CHAPTER I

To the Threshold of Greek Mathematics

Plan of the Chapter

This chapter is introductory. I first survey, in a quick sweep, mathematics before Greece. This is followed by the historical context for the rise of mathematics in Greece itself (a discussion heavy with historiographical problems because so much is speculative). Finally, I conclude with a picture of the earliest known Greek mathematics.

I start with a section titled "Before Greece" – indeed, before any organized science at all. What are the universally shared bits of mathematics known even to simple societies? We find considerable but shallow knowledge. Familiarity with numbers and shapes is nearly universal – but does it amount to mathematics? "Empire and the Invention of Mathematics" brings in the rise of the state and with it, I argue, mathematical knowledge; "Beginning in Babylon" zooms in on the most important antecedent to the Greeks: the mathematics of Mesopotamia.

This, then, provides one kind of introduction. Another has to do with the Greeks themselves. The section titled "The Greeks: Standing Apart?" brings in the basic historical context: the unique characteristics of early Greek civilization. But where and how does mathematics emerge in Greece? "Greek Mathematical Myths" argues against some traditional narratives (most important: Pythagoras the mathematician was, I argue, indeed a myth). Another problematic context is that of Mesopotamian mathematics, and the following section, "Greeks and the Near East: A Historiographical Detour," tries to delineate a possible account of the debt owed by the Greeks to their predecessors.

With all of this in place, we may finally get to "The Threshold of Mathematics," which I identify as the mathematics attested to Hippocrates of Chios, and I conclude with "Assessing the Threshold": the historical meaning of this new Greek invention of mathematics.

Before Greece

Throughout this book, I will argue that Greek mathematicians had achieved something quite unprecedented. But of course, people everywhere know some mathematics, and the Greeks specifically must have owed something to past cultures. They did not start from scratch!

All of this sounds nearly obvious. In fact, we've merely started – and have already entered a minefield. The question of the cognitive universals underlying mathematics is invested with political meaning.

The issue can be stated quite simply, and it should be stated right as we begin. Students from disadvantaged backgrounds do much worse in mathematical tests. The response to this fact varies. Some take comfort. (They see the results of mathematical tests as proof of their belief in their group's superiority over others.) Most, aware of the enormous difference that social conditions make to cognitive growth, are less surprised that the underprivileged are also the mathematical underachievers. The explicit racist position is, frankly, preposterous, but it is stated by some and perhaps harbored by many. And so it is right that I should address it, head-on, right at the beginning. Consider the following two statements: (A) "The Greeks invented mathematics because they were white," and (B) "John is good at math because he is a white boy." If A strikes you as implausible, so should B. And if A does not strike you as implausible. . . . Well, this is, in part, why I've written this book.

So, what to do with mathematical tests? Some would say that they should not matter. Do math for the intellectual satisfaction it brings, not to get a good grade! But mathematical educators do not have the luxury of retreating into such fantasies. They have to go and teach in a world where mathematical tests do matter, and so the urgent task is this: How can we make mathematics more accessible to underprivileged students?

Now, this brings us back to the history of mathematics and to the question of universals. This question – how to make mathematics more accessible to the underprivileged – became especially acute in the global scene in the aftermath of decolonization in the 1960s and 1970s. New states in the Third World aimed to make education universal; however, this newly available education, more often than not, did not empower students but instead instilled in them a sense of helplessness and dependency. The mathematics was alien and forbidding, and so the best educators looked for ways to make it grow directly out of the students' own culture. Paulus Gerdes, for instance, as a young mathematics teacher in Mozambique, noticed that fishermen prepare their haul for sale by drying their fish near a fire built in the sand by the seashore. To make sure all the fish become dry at the same time, they follow a certain procedure. First, plant a stick in the ground, then attach a rope, and with a second stick attached at the other side of the rope, draw a circle in the sand. At this point, place all the caught fish along this drawn line, and finally, build the fire at the center. Gerdes's idea was revolutionary – and straightforward: Instead of starting with some abstract definitions, would it not make more sense to teach the children of those fishermen the concepts of "circle," "center," and "circumference" based on this procedure?[1]

Multiply this kind of example hundreds of times, and you have the discipline of ethnomathematics. Anthropologists, even apart from any application to the education of mathematics, came to be interested in the mathematical ideas available to preliterate societies; cognitive psychologists soon came to appreciate the significance of this research for the study of the universal human mind.

Thanks to the work of the ethnomathematicians, several observations emerged. First, numbers are pretty much universal. To be clear: it has been observed that the Pirahã tribe in the Amazon has no words for numerals. (There is some scholarly debate over this: Do the Pirahã words *hói* and *hoí* mean "one" and "two," respectively, or do they mean – as the best experts now seem to believe – merely something like "small" and "larger"?) It is extremely interesting to cognitive psychologists if, indeed, even a single language could fail to develop numerical terms – and so, perhaps, number is not directly hardwired into the human brain.[2] However, from the point of view of the anthropologist or of the historian, the example of the Pirahã is striking primarily for its freakish rarity. Everywhere you go around the globe, languages possess varied systems of counting. A few might be more impoverished (in particular, the Amazon has a number of less numerical societies, of which the Pirahã are an extreme and relatively well-studied case). But more often, simple societies have highly sophisticated numerical systems, with addition, multiplication, and iteration encoded into language itself. (Only one among these is the base-ten numerical systems now used by nearly all humans; it is nearly universal, perhaps, because it is, if anything, mathematically simpler than many of its alternatives.)

[1] This example and more like it are detailed in P. Gerdes, "Conditions and Strategies for Emancipatory Mathematics Education in Undeveloped Countries," *For the Learning of Mathematics* 5 (1985): 15–20.

[2] For a fascinating account, see M. C. Frank, D. L. Everett, E. Fedorenko, and E. Gibson, "Number as a Cognitive Technology: Evidence from Pirahã Language and Cognition," *Cognition* 108 (2008): 819–824.

Second, geometrical terms are not as universally verbalized, but once again, one of the most persistent features of almost all cultures is some kind of attention to patterns – molded, painted, tattooed, drawn in the sand. Those patterns often display symmetries and occasionally involve more precisely drawn geometrical shapes. Does this amount, in and of itself, to geometry? Is any of this *mathematics*?

Authors in the tradition of ethnomathematics often elide this question, and one sometimes has the impression that they try to impute to indigenous cultures geometrical knowledge concerning figures, where in fact, all that those cultures have is the habit of producing those figures. Some ethnomathematicians probably are overenthusiastic in this sense, but mostly this is a misleading framing. Once again, let us take an example from Paulus Gerdes. He describes the following pattern in Mozambique weaving baskets:

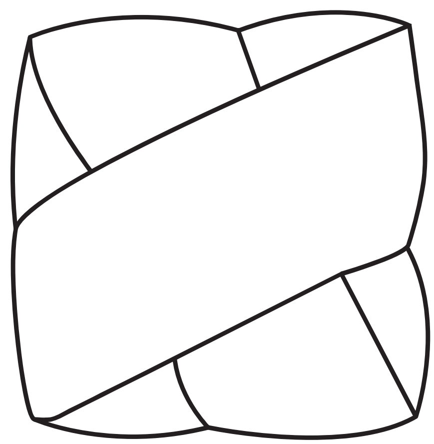

Figure 1

A nice geometrical pattern! But more than this, Gerdes observes, we may share this pattern in class and then proceed to discuss, with our students, how we may find here a relation between the various areas. In fact, with a little manipulation, we can derive, from this pattern, Pythagoras's theorem itself! (The main idea is that we see a big square – composed of four identical right triangles – and a smaller square enclosed in the middle. It is likely, I believe, that Pythagoras's theorem was indeed discovered around such drawings – by Babylonian teachers, working in a very different milieu. We shall return to see this in the discussion that follows.) Now, Gerdes does not mean that African basket-weavers are aware of Pythagoras's theorem; but it is nonetheless likely that the near-universal presence of patterns, of one form or another, is a significant precondition for the rise of geometrical reflection.

However, let us not get carried away. This is not yet reflective of explicit knowledge of geometrical properties, nor is the presence of a numerical vocabulary tantamount to the explicit knowledge of arithmetic. The discipline of ethnomathematics is useful for its scope – as well as for its limits. All humans, everywhere, talk about quantity and operate with shapes. But they almost never reflect on them explicitly, let alone develop a specialized craft of talking about numbers and figures. The discipline of mathematics and the profession of the mathematician are extremely rare.

Ethnomathematics is, of course, part of ethnography, and ethnographers tend to focus on what people *do* – how people interact, form kinship structures, cook, talk, sing. Anthropologists are trained to observe action, and so ethnomathematicians, quite properly, observe actions that are rich in mathematical meaning: counting, calculating, patterning. Those actions are real and form the background for the history we are about to explore in this book. Still, we should try to draw a line between an action that can be explained, *by us*, through our own mathematical understanding and the actors' own mathematical knowledge. Fishermen in Mozambique draw lines in the sand to dry their fish, and it is right and proper that we describe those lines in terms of circle, center, and circumference. It is also important to draw the conclusion that those fishermen have what it takes to create geometry. And finally, it is reasonable to say that the fishermen act in a geometrically intelligent way, without possessing any knowledge of a theoretical field such as geometry.

Many of you would probably agree that drawing a circle in the sand does not display, yet, knowledge of the theoretical field of geometry. I would say that the same is true about drawing a route, from point A to point B, along a straight line. This is a geometrically intelligent practice – but not a display of geometrical theory. I would also say the same about the building of a straight canal of irrigation leading to your fields. If you construct such a canal, then it is still the case that you may, or may not, have some theoretical understanding of "lines." I would also say the same about a straight road, faced by straight walls that form rectangular houses. And I would continue to say the same even if the houses become very imposing and perhaps assume the more complicated forms of various temples and pyramids. A pyramid, in and of itself, implies no more science than a line drawn for drying fish on the sand.

