Chapter 7
Natural deduction

Before turning to *Predicate Logic* in the second part of the book, we will look at one more way to present the logic of propositions. A system of natural deduction gives you a way to develop proofs of formulas, from basic proofs that are known to be valid.

Conjunction, disjunction and negation

The rules tell us how to build up complex arguments from *basic* arguments. The basic arguments are simple. They are of the form

$X \vdash A$

whenever A is a member of the set *X*. We write sets of formulas by listing their members. So $A \vdash A$ and $A, B, C \vdash B$ are two examples.

To build up complex arguments from simpler arguments, you use rules telling you how each connective works. We have one kind of rule to show you how to introduce a connective, and another kind of rule to show you how to eliminate a connective once you have it. Here are the rules for conjunction:

$$\frac{X \vdash A \quad Y \vdash B}{X, Y \vdash A \,\&\, B} \,(\&\mathrm{I})$$

$$\frac{X \vdash A \,\&\, B}{X \vdash A} \,(\&\mathrm{E}_1) \qquad \frac{X \vdash A \,\&\, B}{X \vdash B} \,(\&\mathrm{E}_2)$$

The rules follow directly from the way we use conjunction. The introduction rule (&I) says that if *A* follows from *X,* and if *B* follows from *Y* too, then the conjunction *A*&*B* follows from *X* and *Y* together. The elimination rules ($\&\mathrm{E}_{1,2}$) say that if something entails a conjunction, then it also entails each conjunct.

For implication, we have two rules:

$$\frac{X, A \vdash B}{X \vdash A \supset B} \,(\supset\mathrm{I}) \qquad \frac{X \vdash A \supset B \quad Y \vdash A}{X, Y \vdash B} \,(\supset\mathrm{E})$$

These rules are of fundamental importance for natural deduction systems, as they connect the entailment relation ($\vdash$) to implication. The implication introduction rule $(\supset\mathrm{I})$ states that if *X* together with *A* entails *B,* then *X* entails the conditional $A \supset B$.

So, one way to prove a conditional is to *assume* the antecedent, in order to *prove* the consequent. This is enough to prove the conditional. Conversely, the implication elimination rule $(\supset E)$ states that if some set *X* entails $A \supset B$, and another set *Y* entails *A*, then applying the information in *X* (which gives $A \supset B$) to that in *Y* (which gives *A*) gives us the consequent *B*. So, taking *X* and *Y* together gives us all we need for *B*.

Before going on to see how these rules are used in proofs, we will see the rules for disjunction:

$$\frac{X \vdash A}{X \vdash A \vee B}(\vee I_1) \qquad \frac{X \vdash B}{X \vdash A \vee B}(\vee I_2)$$

$$\frac{X, A \vdash C \quad Y, B \vdash C \quad Y \vdash A \vee B}{X, Y \vdash C}(\vee E)$$

The introduction rules are straightforward enough. If *A* is entailed by *X* then so is $A \vee B$. Similarly, if *B* is entailed by *X* then so is $A \vee B$. The elimination rule is more interesting. If $A \vee B$ follows from *Y*, if *A* (with *X*) gives *C*, and if *B* (with *X*) also gives *C*, then *Y*—which gives $A \vee B$ —also gives *C* (provided you've got both *X* too). This is a form of argument by cases. If you know that $A \vee B$, and if *A* gives you *C* and *B* gives you *C* too, then you have *C*, either way.

Given the rules, we can construct proofs. A proof of $X \vdash A$ is a tree (this time in the usual orientation, with the root at the bottom) with $X \vdash A$ as the root, in which the leaves are axioms, and in which each step is an instance of a rule. Here is an example:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \,\&\, C \vdash A \,\&\, C}{A \,\&\, C \vdash A}(\&E) \qquad A \supset B \vdash A \supset B}{A \supset B, A \,\&\, C \vdash B}(\supset E)}{A \supset B, A \,\&\, C \vdash B \vee D}(\vee I_1)}{A \supset B \vdash (A \,\&\, C) \supset (B \vee D)}(\supset I)$$

In this proof, each step is indicated with a horizontal line, labelled with the name of the rule used. The leaves are all axioms, as you can see. The proof demonstrates that $A \supset B \vdash (A \,\&\, C) \supset (B \vee D)$.

