \end{array} \right. \quad \langle n|(1 + a^\dagger a)|n\rangle = |\rho|^2 \langle n+1|n+1\rangle$$

$$(1+n) = |\rho|^2 \quad \Rightarrow \text{p.x. } \rho = \sqrt{n+1}$$

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$



$$\hat{a}_m^\dagger |n_m\rangle = \sqrt{n_m+1} |n_m+1\rangle$$

$$\hat{a}_m |n_m\rangle = \sqrt{n_m} |n_m-1\rangle$$

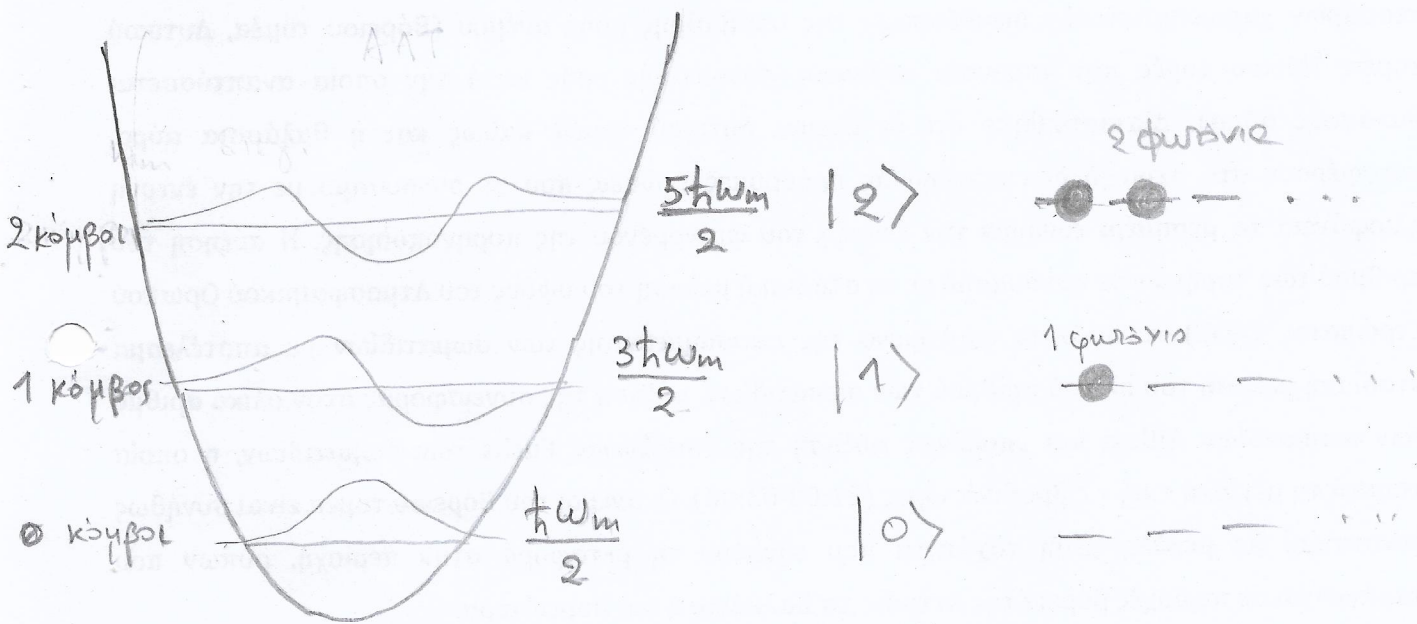
$$\hat{a}_m |0\rangle = 0$$

$$\langle n_m | l_m \rangle = \delta_{nl}$$

Η θεμελιώδης κατάσταση με ιδιοenergeia  $\frac{\hbar\omega_m}{2}$  αντιστοιχεί στο κενό κανένα φωτόνιο (φωτόνιο)

Η 1η διεγερμένη κατάσταση με ιδιοenergeia  $\frac{3\hbar\omega_m}{2}$  αντιστοιχεί σε 1 φωτόνιο (φωτόνιο)

Η 2η διεγερμένη κατάσταση με ιδιοenergeia  $\frac{5\hbar\omega_m}{2}$  αντιστοιχεί σε 2 φωτόνια (φωτόνια)



# φωτονίων = # κόμβων της ιδιοσυναρμόσεως του AAT

Με τη βοήθεια των τελεστών καταστροφής και δημιουργίας  
μπορούμε να γράψουμε

$$\hat{E}_x^m(z,t) = \left( \frac{\hbar \omega_m}{\epsilon V} \right)^{1/2} \sin\left(\frac{m\pi z}{L}\right) (\hat{a}_m^\dagger + \hat{a}_m)$$

$$\hat{B}_y^m(z,t) = \frac{i}{c} \left( \frac{\hbar \omega_m}{\epsilon V} \right)^{1/2} \cos\left(\frac{m\pi z}{L}\right) (\hat{a}_m^\dagger - \hat{a}_m)$$

ΣΧΕΣΕΙΣ ΜΕΤΑΘΕΣΕΩΣ ΜΠΟΣΟΝΙΩΝ

$$[\hat{a}_m, \hat{a}_e] = 0$$

$$[\hat{a}_m^\dagger, \hat{a}_e^\dagger] = 0$$

$$[\hat{a}_m, \hat{a}_e^\dagger] = \delta_{me}$$