All of this is relevant to the question of the rise of mathematics as a theoretical discipline. We can find extremely sophisticated architecture and town planning around the globe – from the imperial cities of China to those of the Aztecs – and it is often assumed, especially by nonspecialists, that such imposing structures must involve theoretical mathematical knowledge. They certainly could, but the buildings, themselves, are not dispositive. And in fact, when we do find mathematics emerging, the context seems to be somewhat different.

Empire and the Invention of Mathematics

We can locate several historical moments where mathematics was independently invented. Taking them together, we may form certain conclusions about the natural context of such an invention – which brings us back to politics.

The Inca empire, ruling over a vast region of the Andes in South America, left behind many monuments – but no writing. From the very beginning, Western observers noted a puzzling and rather humble artifact. Known as the "quipu," this is a system of knotted threads (often made of cotton) that can usually be spread out as pendants – one main thread, with many others hanging on the main one; occasionally, this can become a many-layered object. Each of the threads has a pattern of knots attached to it, and throughout the twentieth century, as more of these artifacts were surfaced and analyzed, the system came to be understood as essentially numerical (and base ten). Roughly speaking, the knots on a cord form clusters. To simplify things a little, it works like this: if you have a cluster of three knots, a space, and then a cluster of two, this can stand for "32." Such individual numbers on the hanging cords are summed up as the number recorded on the main cord. This, then, seems like an accounting device. The research leading up to this basic decipherment, based purely on a mathematical analysis of the extant quipus (of which there are now several hundred), can be found in the work of Ascher, *Code of the Quipu* (1981). I mention this because Ascher is also one of the most brilliant scholars in ethnomathematics and the author of the basic monograph in the field, *Ethnomathematics: A Multicultural View of Mathematical Ideas* (1991). For her, quipus are an example of ethnomathematics: an indigenous culture's preliterate display of mathematical sophistication. We should, in fact, note an ambiguity: Is that display, strictly speaking, *pre*literate? Or was the quipu, instead, simply the Inca form of literacy? As more evidence came to light in the last generation, based on more careful excavations, we came to understand better the original function of the quipu. As was often suspected in the past, it seems to represent a tax system based on geographical allocation through subdivisions. We find that several quipus replicated each other (a guarantee of accounting consistency), and some quipus may be identified as summing up the results in other quipus (apparently, this represents lower and higher layers of the geographical subdivision). Most spectacularly – a veritable Rosetta Stone – a very late set of quipus from after the Spanish conquest was seen to match a Spanish written list of tributes from across many villages. It now seems likely that the colors of the threads were also meaningful, perhaps encoding geographical regions – thus, quipus were an even more informative system than we had ever assumed. The upshot of this research is that the Inca empire produced a specialized class of quipu masters whose job was to maintain information on the tribute required from across the empire. Now, as a matter of fact, we cannot really say how much "mathematics" those quipu masters knew, precisely because the Inca produced no writing. Whatever education was involved in the perpetuation of the quipu-master technique was purely oral and is now lost. But some education of this kind certainly existed, and so we can say this: in the Andes, prior to Pizarro, there must have been some mathematics actively produced, with people explicitly discussing rules of calculation and accounting.

And another remarkable observation: numeracy was so central, in this particular civilization, that it completely supplanted literacy. To explain: the tool that the state needed was some kind of numerical record. This was efficiently achieved by the quipu, and this did not give rise to literacy as a spin-off.

In other places, of course, states did rely much more on writing. Once again, it is useful to start from as far away from Greece as possible: let us get a sense of the entire range of possibilities. We may begin with China, where finally, we see a very clear tradition of theoretical mathematics. Here it is useful to focus on a relatively late work, *The Nine Chapters on the Mathematical Art*, a work that may have reached something like its current form under the Han dynasty (perhaps in the second century CE?). The Chinese court always required a large retinue of scholars, the bulk of whom were masters of religious rituals, but many specialized in fields such as astrology or other forms of scientific knowledge. It seems that at the latest by the Middle Ages, but perhaps even in the very earliest times, some were trained, and examined, based on their knowledge of the *Nine Chapters* – which is appropriately, then, seen to concentrate around accounting-like needs.[3] The measurement of fields and of heaps of rice and grain, taxation, and distribution by proportion – all brought under a set of general, well-understood algorithms, which then become a subject of study in their own right. The needs of the state, generalized – and turned into a mathematical art. Once again, our evidence in this case is late, and it is hard to tell how mathematics first emerged in China. But more recent archeological excavations do provide us with more context and push the evidence further back. One dramatic find is that of "The Book on Numbers and Computation," a set of inscribed bamboo strips that a civil servant took to his tomb, sometime early in the second century BCE. Much earlier, then, in Chinese imperial history – but still well after the formation of the first Chinese states – yet we see here the same kind of material as that found in the *Nine Chapters*. Problems that relate to concrete bureaucratic needs – solved with considerable general sophistication.

Beginning in Babylon

This brings us to the best-documented and most significant emergence of mathematics – and also, much closer to Greece itself. To the extent that the emergence of Greek mathematics was in debt to previous civilizations, it was to Babylonian mathematics.

This begins very early, along the shores of the Tigris and the Euphrates, and especially near their southern marshes.[4] This is one of the origins of urban civilization, and from the beginning, we find a system of accounting – not unlike that of the Quipu, perhaps – based, this time, on clay. (In the steep Andes, transportation is at a premium, and one looks for light tools; in the flat, river-based civilization of Mesopotamia, heavy but durable inscriptions are favored.) Archeologists have noted small, variously shaped pieces of clay found in many sites from the late Neolithic. Schmandt-Besserat was the first to offer a general account of those tools, and although she is not without her critics, very few doubt her basic interpretation (Schmandt-Besserat's critics mostly point out that the

[3] For the relation between mathematics and administration in the early Chinese state, see K. Chemla and B. Ma, "How Do the Earliest Known Mathematical Writings Highlight the State's Management of Grains in Early Imperial China?" *Archive for History of Exact Sciences* 69, no. 1 (2015): 1–53. Chemla and Ma, remarkably, are able to extract detailed information on the working of the administrative state, based on theoretical mathematical writings!

[4] The history of Mesopotamia is complicated: not a single state but a plethora of city-states and kingdoms, whose kaleidoscope kept shifting over millennia. I skip all the details (this is a history of Greek mathematics!), but read, for instance, N. Postgate, *Early Mesopotamia: Society and Economy at the Dawn of History* (New York: Routledge, 1994).

small pieces of clay could have been used for a variety of purposes beyond those she emphasizes; this is a reasonable critique). Most likely, different shapes stood for different commodities – so, for instance, could it have been a particular shape, say, for one head of cattle? Economic obligations – in the form of contracts or even taxation – could have been certified by an archive of such small tokens. This is all still ethnomathematics, a direct reliance on basic calculation and simple tools. And then, Schmandt-Besserat noted, something dramatic happened: it was realized that one could make impressions on clay, whose shape resembled the actual tokens. Late in the fourth millennium BCE, people in Mesopotamia began to use such tracings as economic records. A new idea, then: visual traces to mark numerical quantities. Pretty soon, instead of being tied to particular commodities, symbols emerged to represent *number as such*, and at this point, it took a mere step (or, if you will, a leap of genius) to begin to record other linguistic elements as well – at first, names of the objects counted and, very soon, language itself with its full vocabulary. By the end of the fourth millennium BCE, one of the major Mesopotamian languages – Sumerian – became fully written, the first ever. Literacy emerged, piggy-backing on numeracy.

Skipping many centuries of Mesopotamian history, we may look at the same shores of the Tigris and the Euphrates almost a millennium later. They are now dominated by different people, speaking a different language (Akkadian, a Semitic language that is somewhat similar to Hebrew or Arabic), still using the same script, the same inscriptions on clay. The technical knowledge of the Sumerians was not lost, in this and in other matters. The rivers themselves required constant attention – digging the canals and irrigating the fields. A lot of engineering, planning, and control was necessary, and throughout, Mesopotamia saw the rise of strong central authorities, powerful temple centers, and kings and their retinue. In the late third millennium, we see clear evidence for a specialized bureaucracy. Scribes were trained in writing, keeping accounts, and advising the rulers. What is most important: they did not just use the basic techniques of writing and calculation; they took pride in becoming genuine masters in all of those. Thus, besides simply writing down bureaucratic records in Akkadian, they also transcribed (a much harder task) the old literary legacy in Sumerian. And they did not just calculate, say, how many workers were required to dig a canal or how much tax should be levied on a field – they also invented particular fictional problems of a more abstract character, where one calculated volumes, plane areas, and work rates. In the Chinese *Nine Chapters* (or in the somewhat earlier "The Book on Numbers and

Computation"), we see the end result of, perhaps, a similar trajectory: bureaucratic training becoming its own raison d'être, giving rise to the problem-set version of a mathematics, which, although quite elementary, is already sophisticated. In Mesopotamia, our evidence is much more plentiful (early Chinese writing used a variety of delicate surfaces, such as the bamboo strip; from Mesopotamia, we have the clay tablet, history's most robust writing material). And so we get a closer sense of the entire transition: tokens, then writing, a bureaucracy, and this, finally – sublimated into mathematics. We have massive evidence, from the end of the third millennium to the beginning of the second millennium BCE. The evidence stops quite abruptly a little after 1800 BCE, for reasons we cannot quite fathom (for indeed, we no longer have substantial evidence!). It appears that the same old cities came under different sets of rulers and that the scribal traditions were disrupted. Little is known, then, for over a millennium – but clearly, there was some continuity. Beginning in the eighth century BCE, we find, once again, Mesopotamian palaces – preserving masses of clay tablets and a lot of the ancient culture. There is little mathematics to be found, though, in this later material (but plenty of astronomy; we shall return to this in Chapter 5). The object we study, then, is fantastically distant in time: the mathematics produced early in the second millennium BCE, or roughly four thousand years ago.