Each step of the proof follows from the previous steps by way of the rules. However, I constructed the proof in reverse. I know that I wanted to prove that $A \supset B \vdash (A \,\&\, C) \supset (B \vee D)$. To do this, I knew that I had just to prove $A \supset B, A \,\&\, C \vdash B \vee D$. (To prove $X \vdash A \supset B$, assume *A* with *X*, and prove *B*.) Then it is clear that we can prove $A \supset B, A$ $A \& C \vdash B$, since $A \& C \vdash A$, and we are done.

We can also present proofs in *list* form, in which each line is either an axiom or follows from earlier elements in the list by way of the rules:

1	$A \supset B \vdash A \supset B$	Ax.
2	$A \supset (B \supset C) \vdash A \supset (B \supset C)$	Ax.
3	$A \vdash A$	Ax.
4	$A \supset (B \supset C), A \vdash B \supset C$	2,3($\supset$E)
5	$A \supset B, A \vdash B$	1,3($\supset$E)
6	$A \supset (B \supset C), A \supset B, A \vdash C$	4,5($\supset$E)
7	$A \supset (B \supset C), A \supset B \vdash A \supset C$	6($\supset$I)
8	$A \supset B \vdash (A \supset (B \supset C)) \supset (A \supset C)$	7($\supset$I)

In this proof, we have annotated each line with an indication of the lines to which it appeals and the rules used in the derivation. This presentation encodes exactly the same information as the tree. It is *often* easier to produce a proof in list form at first, as you can go 'down the page' as opposed to horizontally across it as the proof gets more complex. Also, in a list proof, you can make assumptions that are not used further in the proof, which can be helpful in producing the proof. However, once the proof is produced, representing it as a tree provides a more direct representation of the dependencies between the steps.

Let's see another example to give you some more ideas of how proofs are produced. Let's prove

$$A \supset ((B \& C) \supset D) \vdash (A \& C) \supset (B \supset D)$$

To do this, we know that we will have to assume $A \supset ((B \& C) \supset D)$, and also $A\&C$ and B to deduce D. But these together will give us D rather simply. By $A\&C$, we get A, and this gives us $(B \& C) \supset D$. $A\&C$ also gives us C and B, which then give $B\&C$, which then gives D. So, let's wrap this reasoning up in a proof:

1	$A \supset ((B \& C) \supset D) \vdash A \supset ((B \& C) \supset D)$	Ax.
2	$A\&C \vdash A\&C$	Ax.
3	$B \vdash B$	Ax.
4	$A\&C \vdash A$	2(&E)
5	$A \supset ((B \& C) \supset D), A \& C \vdash (B \& C) \supset D$	1,4($\supset$E)
6	$A\&C \vdash C$	2(&E)
7	$A\&C, B \vdash B\&C$	3, 6(&I)
8	$A \supset ((B \& C) \supset D), A \& C, B \vdash D$	5,7($\supset$E)

9 $A \supset ((B \,\&\, C) \supset D), A \,\&\, C \vdash B \supset D$ 8($\supset$I)

10 $A \supset ((B \,\&\, C) \supset D) \vdash (A \,\&\, C) \supset (B \supset D)$ 9($\supset$I)

This proof explicitly represents the informal reasoning in the paragraph above.

Negation

The rules so far only give us a fragment of our language. We have conjunction, implication and disjunction. To add the negation rules, it is helpful to first add a proposition $\bot$ (called the *falsum*), which is always evaluated as false. This is governed by the simple elimination rule

$$\frac{X \vdash \bot}{X \vdash A} (\bot\text{E})$$

Since $\bot$ is always false, it does not have an introduction rule. Given $\bot$, we can define negation rules simply:

$$\frac{X, A \vdash \bot}{X \vdash \sim A} (\sim\text{I}) \qquad \frac{X \vdash \sim A \quad Y \vdash A}{X, Y \vdash \bot} (\sim\text{E})$$