Just what is this mathematics? Let me paraphrase a very simple tablet (BM 13901 #1):

> I have it that the surface of the square, and its side, taken together, are three quarters.
>
> [Implicitly, our task becomes to find the numerical values of the side and area of this square. We're no longer just calculating taxes on fields; we're doing clever problems that build off such calculations! I attach Figure 2; notice that here, as in most cases, we do not have a figure on the clay tablet itself.]
>
> Here is what you should do. Make one as a projection to the side.
>
> [We now have in Figure 2 an elongated rectangle, divided into two parts, of which the right one is the original square, and the left one is a rectangle, one of whose sides is the original side of the square, its other side – one. The area of this left rectangle, then, is equal to 1 × the original side of the square, so its area is taken to be equal to the original side of the square. At this point, we can say that the entire elongated rectangle is equal to the original square plus the original side of the square. This is all equal to three-quarters, then.]

Break [the left rectangle] into two equal parts.

[And it is also implicitly understood that the broken left rectangle is now rearranged as in Figure 2, in the shape of a gnomon, a square minus a square. This gnomon, too, is equal to three-quarters].

Multiply half by half [to get a quarter].

[This is the area of the small square "implied" inside the gnomon because its side is the broken-into-two one, that is, 1/2].

Add the quarter to three-quarters, so you get 1.

[This, 1, is the area of the big square we would form from "completing the gnomon" and is also therefore the side of the big square.]

Take away the one-half in the inside, and one-half remains.

[Take away the side of the small square "implied" inside the gnomon, and you have, obviously, the side of the original square that we set out to find.]

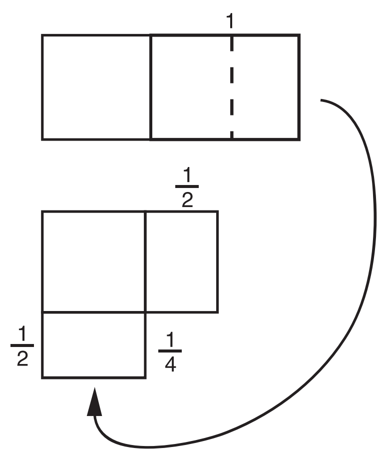

Figure 2

And this is how mathematics first emerges in the historical record: the simple, clever games accompanying the education of bureaucrats.

We have come a long way from the fishermen drawing a circular line on the sand. Here, surely, is mathematics. And it is precisely *here* that mathematics is to be found, in this particular variation on bureaucratic education. The Egyptian builders of pyramids may have been no more than the makers of glorified lines in the sand. Babylonian schoolmasters, however, created theoretical knowledge. And the difference is clear. As one builds a pyramid, one engages, throughout, in a concrete endeavor. There is no occasion to abstract away from the actual slabs of limestone to purely geometrical prisms: the slabs are what you handle throughout. But in the schoolroom, for instance, in the calculation example cited earlier, one no longer deals with actual measured fields. One deals with rules of calculation for the measurement of objects *such as* a square field. The schoolroom is at a remove from the field itself, and so its subject matter is not the concrete objects under calculation but, rather, the terms for calculation itself. Paulus Gerdes's project, setting out on his campaign for ethnomathematics, was to use the concrete knowledge of the fishermen of Mozambique as a starting point for the teaching of theoretical knowledge in the classroom. And in this, he traced the very same movement through which mathematics first emerged, four thousand years before.

The Greeks: Standing Apart?

This is nearly universal to humans: an ability to calculate with integers, the manipulation of shapes and patterns. Complex states give rise to bureaucracies, and this, occasionally, may give rise to the training of a scribal elite, which, finally, provides the context for the explicit statements of mathematical facts. And so, at last, you do not just calculate or draw patterns without reflection. Instead, you produce rules for calculation and for the measurement of areas.

This is a valid broad outline of the rise of mathematics in many parts of the world, but it has to be qualified. Even the human universals have a great deal of variety in them – perhaps the Pirahã don't even have numbers! – and the same may be true for the rise of state bureaucracies. In fact, even the three cases just mentioned – the Andes, China, and Mesopotamia – show considerable variety. We have no evidence for a more reflective geometry in the Andes. In China, reflective mathematics seems to postdate empire; in Mesopotamia, mathematics emerged almost simultaneously with the market economy itself. It might be argued that all of this is a matter of the different sources of our evidence. In the Andes, one relied less on bulky artifacts, and so we have merely the threads of the quipu to tell our stories; presumably, much more oral lore circulated and is now lost. The Chinese bamboo strip is only slightly more robust than the quipu; we must have lost a lot from the initial stages of Chinese mathematics. Mesopotamian clay, finally, is extremely durable, providing us, in such a way, a much more detailed panorama of early Mesopotamian civilization. All of this is true but perhaps misses the point. The various societies used different media *because* state bureaucracy was not always the same. Mesopotamia really was – at least at times and in certain places – a heavily regimented society, recording the tiniest details of property and labor. It used an abundant, robust form of writing because this is what it required. Writing is not some kind of ornament; it may be built into the very fabric of society, defining its overall practices and achievements.

And so let us approach the Greek evidence in an open-minded way. Did the Greeks have bureaucratic state mathematics? Were they like Mesopotamians? At first, they seem so. Indeed, once again, we find the very same medium already familiar from Mesopotamia, that of the clay tablet.

Since the beginning of the twentieth century, excavations have discovered extremely ancient temples and palaces in Greece and, in particular, in Crete. Dating from the fifteenth to twelfth centuries BCE, several of those sites also yielded written tablets. For a long time, it was not even clear which civilization – or language group – occupied, in such ancient times, the lands now known as "Greek." It was only after the decipherment of Linear B in the 1950s – a triumph of linguistic deduction made by Alice Kober and Michael Ventris[5] – that the language of the tablets was identified as Greek. In some other ways, we can say, the decipherment produced an anticlimax. The contents of the tablets, themselves, were very mundane pieces of accounting. All this linguistic brilliance put in by Kober and Ventris – and then: "two tripod cauldron of Cretan workmanship . . . One tripod cauldron with a single handle . . ."

An anticlimax, perhaps, but also a meaningful result. Early Greek civilization, in the Bronze Age, blended together with that of the ancient Near East as a whole. Not just the tool of the clay tablet – we find an entire cultural practice shared and perhaps transmitted: centers of political authority, where detailed numerical accounts are written down and stored. The implied sociology – the rise of some kind of state power (kings or king-priests?), with its bureaucrats, is clear enough. Did this go together with a more explicit training in numeracy? Was there, then, Bronze Age Greek mathematics? If so, it left no traces in writing. This is not where our story begins, and – the more general point – this is not, really, where the

[5] In the Suggestions for Further Reading, I go on to recommend John Chadwick's book from 1958, detailing the decipherment of Linear B. John Chadwick is a very fair and reliable narrator, and he does give proper credit to Alice Kober, but reception at the time was such that only Ventris's contribution was etched into shared historical memory (this is all very similar to Rosalind Franklin's place in the discovery of DNA). Kober's major contribution was A. E. Kober, "The Minoan Scripts: Fact and Theory," *American Journal of Archaeology* 52 (1948): 82–103.

Greek story begins, in general. What the archeological evidence suggests, instead, is dramatic rupture. At about 1200, early Greek civilization comes nearly to a halt. Palaces fall down; plunderers set in. The writing stops, seems to be forgotten; cities and their civilization shrink and disappear. This rupture is not isolated and is instead seen across the Near East, where many states seem to fail at about the same time: the rare event of a civilizational near-extinction. Well, elsewhere in the Near East, the habits of the state were perhaps more powerful, and the rupture was not as total. But in the Greek-speaking world, the year 1200 could well have been a kind of year zero. The culture of Linear B would have been as puzzling to the Greeks of the year 800 as it would be, many centuries later, to Kober and Ventris.

But then, at about the year 800, something new began to stir. And here, finally, we come to our proper topic. To clarify the contrast: earlier, in the second millennium BCE, we clearly found state formation in the Greek-speaking world, along the familiar lines of kings or king-priests setting up strong centers of power. This early state formation is knocked out by the crisis of the twelfth century, and when the state begins to reemerge, it seems to take a different shape from that of the previous temples and palaces and that of the other early empires noted in this chapter.

What was remarkable about the reemerging Greek state? That it was weak and small. The usual tale of state formation is that as societies become more complex, they become more unequal. A few individuals emerge as the powerful rulers, and if lucky, they manage to consolidate power over ever-larger groups. Eventually, state becomes empire. This was definitely the case in the pre-Columbian Andes and in China, and although Mesopotamian states often controlled no more than a mere region (sometimes, not much more than a single city), control was often centralized, a king, priests, their retinue – and a mass of subjects. Greek cities just did not go along such a route. Emerging from the rubble of the post-1200 collapse, villages gradually grew in size, but they never reached a very large extent. Greece, quite simply, was not a great river civilization. It was always marked by steep mountains, sharply cutting into the sea, isolating small islands and tiny valleys. At its height, centuries later, Athens would have hundreds of thousands of residents; but for now, in the eighth or seventh century, even larger settlements would have no more than a few thousand residents. The people whose voice mattered for any kind of political or cultural life – the more or less self-sufficient male adults – would number much fewer, surely no more than a few hundred.

To start with, those settlements were not very rich, and thus even the rulers could not be very rich themselves. (If you rule over a couple hundred small-scale farmers, how much wealth can you amass?) And so: scale creates habits, which can then become self-sustaining. Instead of relying on smaller armies of noble horsemen and charioteers, one relied on the larger standing army of the entire polis (*polis* is the Greek word for "city," which we will use from now on; the Greek term evokes a distinctive cultural model). It is fitting that those soldiers are armed with the cheaper and more widespread material of *iron* – not the expensive, specialized bronze of predecessor states. Kings in the ancient Near East counted their wealth in heavy ingots of precious metals. Citizens of the Greek polis would begin to use a more manageable currency, the coin. Invented in Asia Minor, or present-day Turkey, in the seventh century, the coin would become a hallmark of the Greek world: small pieces of metal, widely owned across the social strata, easy to transport and manipulate. Also, in Mesopotamia and in Egypt, literacy was a scribal specialty, sometimes monumental, always based on arcane, difficult systems of writing. The Greeks borrowed the cheapest and most portable writing surface available from Egypt – the papyrus roll – and they borrowed the simplest and easiest-to-learn writing system from Phoenicia: the alphabet. Writing, for the Greeks, was never a matter of some rarified scribal elite.[6] Not that it mattered all that much in the eighth or seventh century. Culture, quite simply, did not belong to any inward-looking court, with its established retinue of bureaucrats. There was nothing like that in early Greece. Culture belonged to the open spaces of the polis – perhaps in a festival, the youth singing together as they walk in procession; perhaps in a public square, a professional bard reciting an epic; sometimes, the richer folk, relaxing together and singing in a symposium with their guests from other poleis (because, in such a world of small poleis, there are always many other poleis, not far away, their people coming and going through your own).

The thread running through all of this is *being spread out*. It is the opposite of the concentration of Mesopotamia. The ancient Near East had a few expensive bronze chariots; hard-to-decipher hieroglyphs and cuneiform scripts; and rare, heavy ingots of gold; its culture was reproduced among a tiny group of trained scribes. Greece had plentiful iron spear tips

[6] It seems likely that the alphabet was, to begin with, the invention of subaltern groups that appropriated to themselves, in a much-simplified form, the complex systems of scribal elites. See O.Goldwasser, "How the Alphabet Was Born from Hieroglyphs," *Biblical Archaeology Review* 36 (2010): 40–53.

and relatively accessible alphabetic writing on papyrus, and eventually, it would have plenty of tiny silver coins – and it had the culture of the open polis, which could be shared with almost all members of the community. The political and demographic forces that made for shared culture became entrenched in tools that had a way, in turn, of maintaining the features of this culture, even as states did become, eventually, somewhat more powerful. Silver coins would tend to preserve a market economy, just as iron spear tips would tend to preserve military and political order based on the citizen body, fighting side by side. Public performance – recorded in widely accessible script – would become the most stubbornly entrenched of all those cultural habits. We still read Homer.

Without any specialized bureaucracy, archaic Greeks surely did not develop any specialized scribal schools. It is, in fact, very hard to reconstruct anything about Greek mathematical education in the archaic era. (I shall return to this topic in Chapter 4, which discusses Greek mathematical education in the Hellenistic era and later, where some evidence is available.) All we can say is that the Greeks knew little, and what they knew must have come from elsewhere. But surely, if we want to understand the rise of Greek mathematics, we ought to make some guesses concerning the contents of such knowledge!

To get there will require a double detour, into the historiography of early Greek science and philosophy and into the historiography of the ancient Near East. Why "historiography"? Because in those early mists of history, so little is known, and so much is based on speculation, that it is impossible to discuss the past in separation from the way in which modern historians have interpreted it. And so, Thales and Pythagoras, and then, Babylon.

Greek Mathematical Myths

When Heath wrote his own history of Greek mathematics a century ago, he thought he could give a very detailed survey on the question of origins, of first steps. Following his initial introductory chapter and a second chapter, "Greek Mathematical Notation and Arithmetical Operations," the titles of his Chapters 3, 4, and 5 are, respectively, "Pythagorean Arithmetic," "The Earliest Greek Geometry. Thales," and "Pythagorean Geometry." Heath's picture was essentially of a few isolated geniuses, such as Thales and Pythagoras, coming up with a new kind of science that eventually led to the great achievements of Greek mathematics.

When I lecture on Greek mathematics, I am often asked about Pythagoras (Thales, not so much), who still occupies a central role in the cultural imagination, and so this is a point I need to emphasize. Thales and Pythagoras were real historical persons, both active in the sixth century BCE (Thales somewhat earlier, Pythagoras somewhat later). However, although different scholars take different views on this question, the standard view is that *Thales and Pythagoras did no mathematics whatsoever*. This is all a myth. To be sure, it was a myth started by the Greeks themselves – who did like to invent stories, projecting their contemporary achievements into the distant past. But those later stories tell us more about the agenda of later Greeks than they do about the Greek culture of the Archaic era.

I have already mentioned that early Greek civilization was all based on public performance, and what comes to mind, first of all, is the public performance of recitation and song. Alongside the *bard*, the early Greeks also had the *sage*: the man respected for his words of wisdom and advice, words that might ring paradoxical and yet impress for their kernel of truth, often touching on the political life of the community but sometimes reaching beyond that, to speculation about the cosmos, about the human condition, about truth itself.[7] Such wise words could sometimes be noted and commemorated, and eventually – in the fifth century – a wise man even could decide, occasionally, to write down a book. But it was all about the public performance of wisdom – as it would still be all the way down to Socrates. Thales and Pythagoras certainly put nothing down in writing, and it is not clear that they had a "something" – a clearly articulated body of doctrines – for which "putting down in writing" would be the appropriate exercise. The sense that they had such a thing is a later construct that we can see emerging very clearly from the works of Aristotle. One of Aristotle's favorite techniques was to go through past views on a particular topic so as to see them as approximations – but no more – of Aristotle's own views. Thus, past philosophers, according to Aristotle, only looked for the material cause, "the stuff out of which things are made," unlike Aristotle, who developed a complex, nuanced understanding based on different kinds of causes. And so past sayings attributed to past figures were shoehorned into the model of material cause, and so also, whatever

[7] The claim here – that Archaic Greek sages should be understood alongside other public performers – is based on R. P. Martin, "The Seven Sages as Performers of Wisdom," in C. Dougherty and L. Kurke (eds.), *Cultural Poetics in Archaic Greece: Cult, Performance, Politics* (Cambridge: Oxford University Press, 1993), pp. 108–128.

sage words were attributed to Thales concerning the cosmos got transformed, in Aristotle's telling, to "Water is the Material Cause," which still survives as the first thing one usually reads in a general history of philosophy: "Thales was the first philosopher, and he said that all is water."[8]

It is reasonable enough that Thales, the wise man, would talk about the cosmos, and so we may perhaps be justified in saying that he probably did like to mention water in whatever it was he was saying. It is not entirely clear if he, or Pythagoras, had anything to say specifically about mathematics. Consider, for instance, the tradition concerning the most famous piece of such early mathematics: Pythagoras's theorem itself. It all goes back to a story of Pythagoras's pride in the finding of this theorem, a story that we find in many sources. The earliest of these, however, is Cicero.[9] A first-century BCE Roman figure, Cicero was active about five hundred years later than Pythagoras. Cicero and others later than him all seem to rely on a piece of verse, written about a century or two earlier than Cicero – so we are getting closer – but this piece of verse, by a certain Apollodorus, was apparently intended to be *funny*. The background is the following (it's an awful thing – I have to explain a joke now; by the time I'm done, there will be nothing funny left). Pythagoras, the wise man, did, in fact, promulgate a code of conduct demanding a certain kind of purification, of which one of the key demands was vegetarianism. Further: Greeks often thought of a remarkable achievement in terms of the religious ritual to celebrate it. The more remarkable the achievement, the more spectacular the ritual. And so the anecdote relayed by Apollodorus in his verse: Pythagoras celebrated the discovery of the most famous theorem by slaughtering one hundred bulls. This, I repeat, is meant to be *funny*. (A vegetarian – slaughtering bulls!) When Apollodorus writes it down, his audience perfectly understands the intention, and no one thinks that any of this is the literal historical truth. It does show two things, indeed.

First, in the time of Apollodorus, what we know as "Pythagoras's theorem" was already a well-known result, perhaps the most well-known mathematical result of all. This is not surprising: this result is widely

[8] The argument that Aristotle is an unreliable narrator of the earlier history of philosophy was made forcefully already by H. F. Cherniss, *Aristotle's Criticism of Pre-Socratic Philosophy* (Baltimore, MD: John Hopkins Press, 1935). Most scholars today believe that Aristotle aimed to be faithful to his sources while being completely shaped by the assumptions and agendas of his own time and place. Is this not true of all historians?

[9] Cicero alludes to the story in *On the Nature of the Gods* III.88. (He comments, pedantically, or perhaps mock-pedantically, that the story is impossible.) The verse itself is cited by several sources, beginning with Plutarch (first century CE), *Moralia* 1094b.

assumed in the educational documents we have, and it is referred to as the "schoolchildren result" by Polybius, a historian writing at about the same time as Apollodorus (I return to all of this in Chapter 4). In the Hellenistic era, and probably even before, Greeks knew "Pythagoras's theorem" as a basic part of their education.

Second, in the time of Apollodorus, it was already widely assumed that Pythagoras made contributions to mathematics. To cut a long and complicated story short: in the late fifth and early fourth centuries, there were authors, such as Philolaus and Archytas, active in the same geographical region that resonated with Pythagoras's fame (we will see much more of those authors in the next chapter). Such authors were all interested in mathematics, and they also shared something of the otherworldly, purity-seeking approach of Pythagoras's original type of wisdom. Aristotle (who, once again, is our main source) thought of Philolaus and Archytas as "Pythagorean"; they probably did so themselves. We shall revisit all of that in the following chapter. The point, for now, is that it is through association with those authors that the image of Pythagoras as a mathematician likely emerged.