If *X* together with *A* entails $\bot$ then we know that *X* and *A* can't be true together, so if *X* is true, *A* is false: that is, ~*A* is true. That is the introduction rule for negation. To exploit a negation, if *X* entails ~*A,* and if *Y* entails *A,* then it follows that *X* and *Y* cannot be true together. That is, they jointly entail $\bot$. These rules govern the behaviour of negation. (You might have noticed that these rules just define ~*A* as equivalent to $A \supset \bot$. It is instructive to check using truth tables that ~*A* and $A \supset \bot$ are equivalent.) These rules allow us to prove a great many of the usual properties of negation. For example, it is simple to prove that a double negation of a formula follows from that formula, $A \vdash \sim\sim A$:

1	$A \vdash A$	Ax.
2	$\sim A \vdash \sim A$	Ax.
3	$A, \sim A \vdash \bot$	1, 2(~E)
4	$A \vdash \sim\sim A$	3(~I)

Similarly, the rule of contraposition, $A \supset B \vdash \sim B \supset \sim A$, has a direct proof:

1	$A \supset B \vdash A \supset B$	Ax.
2	$A \vdash A$	Ax.
3	$A \supset B, A \vdash B$	1,2($\supset$E)
4	$\sim B \vdash \sim B$	Ax.

5	$A \supset B, A, \sim B \vdash \bot$	3, 4(~E)
6	$A \supset B, \sim B \vdash \sim A$	5(~I)
7	$A \supset B \vdash \sim B \supset \sim A$	6($\supset$I)

However, the rules for negation cannot prove *everything* valid in truth tables. For example, there is no way to prove a formula from its double negation. (It is instructive to *try* to prove $\sim\sim A \vdash A$. Why is it impossible with these rules? Similarly, the converse of contraposition, $\sim B \supset \sim A \vdash A \supset B$ is valid in truth tables, but cannot be proved using the natural deduction rules we have so far.) To give us the *full* power of truth tables, we have an extra rule

$$\frac{X \vdash \sim\sim A}{X \vdash A}\ (\text{DNE})$$

This is called the double negation elimination rule (DNE). With the rule, we get $\sim\sim A \vdash A$ (apply the rule to the axiom $\sim\sim A \vdash \sim\sim A$ and you have your result) and much more. The full power of truth tables is modelled by these rules. As another example, here is a tautology, the *law of the excluded middle*. It is straightforward to show that $A \vee \sim A$ is a tautology in truth tables. It is a great deal more difficult to prove it using our natural deduction system.

1	$A \vdash A$	Ax.
2	$A \vdash A \vee \sim A$	2($\vee$I)
3	$\sim(A \vee \sim A) \vdash \sim(A \vee \sim A)$	Ax.
4	$A, \sim(A \vee \sim A) \vdash \bot$	2, 3(~E)
5	$\sim(A \vee \sim A) \vdash \sim A$	4(~I)
6	$\sim(A \vee \sim A) \vdash A \vee \sim A$	5($\vee$I)
7	$\sim(A \vee \sim A) \vdash \bot$	3, 6(~E)
8	$\vdash \sim\sim(A \vee \sim A)$	7(~I)
9	$\vdash \sim A \vee \sim A$	8(DNE)

Typically, proofs that require (DNE) are more complex than proofs that do not require it. In general, if you are attempting to prove something that requires the use of (DNE), you should try to prove its double negation first, and then use (DNE) to get the formula desired.

Natural deduction systems provide a different style of proof to that constructed using trees. In a tree for $X \vdash A$, we attempt to see how we could have X and $\sim A$. If there is some consistent way to do this, the argument is not valid. If there is none, the argument is valid. On the other hand, natural deduction systems construct a derivation of *A* on the basis of *X*. The resulting derivation is very close to an explicit 'proof such as you will see in mathematical reasoning. But natural deduction systems have deficiencies. If something is not provable, the natural deduction system does not give you any guidance as to how to show that. Trees give you worthwhile information for both valid and invalid arguments.

We will not pursue these systems of proof theory any further. To find out more about natural deduction, consult some of the readings mentioned below.