We can go, in such a way, through the traditions concerning the various pieces of mathematics attributed to early mathematicians prior to the end of the fifth century. The evidence is extremely late. Very often, our earliest sources are from Late Antiquity (when, indeed, a mathematizing version of Platonism becomes very widespread and is often considered "Pythagorean"; we return to this in Chapter 6). Aristotelian guesses, amplified by later readers, Roman, late ancient – all constructing a myth that remains embedded in the Western historical narrative, all the way down, indeed, to Heath's *History of Greek Mathematics.*

Aristotle had very few sources to work from as regards Thales and Pythagoras, but the point, concerning the nature of the evidence, is not merely methodological. The point has to do with the underlying historical reality itself. The world of Thales and Pythagoras had relatively little use for extensive writing, and so whatever knowledge they uttered would have to be oral. This is why they were "wise men." Preliterate knowledge, preliterate science – wisdom. Thus, the same must be true for mathematical knowledge as well. Whatever the Greeks knew, back then, in the field of mathematics, would have to be understood along the terms of oral knowledge. This is not a matter of scientific doctrine but of shared cultural lore.

But where did this shared cultural lore first emerge? I believe it did, ultimately, in Babylon. But for this, yet another detour is called for.

Greeks and the Near East: A Historiographical Detour

So far, we have considered the mathematics of the ancient Near East essentially as a contrast to the Greeks (Mesopotamia had centralized bureaucracies; the Greeks did not). But can there be, instead, some kind of a *link*? That we even raise this question puts us, already, ahead of Heath. In 1921, historians of Greek mathematics knew essentially nothing of the mathematics of the earlier civilizations of the ancient Near East. It is significant that this changed – and changed so rapidly.

Otto Neugebauer was born in Innsbruck, Austria, in 1899. In 1922, he became a graduate student in, arguably, the best department of mathematics in the world: Göttingen. It was a time and a place where mathematicians were intensely engaged with the very foundations of their discipline. This concern with foundations brought Neugebauer, as a student, to a concern with *origins*. Finding himself fascinated by Egyptian archeology (a common enough fascination), he realized that there was something deeply significant about the fact that the very early civilizations developed abstract, symbolic systems for the representation of number. Mathematics – always looking for deeper abstractions. Throughout the 1920s, he produced important interpretations of the arithmetical systems in Egyptian papyri, followed by an account of the origins of the base-sixty system in Mesopotamia. Neugebauer stayed on in Göttingen after completing his studies, and he became the first lecturer, ever, in a mathematics department, to specialize in the history of mathematics itself. In Neugebauer's hands, the history – even the most distant one – looked forbiddingly impressive, and relevant, to contemporary mathematicians, in its abstraction. Then, in January 1933, the Nazis came to power. Not Jewish himself, Neugebauer could simply go on as before, but when German university teachers were required to declare loyalty to the regime, he refused – very few had this courage, made this sacrifice – thus giving up his position at the best department in the world and leaving Germany for good.

The University of Copenhagen saw the opportunity and offered him a professorship. (Eventually, he would get, in 1939 – another lucky break – to Brown University, which, for half a century, would become the center for research in the ancient exact sciences.) And so, in the year 1935, Otto Neugebauer, a Copenhagen professor, began to publish his *Mathematische Keilschrift-Texte* (*Mathematical Cuneiform Texts*).[10] This was the first

[10] For more on Neugebauer's fascinating intellectual trajectory – much more complicated than my brief outline suggests – see A. Jones, C. Proust, and J. M. Steele (eds.), *A Mathematician's Journeys: Otto Neugebauer and Modern Transformations of Ancient Science* (Berlin: Springer, 2016).

significant scholarly resource with which one could begin to make sense of the mathematics of the ancient Near East. Bear in mind: it was only in the nineteenth century that even the script of the clay tablets – cuneiform – was deciphered. At first, scholars were busy looking for the literary evidence for the biblical world, and only gradually did they come to consider the great bulk of numbers in cuneiform. Even here, the emphasis, to begin with, was on metrology: What were the units? What do these tell us about the society, the economy, and the history of antiquity? Some tablets did not yield that many interesting economic details and instead were more purely numerical – and these, in turn, were very obscure. I cited earlier a Babylonian text, but I did so through the useful interpretation now produced by Jens Høyrup. Imagine how obscure this was without any interpretation. I cite it again, now trying to preserve the sense of the original difficulty:

> My surface and the-equal-to-itself I added: 45′. 1, the beyond, you posit. Half of 1 you break, 30′ you make hold. 15′ to 45′ you add: 1. 1 is side. 30′, which you made hold, take away inside 1: 30′, the-equal-to-itself.

In this passage, I simplified the fractions to modern form, but in the original, they were sexagesimal, or in base sixty, so that 45′ is three quarters, 30′ is a half, and 1 can be thought of as 60′: we may comprehend this easily enough by thinking of minutes as fractions of an hour. (As an aside, this is not an accident: our minutes and hour reach us from Greek astronomy, which, in turn, depended on its Babylonian antecedents; more in Chapter 5.)

But the sexagesimals are relatively easy! The hard part is to make sense of the very point of the mathematical exercise. Neugebauer realized that the surface in question must be that of a square; "the-equal-to-itself" must be its side. However, there is no geometrical meaning to adding together a square and its side. This, then, so Neugebauer understood, must be a more algebraical formulation, dressed in geometry: *geometrical algebra*. The problem, at its core, takes the following form:

$x^2 + x = a$. (In this case, the value is $a = 45'$, and so the value of x is 30′.)

And Neugebauer argued that the Babylonians knew how to solve this algebraic problem in general, using particular numerical examples for the display of their general algorithms.

And so, according to Neugebauer, mathematics was always looking for deeper abstractions. The earliest mathematics, of the cuneiform tablets, was also essentially a piece of sophisticated science. Those individuals were, quite simply, professional mathematicians, pursuing the numerical solutions of algebraic equations. It was Göttingen on the Euphrates.

It is extremely rare that a field of scholarship passes so dramatically, so quickly, from obscurity to clarity. Babylonian mathematics barely existed as a field of study before 1935. Following Neugebauer's publication, it was grounded in a substantial number of texts, now thoroughly understood. We can readily imagine the authority that Neugebauer rightly assumed over his field, and indeed, for many years, no one would doubt his interpretation.

Now add this: in the 1930s and for many decades hence, the scholarly consensus was also to accept, at face value, the testimonies, such as those of Proclus and Aristotle, concerning the earliest Greek mathematics. Heath, we recall, had three full chapters on the mathematics of Thales and Pythagoras! The implication of that consensus would be, essentially, that even early on, the Greeks had the equivalent of professionalized mathematics, authors whose goal was to promote theoretical mathematical understanding. Put this side by side with Neugebauer's theoretical mathematicians of Babylon, and you have a continuity.

More than this: an interpretation presents itself almost immediately, indeed, was assumed already by Neugebauer himself, even as he was making his very first proposals concerning Babylonian mathematics. It has been central to later reconstructions of early Pythagoreanism that this philosophy somehow involved an early mysticism of number, as if somehow, integer numbers underlay the very structure of the cosmos. It is evident that such a mysticism would sit awkwardly with the discovery of irrationality. If the side and diagonal cannot both be expressed with rational numbers (for which, see pages 78–81), then it becomes impossible to describe the entire cosmos with such rational numbers. But it is clear that very early on, the Greeks did discover irrationality! So this should be difficult for Pythagoreanism, if indeed we believe in its existence as an early mathematical doctrine based on rational numbers. Such a belief began in antiquity itself – perhaps as early as Aristotle – and by Late Antiquity, some authors speculated that the discovery of irrationality could cause a crisis for Pythagoreanism.

The Göttingen era of Neugebauer's mathematical education was consumed by the crisis of foundations. If you try to base mathematics on set theory, this could easily lead to paradoxes; thus, one required special axiomatic assumptions to make a consistent mathematics even possible. The debate ranged further: Does mathematical existence require explicit construction? If so, are we to allow infinite constructions? Is mathematics purely formal? Is it even theoretically possible, formally, to axiomatize all of mathematics? (In 1931, Kurt Gödel – an Austrian mathematician working in Vienna – proved certain surprising results that made such a goal appear impossible, but throughout the 1920s, while Neugebauer was formed as a mathematician, Göttingen was consumed by the dream of a fully grounded, axiomatic mathematics.)

And so, it was tempting and natural to project a crisis of foundations, with its consequences, on the earliest history of mathematics. Babylonians came up with the algebraical study of numerical relations. But then their Greek followers discovered the phenomenon of irrationality and therefore realized that it is impossible to describe all relations in purely numerical terms. One therefore needed to formulate algebra in a strictly geometrical way – hence the mature geometry of Euclid's *Elements*.[11]

This account, then, in outline, was shared by scholars for decades. It came under pressure only very gradually. This happened in many stages. First, in 1962, the German philologer Walter Burkert published *Weisheit und Wissenschaft: Studien zu Pythagoras, Philoloas und Platon*, which – especially following its English translation from 1972, *Lore and Science in Early Pythagoreanism* – would come to define scholarship into Pythagoras. Here was a more careful, professionalized classical philology, keen to understand the authors we read not as mere parrots, repeating their sources, but instead as thoughtful agents who shape and retell the evidence as suits their agenda. Pythagoras, under such a reading, crumbles to the ground: almost everything – as noted earlier – comes to be seen as the making of later authors from Aristotle on.

Never mind: the historians of mathematics went on as before (and Burkert, with all his skepticism, did not consider himself an expert in the history of mathematics; he did not engage directly with this part of the Pythagorean legend). But there were other considerations, from within the discipline of the history of mathematics itself. The history of science, in general, was becoming more professional. It was no longer acceptable to pursue the history of science as an abstract, disembodied history of ideas, completely at a remove from any sense of historical context. In 1975, in his article "On the Need to Re-Write the History of Greek Mathematics," Sabetai Unguru made a very modest plea: mathematical texts should be understood on their own terms, not in some kind of translation to an abstract, ahistorical symbolism. The Greeks did not study equations; they studied geometrical configurations inside diagrams. The plea was modest, but Unguru framed it as a radical critique – which it was – of the dominant

11 For this supposed crisis of foundations – the evidence and a spirited discussion – see D. H. F. Fowler, *The Mathematics of Plato's Academy* (Oxford: Clarendon Press, 1987), pp. 294–308.

approach to the history of mathematics. It touched a nerve. He was savagely attacked by some of the most prominent historians of mathematics, even mathematicians, of the time. Well, the discipline paid attention. And in the end, the authority of Unguru's detractors made no difference. The need to historicize, to read texts in context, was much too powerful.

Burkert was circumspect and did not seek to dethrone the early Pythagoreans from their mathematical pedestal, but later historians of mathematics, following his lead, had fewer compunctions. In general, it became clear that the very effort to reconstruct the earliest stages of Greek mathematics was fraught with speculation. In Unguru's wake, this endeavor largely went out of fashion. Indeed, until the 1980s, most scholarship in Greek mathematics went into the reconstruction of the earliest mathematics and its interaction with philosophy. Since the 1980s, almost no one kept writing on this topic. Scholarship moved on to the interpretation of later mathematics, based on the firmer grounds of our extant texts from the fourth century BCE onward, texts that one could fully understand in historical context. The tendency was to be agnostic about any earlier Greek mathematics; through the years, scholars became more comfortable with the idea that Greek mathematics, in fact, did not start till quite late. While no one looked, Pythagoras and Thales were quietly removed from the museum.[12]

Meanwhile, things changed in Babylonian studies, as well. Professionalization would, at the end, overrun the authority even of Neugebauer himself. In a series of publications from the 1980s onward, Høyrup went back to the cuneiform texts, paying closer attention to the fine detail of vocabulary. For instance, why does the tablet quoted earlier talk about multiplication as "make hold"? Why is the subtraction it mentions taking from "inside"? Paying attention to such detail, Høyrup succeeded in reconstructing a concrete reference to all those seemingly abstract manipulations of numbers. In fact, Neugebauer had it upside down. This was not Babylonian algebra, later to be turned, by the Greeks, into geometry. The Babylonians, already, dealt with concrete geometrical configurations. It was all about measurement, calculation, patterns. It is simply that in the Babylonian case, the diagrams (or their more concrete, manipulated equivalent) were lost. The clay tablet contained directions for operation, referring to an external object that was

[12] This transformation was noted already by K. Saito, "Mathematical Reconstructions Out, Textual Studies In: 30 Years in the Historiography of Greek Mathematics," *Revue d'histoire des mathématiques* 4 (1998): 131–142.

operated upon concretely. Høyrup evoked a vivid sense of Babylonian mathematics in action – moving stuff about, tearing pieces of fabric or wood apart. He still wrote, however, strictly as a historian of science (albeit with extraordinarily wide-ranging knowledge and curiosity!). More recently, the study of Babylonian science has become even more professionalized, and scholars such as Eleanor Robson, active since the 1990s, are now fully trained not just as historians of science but also as social historians and archeologists. Robson is now able to put the vivid practice, recovered by Høyrup, in the context of the Babylonian school. And so, we now understand how those tablets came about. These are not at all professional mathematicians à la Göttingen, pursuing theoretical knowledge for its own sake. These are teachers in scribal schools, engaging with problems that are at a certain remove – but never entirely divorced from – their practical application.

But if Babylonian mathematics was always strictly a scribal school practice, how exactly did it even endure? The implicit assumption underlying Neugebauer's account of the ancient exact sciences was that there was something, such as a theoretical understanding of algebra, perhaps written down and since lost but, at any rate, circulating, in some form, through the generations, crossing eventually from Babylon to Greece. But this is clearly wrong and, in fact, directly contradicted by the pattern of our evidence. We do not see a theoretical continuity anywhere. What we see is that in several cities across Mesopotamia, centuries and even millennia apart, we find new foundations of scribal schools with their own flavor of mathematical education. And here is the thing: we can trace a continuity between problems set out in tablets from the twenty-first century BCE and, say, the eighteenth century BCE, and this is of a kind that can be traced even between distant cities. But the schoolmasters of a given city did not learn about their distant predecessors by the reading of tablets (which, by the eighteenth century, were already covered with dust, to be dug up only four millennia later!). Although, of course, tablets did travel around to be copied, this was not their main function. These were local tools of the trade, not at all the same as a book. No: whatever tradition there was had to be *oral*. At the first instance, it was not tablets that traveled, but *people*.

More than this: there is a set of numerical values and problem types that can be traced across an even wider geographical and chronological range. Once again, we owe the following observation to Jens Høyrup. Now, in the simple example of Babylonian mathematics quoted earlier, we are given the sum of a square area and its side. A much more specific type of question is where we are given the sum of the square area and its *four* *sides, taken together*. Remarkably, Høyrup found the exact same problem asked, and solved, in nearly the exact same manner, in Babylonian, Greek, Arabic, and Italian sources. How do problems travel? Well, the story of Cinderella, too, is told far and wide. When people meet, Høyrup explains, they share stories. A mathematical riddle is just that – a riddle, something to tell and to solve. Apparently, this happened frequently enough, throughout history, so as to preserve a certain minimal layer of shared mathematical knowledge. The likeliest account of mathematical transmission between cultures, then, is nearly the opposite of that implied by Neugebauer's account. It is not that some theoretical understanding was transmitted from the Babylonians to the Greeks. Instead, it was the problems themselves – puzzles and riddles, freed from their original school context – that came to be more widely shared. And the very idea of a "Babylonian influence over Greece" is misleading. Not that it ever made any sense to seek the impact of eighteenth-century BCE clay tablets from southern Mesopotamia on the words of a wise man in southern Italy twelve centuries later. There was a continuity, but it must have been, precisely, *along a continuum*. Somehow – even through the major disruption of Mesopotamia from 1800 BCE onward – some kind of mathematical lore survived. Some teaching must have continued; there must have come about a kind of folk koine shared by practitioners of various crafts. It owed its birth to the scribal schools of past empires. But something did survive their demise, a more humble practice – but one freed, perhaps, from the court?

This is all lost: away from the courtly scribal schools, we just do not have the evidence. But we can surmise the presence of real mathematical progress. Let me outline how this could have come about.

The riddles that Høyrup detects, permeating across cultural and linguistic borders, generally take the form of a challenge: If you know X, can you also know Y? He looks specifically at the following riddle:

> If you know the sum of the square area with the square's four sides, can you tell the side of the square?

And we have earlier cited the following riddle:

> If you know the sum of the square area with one of the square's sides, can you tell the side of the square?

A slightly more complicated puzzle is the following:

> If you know the difference between the two sides of a rectangle, and you also know the area of the rectangle, can you tell what the two sides are?

This riddle was solved by the Babylonians with a technique directly analogous to that cited on pages 10–11: Because you know the difference between the two sides, you can mark the difference on the longer side as a projection beyond a square. And now you have the situation where you know the area of a square plus its projection. For instance, suppose the area is 60, and the difference between the sides is 7 (yes, you know the answer already, but please wait till the end of the example).

Because the difference is 7, I can break it into two equal parts and transpose one of the resulting triangles. I get a gnomon, whose area is *still* 60.

This gnomon has a small gap, a square, whose side is 3.5. The area of this small square, then, is $3.5^2 = 12.25$.

Add this to the gnomon, and we have a larger square whose area is 72.25.

So, its side is 8.5.

This is simply the middle between the greater and smaller sides of the rectangle, when their difference is 7. Add 3.5 and remove 3.5, and you have the two sides: 5 and 12.

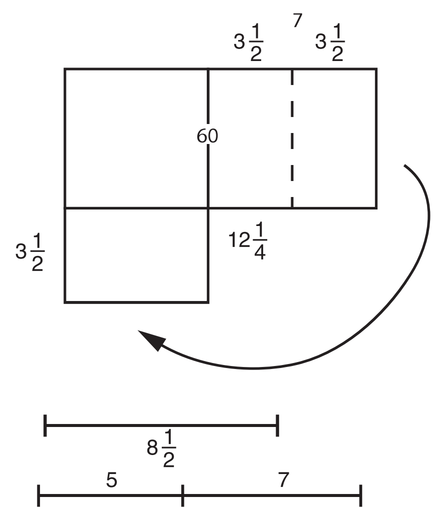

Figure 3

To repeat, what this approach shows us is how we can solve a riddle: given the difference between the two sides of a rectangle, and the area of the rectangle, to find what the two sides are.

At some fairly late stage in the history of Babylonian mathematics, the thought suggested itself to some schoolmaster to improve on this type of riddle by considering the pattern – noted by Gerdes in Mozambique weaving! – such as this:

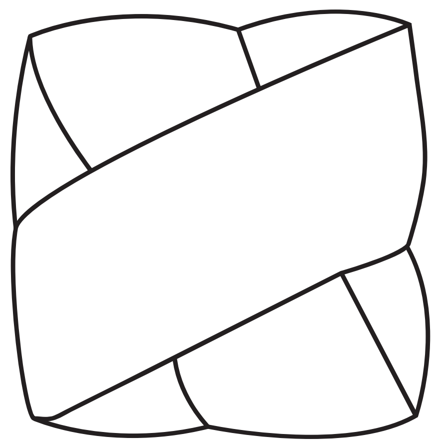

Figure 4

That is, we may consider a square and the diagonals of four equal rectangles drawn on it obliquely. Note that if we draw all four rectangles, we have a small gap, a square, in the middle. This small square stands on a side that is, in fact, the difference between the two sides of the rectangle:

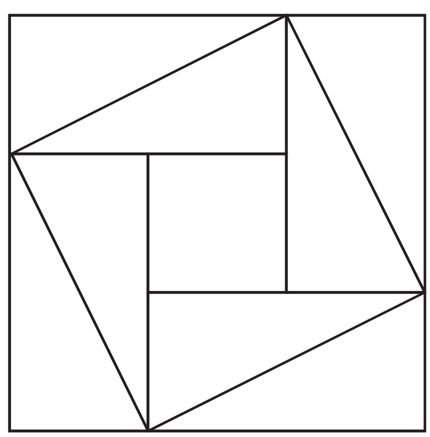

Figure 5

Suppose you have taught your students a hundred times already how, if they are assigned the difference between the two sides, as well as the area of the rectangle, they can find the two sides. It dawns on you that you can have a new way to find this difference because the small gap, the small square, is, in fact, the difference between the slanted square across the figure and *four triangles* – which are the same as *two rectangles*.