Further reading

Lemmon [15] is still the best basic introduction to this form of natural deduction. Prawitz's original account [20] is immensely readable, and goes into the formal properties of normalisation. For a more up-to-date summary of work in natural deduction and other forms of proof theory, consult Troelstra and Schwichtenberg's *Basic Proof Theory* [30].

The system of logic without (DNE) is called *intuitionistic logic.* The philosophical underpinning of intuitionistic logic was developed by L.E.J.Brouwer in the first decades of the twentieth century. For Brouwer, mathematical reasoning was founded in acts of human construction (our intuition). Brouwer did not think that a formal system could capture the notion of mathematical construction, but nevertheless a logic of intuitionism was formalised by Heyting (see *Intuitionism: An Introduction* [11]). The rule (DNE) fails, since ~*A* holds when you have a construction showing that *A* cannot be constructed or proved. This is a refutation of *A*. Thus ~~*A* means that you have a demonstration that there cannot be a refutation of *A*. This does not necessarily give you a positive construction of *A*. Similarly, you may have neither a refutation of *A* nor a construction of *A,* so $A \vee {\sim}A$ fails as well. Intuitionistic logic is important not only in the philosophy of mathematics and the philosophy of language (see the work of Michael Dummett [4], who uses the notion of verification or construction in areas other than mathematics), but also in the study of the computable and the feasible.

These natural deduction systems are easy to modify in order to model *relevant* logics. We modify the rules to reject the axiom *X* £*A,* and accept only the instances *A* £*A* (the other elements of *X* might be *irrelevant* to the deduction of *A*). Then, there is no way to deduce *A*£*B*→*A,* since the *B* was not *used* in the deduction of *A*. It is also possible to make the number or the order of the use of assumptions important. Slaney's article 'A general logic' [28] is a short essay on this approach to logic, and it is taken up and explored in my *Introduction to Substructural Logics* [22].

Exercises

Basic

{7.1} Prove these, not using (DNE):

$$A \supset \sim B \vdash B \supset \sim A \quad \sim\sim\sim A \vdash \sim A \quad \sim A \vee \sim B \vdash \sim(A \,\&\, B)$$
$$\sim(A \vee B) \vdash \sim A \,\&\, \sim B \quad A \,\&\, \sim B \vdash \sim(A \supset B)$$

{7.2} Prove these, using (DNE):

$$\vdash ((A \supset B) \supset A) \supset A \quad \sim(A \,\&\, B) \vdash \sim A \vee \sim B \quad \vdash A \vee (A \supset B)$$
$$\sim A \supset \sim B \vdash B \supset A \quad (A \,\&\, B) \supset C \vdash (A \supset C) \vee (B \supset C)$$

Advanced

{7.3} Show that the natural deduction rules without (DNE) (that is, the rules for intuitionistic logic) are sound for the following three-valued truth tables:

p	q	$p\&q$	$p \vee q$	$p \supset q$	$\sim p$
0	0	0	0	1	1
0	*n*	0	*n*	1	1
0	1	0	1	1	1
n	0	0	*n*	0	0
n	*n*	*n*	*n*	1	0
n	1	*n*	1	1	0
1	0	0	1	0	0
1	*n*	*n*	1	*n*	0
1	1	1	1	1	0

To do this, show that if $X \vdash A$ can be proved by the natural deduction system then $X \models A$ holds in the three-valued table. (For $X \models A$ to hold, we require that in any evaluation in which the premises are assigned 1, so is the conclusion.) Show, then, that none of the argument forms in Exercise 7.2 are provable without (DNE), by showing that they do not hold in these three-valued tables.

{7.4} Show that these three-valued tables are not *complete* for intuitionistic logic. Find an argument that cannot be proved valid in the natural deduction system, but that is valid according to the three-valued tables.

{7.5} Show that the three-valued Łukasiewicz tables are not sound for intuitionistic logic. Find something provable in intuitionistic logic that is not provable in the three-valued Łukasiewicz tables.

Nothing prevents us from being natural
so much as the desire
to appear so.
—Duc de la Rochefoucauld