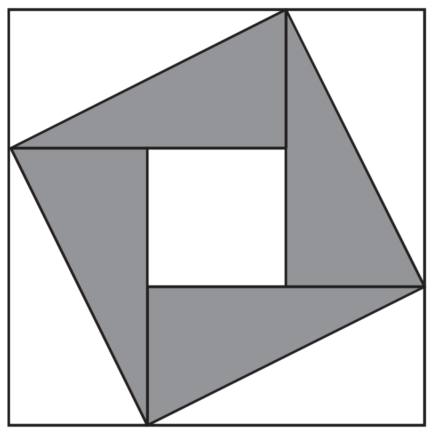

Figure 6

The four triangles are equal to two rectangles, whereas the slanted square is simply the square on the diagonal of the rectangle. Now, in the riddle we are most familiar with, we are given the area of the rectangle and the difference between the two sides of the rectangle. We have now realized that if we are given the area of the rectangle, and we are also given the length of the diagonal, we can find the difference between the two sides!

This will provide us with a much neater, much more impressive riddle:

> If you know the area of the rectangle, and you also know its diagonal, can you tell its two sides?

Wow! And in fact, this is directly doable: take the square on the diagonal, and subtract from that 2 × the area of the rectangle. You come up with the area of the small square in the middle. Find the root of this square. This is the difference between the two sides. So now we've made it to the position where we know the area of the rectangle, and we also know the difference between its two sides – and this, we know already!

This is as far as our evidence leads us. Babylonian geometrical problems are typically related to the measurement of surfaces. They are never entirely detached from, well, the measurement of fields – from which it all began.

However, this riddle did exist, and apparently someone, in the lost mists of time, thought of an extension of the same technique to derive a related riddle, one no longer connected to the measurement of the areas themselves:

> If you know the two sides of the rectangle, can you tell the length of the diagonal?

The approach to this riddle should now be obvious. By multiplying the two sides, we get the area of the rectangle. Double this, and we get two rectangles, or four triangles. Next, by subtracting the smaller from the greater side, we get the side of the small square in the middle. Add up the four triangles and the small square, and we have the slanted square; find its root, and we get the length of the diagonal.

Finally, if you have some experience with the numerical finding of roots and squares (a technique quite central to this schoolmasterish tradition), you are aware that if you have two values, then the result of multiplying them, and doubling that, and then adding to this the square on their difference, is nothing else than the sum of their squares. As we would put this:

$$2ab + (a - b)^2 = a^2 + b^2.$$

And so, this riddle is rather elegantly solved: if you are given the two sides of the rectangle and you want to know the length of the diagonal, simply square each side, add them up, and look for the root of the sum – and there, you have the length of the diagonal. To repeat: this last riddle is a reconstruction. It was probably not offered by the year 1800 BCE. But it appears that it did enter the mathematical koine, the shared template of familiar mathematical riddles, out of which Mediterranean civilizations could – if they wanted to – build up their education. And so, Pythagoras's theorem. Not the fruit of some theoretical reflection by an isolated Greek genius. Rather, the outcome of the school practice and the mathematical folklore of riddling, through the centuries. As we imagine the beginnings of Greek mathematics, we must imagine it against the background of this kind of widely shared, elementary knowledge.

The Threshold of Mathematics

There is no turning back from the empire to the tribe. In simple societies, there is usually a language for counting and practices of calculation, as well as pattern-making. People *do* things, but they do not produce theoretical reflections on such activities. Empires give rise to bureaucrats, and those (often) give rise to teachers, and teachers, finally, do produce theoretical reflections. And at this point, they begin to do something new. They engage in practices of writing, they impart rules, and they share riddles. And these, finally, will tend to reproduce outside the closed environment of the classroom. Babylon was gone. But the ancient Near East, *even post-Babylon*, would not have the same mathematical practice as that of an isolated tribe in the Amazon. The riddles were already out in the wild, reproducing.

What is described here is oral knowledge that, by definition, can no longer be uncovered. How many people knew just what? All we can do is point at the overall parameters. Certainly, as Greek democracy grew, more Greeks had access to education. From what we have seen in the foregoing discussion, it is likely that such an education included a modicum of mathematics. We may then follow Høyrup's lead and suggest that mathematical knowledge spread through a few orally shared examples/riddles, and we might also suggest, then, that this knowledge, as it spread through the Greek world and became entrenched in elementary mathematical education, included the basic rules of area measurement, up to and including Pythagoras's theorem.

Peering back into their earliest history, Greeks felt that they had to attribute the beginnings of their mathematical knowledge to particular named Greek authors: hence, the invention of Thales and Pythagoras as mathematical authors. But this was a category error, on the behalf of Greek historians as well as their modern followers. The earliest mathematical knowledge shared by the Greeks was neither authored nor Greek. It was, instead, the reflection in oral teaching of past Mesopotamian scribal schools. (Which, to be fair, many Greeks sensed, as well; Herodotus, for instance, was quite explicit that Greek knowledge of mathematics came from the East!)

And yet, the very need to attach cultural achievements to the names of their authors is telling – and specifically Greek. We do not pause to consider this as such because this habit – the reliance on named authors – has become so natural to us. But it is, in fact, a specific historical invention, happening only in some civilizations. We do not have an author for the epic of Gilgamesh, for the Chinese Book of Odes, for the Mesoamerican Popol Vuh. Indeed, Greek bards, too, recited epic stories of the siege of Troy, never attributing them to any named author. As in so many other civilizations, these were just the stories being told.

A Babylonian teacher, imparting the rule whereby one finds the sides of the rectangle from the combination of area and diagonal, did not invoke the "author of the rule." It was just a rule: teachers knew how to apply it. Similarly, a Greek, singing the song of Achilles's wrath, did not invoke the "author of the poem." It was just an epic poem: bards knew how to sing it.

And then, a few Greeks started to *make a name for themselves*. Instead of just reciting a well-known set of epic poems, widely shared among professional bards, some Greeks started to sing more personal songs, produced from a personal angle. Others repeated those poems, still attributing them to their original authors. In this new tradition, what made a particular poem effective was precisely that one could imagine, in concrete detail, a particular person – say, Sappho or Hesiod – in a particular place, say, Lesbos or Boeotia, authoring the song. So much so that soon enough, people began to imagine a concrete person, the author of the epic poems themselves! Perhaps at some point in the sixth century, "Homer" was invented. The Greeks started to believe that there was one particular person who was responsible for just those two epics, the *Iliad* and the *Odyssey*.[13] And now we can see: Thales and Pythagoras became, as it were, the "Homer" of mathematics. They were retrospectively reinvested with a new identity; they were reidentified as the authors of what was, in fact, an unauthored oral tradition.

For indeed, there was no turning back now. All Greek culture, from now on, would have a named author. This was a new, exciting departure, in many ways specific to the Greeks: through literature, one could make a lasting reputation for oneself.

This invention of the author is also the direct background for a new development in the late fifth century: a proliferation of writing. The real – that is, historical – Thales and Pythagoras were sages, performing orally. Sages do not write; they proclaim. Throughout the fifth century, a handful of sage-like figures began to add a book to their résumés. The proclamations, attached to a name, would spread far and wide. Such are the wise, surprising proclamations of Parmenides, Anaxagoras, Philolaus: a single book, distilling a life of wisdom. Right near the end of the fifth century, this practice of book writing becomes an avalanche. Suddenly, many authors try to make their name in prose, and several produce not one book but many. "Author" becomes, in such cases, the key identity. And so, more and more figures of the author emerge. The genres multiply. These

[13] This reconstruction must be speculative, but see M. L. West, "The Invention of Homer," *Classical Quarterly* 49 (1999): 364–382.

were exciting times, and many turned to the writing of history. Political speech mattered a lot: many wrote speeches, and some produced manuals of rhetoric; teachers of speech-making, they wrote about the practice of speech – as many others now did, writing about their craft. Physicians wrote about health and disease; artists wrote about the proportions of their statues; architects, about the proportions of their buildings. And so, a few wrote on mathematics, as well. Our history proper begins here.

Let us, then, lay out the evidence. So far in this chapter, we have measured our narrative in centuries, even millennia. Now, decades matter. It is the invention that we now need to understand, and so we try to isolate a very precise group: authors likely to have circulated mathematical texts throughout the last few decades of the fifth century.

The list of such attested authors is, in fact, very small. A couple of astronomers are mentioned for Athens – Euctemon and Meton – and we will return to discuss them in Chapter 5. As I will explain there, it is at least possible that their brand of astronomy was not the kind we would recognize as mathematical. Meton, at least, is well dated: he is mocked by Aristophanes in a comedy dated to the year 414. Others are much less well dated. Theaetetus was commemorated by Plato, in the dialogue carrying his name, and we get to know the precise historical circumstances of his death in battle. We can independently date this to 369 BCE. In the same dialogue, we find that as a youth, Theaetetus studied with the mathematician Theodorus of Cyrene. Theaetetus, apparently, did not die in old age, so he could not be Theodorus's student that many years before 369. The dialogue does have Socrates conversing with both Theodorus and Theaetetus, a conversation set just as Socrates's trial is about to begin, in 399; but perhaps not too much is to be made of this because Plato did allow himself considerable historical license, and the moment of the trial was one he returned to for dramatic effect. The likelihood remains that Theodorus was active very late in the fifth century or very early in the fourth.

Another name we must mention is that of Democritus: the first genuinely prolific prose author in Greek. Many dozens of works are attributed to him, and ancient catalogues identified an entire set of works on mathematical topics, for instance, *On the Contact of Circles and Spheres*. This, presumably, was a study not of the tangency of a circle with a sphere but, rather, a study of the general question of how straight lines touch curvilinear lines. There is one concrete mathematical result that we can certainly ascribe to Democritus – because we can do so on the authority of Archimedes himself. What we learn from Archimedes is that Democritus asserted – although without proof – that cones are one-third of the cylinder in which they are enclosed, and also that pyramids are one-third of the prism in which they are enclosed.

We can pursue this report a little further. First of all, the key piece of background: the one thing we know for sure about Democritus is that he was the central figure in early atomism. One of the main topics for paradoxical, wise statements about nature, among early authors, was the very makeup of the universe: What are its ultimate constituents? Atomists contended that the world is made of microscopic, unbreakable pieces.

Now, there is a well-known passage where Plutarch, a philosopher and belletrist of the imperial era, quotes Chrysippus, a Hellenistic Stoic philosopher. Chrysippus cites a puzzle raised by Democritus: if you cut a cone by a plane parallel to the base, you get two circular faces, one at the top of the lower cut, the other at the bottom of the upper cut. The interpretation of this passage is contested, but my own understanding is that Chrysippus implies that Democritus said as follows: surely the two faces are not equal to each other – for otherwise, we would have not a cone, but a cylinder. Hence, what appears to us as the smooth surface of a cone must, in reality, be a terraced, jagged surface, made by microscopic steps. This, then, is consistent with atomism. The world is not a continuum but is instead, so to speak, rough at the (invisible) edges: our impressions of smoothness are no more than an illusion. We now see how discussions of the surface of the cone, or of the sphere, can easily belong to such philosophical debates.[14] It seems that Democritus did assert an intuition – which is, in fact, mathematically correct: that the cone in the cylinder is essentially the same as the pyramid in a prism. He probably meant, however, not some kind of theoretical mathematical observation, but instead a philosophical one: his argument was that the cone in the cylinder simply *was* a pyramid in the prism. It is possible, then, that the evidence for Democritus does not imply any original mathematical activity, although it does imply familiarity with a mathematical lexicon and probably also some mathematical results (it seems that Democritus did know, already, that a pyramid is one-third of the prism containing it – an elementary, although not a trivial, result). But this does not add much to our knowledge of the relevant chronology. Democritus was probably born around 460, but he certainly had a very long and very productive life. The implication, then, is that if we consider

[14] For an account of Democritus's discussion of the cone – and of Democritus's atomism as a whole – see N. D. Sedley, "Atomism's Eleatic Roots," in P. Curd and D. W. Graham (eds.), *The Oxford Handbook of Presocratic Philosophy* (Oxford: Oxford University Press, 2008), pp. 305–332.

an arbitrary work by Democritus, it likely originates in the late years of the fifth century or even the beginning of the fourth. Can we not push the evidence any earlier?

We may, in all likelihood – with just two outstanding names. These are Oenopides of Chios and Hippocrates of Chios (not to be confused with that other Hippocrates – of Cos! – the father of medicine). That the two come from the same island may or not may be significant: our evidence hangs on a thin thread. But could mathematical literature – at the moment of its very inception – perhaps be a local phenomenon?

The evidence for Oenopides is tantalizing, meager – and significant. Three observations stand out.

First, Oenopides is consistently credited with some astronomical discovery having to do with the ecliptic. As I will return to explain in Chapter 5, the motion of the sun, moon, and planets takes place on a thin strip, a circle located on the sphere of the fixed stars, called the "ecliptic," and it happens to be set at a particular oblique angle to the equator: roughly twenty-three degrees. It seems likely that Oenopides made some statement concerning the obliquity of the ecliptic. This suggests an interest in some kind of explicit geometrical model of the sky.

Second, this interpretation can be supported by a couple of very late citations produced by Proclus, the fifth-century author mentioned earlier concerning his testimony for Thales and Pythagoras.[15] What Proclus claims, in his commentary to Euclid's *Elements* 1, is that Oenopides first discovered the construction of a perpendicular to a given line (*Elements* 1.12), as well as the construction of an angle equal to a given angle (*Elements* 1.23). As usual, everything has to be taken with a grain of salt, but the report seems significant for several reasons. Concerning 1.23, Proclus claims that his testimony is based on Eudemus – a follower of Aristotle who composed a history of geometry (which is now, unfortunately, lost; it is our best source for early Greek mathematics, but we rely on a few scattered quotations by authors later than Eudemus). Concerning

[15] I will frequently refer, in this book and especially in this chapter, to Proclus's commentary to Euclid's *Elements* (discussed in the "Proclus and the Philosophical Schools" section in Chapter 6). This has an excellent English translation: G. R. Morrow, *Proclus: A Commentary on the First Boom of Euclid's Elements* (Princeton, NJ: Princeton University Press, 1970/1992). Note that it is customary to make references inside the work not according to the page numbers in Morrow's translation but according to the scientific edition from the nineteenth century. Pages 65–68 are especially useful because they contain a brief history of early Greek mathematics, which Proclus apparently had on the authority of the early historian Eudemus.

1.12, Proclus claims that this result was used by Oenopides for the sake of astronomy, and he also adds – which provides the report with extra credibility – that Oenopides used a particular archaic term for "perpendicular" ("at a gnomon," which in itself may or may not carry a specific astronomical implication). The reference to the precise language used suggests quite clearly that Proclus, ultimately, relies on a source who could read Oenopides's own writing. Oenopides, then, was a writer! And in all likelihood, this report, too, ultimately went back to Eudemus.

To bring all this into an appropriate context, it should be noted that scholars have long concluded that Eudemus may have based his history, in part, on his own rational reconstructions. That is, Eudemus would ascribe to past mathematicians knowledge of such things that was required for the sake of what he knew they actually did know. If result A is logically demanded by result B, and Eudemus found evidence that a mathematician asserted B, Eudemus would then claim also that the mathematician knew A. Eudemus, further, seems to have structured his own work primarily as a survey of "first discoveries"; that is, he looked for evidence, however indirect, concerning the identity of the first authors who knew about particular results.[16] Add to this Proclus's own purpose. He does not write a summary of the history of Eudemus; he takes from it such tidbits that are relevant for Proclus's own commentary to Euclid's *Elements* 1.

Bring all of the evidence together, and a likely account emerges: Oenopides may have been the first author to produce a geometrically motivated account of the sky, for which – quite naturally – he relied on various assumptions concerning the construction of angles. All of this is important, and we shall return to it in Chapter 5.

But why do I even trust any of this? Why should I not dismiss the evidence, as I did for Thales and Pythagoras? Why assume that Oenopides was a fully fledged author and not just a wise man, proclaiming his thoughts concerning the stars and the earth?

The main reason has to do with dates. Oenopides is said, by Proclus, to have been somewhat younger than Anaxagoras. This is in the context of a quasi-chronological list of early Greek geometry, and Oenopides is followed by Hippocrates of Chios, who is in turn followed by Theodorus of

[16] I won't go into this historiographical detail at length, but Eudemus is indeed a central pillar to our history. See L. Zhmud, "Eudemus' History of Mathematics," in I. Bodnár and W. W. Fortenbaugh (eds.), *Eudemus of Rhodes* (New Brunswick, NJ: Routledge, 2002), pp. 263–306.

Cyrene. Nothing here inspires huge confidence (the pairing of Anaxagoras and Oenopides seems to depend on a forged dialogue by Plato, *The Lovers*, where the two are mentioned side by side). But we recall that Theodorus was likely active near the very end of the fifth century; Anaxagoras was a philosopher reported to have been close to Pericles, and thus he was active not long after the middle of the fifth century. With Oenopides and Hippocrates of Chios, then, we likely find authors active in the last few decades of the fifth century: two authors, close to each other in time and in place. And so, to the extent that we find that Hippocrates of Chios was likely the fully fledged author of a mathematical book, we are probably justified to assume the same for Oenopides.

And there is, in fact, very little room for doubt: Hippocrates of Chios must have been a fully fledged author because – finally! – we have a very substantial fragment of his original writing still extant. Well, "extant" is – as you would expect by now – a relative term. We have a report from a late commentator to Aristotle, called Simplicius (very late, from the sixth century CE!). He tries to account for a passage in Aristotle where Hippocrates of Chios is briefly mentioned. And to do so, Simplicius makes a lengthy quotation from the very same Eudemus, the early historian of geometry. The quotation certainly includes many interpolations by Simplicius himself, but past scholars have identified certain linguistic hints that allow us to separate Eudemus's original text from that of Simplicius. The result is that we can read, at the very least, a nearly unadulterated citation from Eudemus, which seems to hew closely, at least in parts, to Hippocrates's own words.

This, then, is our first substantial glance at Greek mathematics. It is our first extended close-up, and it is a pivotal moment; we should linger here for a while.

In Eudemus's own presentation, Hippocrates produced four separate results. I will cite the first two in full and then only briefly mention the third and the fourth. In what follows, the assumption is that we translate a text by Eudemus. The word *he* refers to Hippocrates of Chios:[17]

> He first proved by what method a quadrature was possible, of a lunule having a semicircle as its outer circumference. He did this after he circumscribed a semicircle about a right-angled isosceles triangle and, about the

[17] What follows is my own translation. The entire passage may be read in I. Thomas, *Greek Mathematical Works* (Cambridge, MA: Harvard University Press, 1939), pp. I.234–I.252. (This is a translation of selected passages of Greek mathematics, designed to complement Heath's history.)