Beginning Logic

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in brackets refers either to a line of a proof or to a sentence or formula so numbered earlier in the same section; context will always determine which.

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I should like to dedicate this book to Arthur Prior, without whose encouragement and enthusiasm I would never have entered logic, and to the memory of my father, who I hope would have enjoyed it.

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CHAPTER 1

The Propositional Calculus 1

1 THE NATURE OF LOGIC

It is not easy, and perhaps not even useful, to explain briefly what logic is. Like most subjects, it comprises many different kinds of problem and has no exact boundaries; at one end, it shades off into mathematics, at another, into philosophy. The best way to find out what logic is is to do some. None the less, a few very general remarks about the subject may help to set the stage for the rest of this book.

Logic's main concern is with the soundness and unsoundness of arguments, and it attempts to make as precise as possible the conditions under which an argument—from whatever field of study—is acceptable. But this statement needs some elucidation: we need to say, first, what an argument is; second, what we understand by soundness; third, how we can make precise the conditions for sound argumentation; and fourth, how these conditions can be independent of the field from which the argument is drawn. Let us take these points in turn.

Typically, an argument consists of certain statements or propositions, called its *premisses*, from which a certain other statement or proposition, called its *conclusion*, is claimed to *follow*. We mark, in English, the claim that the conclusion follows from the premisses by using such words as 'so' and 'therefore' between premisses and conclusion. Instead of saying that conclusions do or do not follow from premisses, logicians sometimes say that premisses do or do not *entail* conclusions. When an argument is used seriously by someone (and not, for example, just cited as an illustration), that person is asserting the premisses to be true and also asserting the conclusion to be true *on the strength of* the premisses. This is what we mean by *drawing* that conclusion from those premisses.

Logicians are concerned with whether a conclusion does or does not follow from the given premisses. If it does, then the argument in question is said to be *sound*; otherwise *unsound*. Often the terms 'valid' and 'invalid' are used in place of 'sound' and 'unsound'. The question of the soundness or unsoundness of arguments must be carefully distinguished from the question of the truth or falsity of the propositions, whether premisses or conclusion, in the argument. For example, a true conclusion can be soundly drawn from false premisses, or a mixture of true and false premisses: thus in the argument

(1) Napoleon was German; all Germans are Europeans;
therefore Napoleon was European

we find a true conclusion soundly drawn from premisses the first of which is false and the second true. Again, a false conclusion can be soundly drawn from false premisses, or a mixture of true and false premisses: thus in the argument

(2) Napoleon was German; all Germans are Asiatics;
therefore Napoleon was Asiatic

a false conclusion is soundly drawn from two false premisses. On the other hand, an argument is not necessarily sound just because premisses and conclusion are true; thus in the argument

(3) Napoleon was French; all Frenchman are Europeans;
therefore Hitler was Austrian

all the propositions are true, but no one would say that the conclusion followed from the premisses.

The basic connection between the soundness or unsoundness of an argument and the truth or falsity of the constituent propositions is the following: an argument cannot be sound if its premisses are *all* true and its conclusion false. A necessary condition of sound reasoning is that from truths only truths follow. This condition is of course not sufficient for soundness, as we see from (3), where we have true premisses and a true conclusion but not a sound argument. But, for an argument to be sound, it must *at least* be the case that if all the premisses are true then so is the conclusion. Now the logician is primarily interested in conditions for soundness rather than the actual truth or falsity of premisses and conclusion; but he may be secondarily interested in truth and falsity because of this connection between them and soundness.

What techniques does the logician use to make precise the conditions for sound argumentation? The bulk of this book is in a way a detailed answer to this question; but for the moment we may say that his most useful device is the adoption of a special symbolism, a logical notation, for the use of which exact rules can be given. Because of this feature the subject is sometimes called *symbolic* logic. (It is sometimes also called *mathematical* logic, partly because the rigour achieved is similar to that already belonging to mathematics, and partly because contemporary logicians have been especially interested in arguments drawn from the field of mathematics.) In order to understand the importance of symbolism in logic, we should remind ourselves of the analogous importance of special mathematical symbols.

Consider the following elementary algebraic equation:

(4) $x^2 - y^2 = (x + y)(x - y)$,

and imagine how difficult it would be to express this proposition in ordinary English, without the use of variables 'x', 'y', brackets, and the minus and plus signs. Perhaps the best we could achieve would be:

(5) The result of subtracting the square of one number from the square of a second gives the same number as is obtained by adding the two numbers, subtracting the first from the second, and then multiplying the results of these two calculations.

Comparing (4) with (5), we see that (4) has at least three advantages over (5) as an expression for the same proposition. It is briefer. It is clearer—at least once the mathematical symbols are understood. And it is more exact. The same advantages—brevity, clarity, and exactness—are obtained for logic by the use of special logical symbols.

Equation (4) holds true for *any* pair of numbers x and y. Hence, if we choose x to be 15 and y to be 7, we have, as a consequence of (4):

(6) $15^2 - 7^2 = (15 + 7)(15 - 7)$.

If we now compare (6) with (4), we can see that (6) is obtained from (4) simply by putting '15' in place of 'x' and '7' in place of 'y'. In this way we can check that (6) does indeed follow from (4), simply by a glance to see that we have made the right substitutions for the variables. But if (6) had been expressed in ordinary English, as (4) was in (5), it would have been far harder to see whether it was soundly concluded from (5). Mathematical symbols make both the doing and the checking of mathematical calculations far easier. Similarly, logical symbols are humanly indispensable if we are to argue correctly and check the soundness of arguments efficiently.

If in the sequel it seems irritating that a special notation for logical work has to be learned, the reader should remember that he is only mastering for argumentation what he masters for calculation when he learns the correct use of '$+$', '$-$', and so on. This device, which logic has copied from mathematics, is the logician's most powerful tool for checking the soundness and unsoundness of arguments.

Our final question in this section is how the conditions for valid argument can be studied independently of the fields from which arguments are drawn: if this could not be done there would be no separate study called logic. A simple example will suffice for the moment. If we compare the two arguments

(7) Tweety is a robin; no robins are migrants;
therefore Tweety is not a migrant

and

(8) Oxygen is an element; no elements are molecular;
therefore oxygen is not molecular,

both of which are sound (one drawn from ornithology, the other from chemistry), it is hard to escape the feeling that they have something in common. This something is called by logicians their *logical form*, and we shall have more to say about it later. For the moment, let us try to analyse out in a preliminary way this common form. The first premiss of both (7) and (8) affirms that a certain particular thing, call it m (Tweety in (7), oxygen in (8)), has a certain property, call it F (being a robin in (7), being an element in (8)). The second premiss of (7) and (8) affirms that nothing with this property F has a certain other property, call it G (being a migrant in (7), being molecular in (8)). And the conclusion of (7) and (8)

affirms that therefore the object m does not have the property G. We may state the common pattern of (7) and (8) as follows:

(9) m has F; nothing with F has G;
therefore m does not have G.

Once the common logical form has been extracted as in (9), a new feature of it comes to light. *Whatever* object m is picked out, *whatever* properties F and G are chosen to be, the pattern (9) will be valid: (9) as it stands is a pattern of a valid argument. For example, take m to be Jenkins, F and G to be the properties respectively of being a bachelor and being married: then (9) becomes

(10) Jenkins is a bachelor; no bachelors are married;
therefore Jenkins is not married,

which, like (7) and (8), is a sound argument. Yet (9) is not tied to any particular subject-matter, whether it be ornithology, chemistry, or the law; the *special* terminology—'migrant', 'molecular', 'bachelor'—has disappeared in favour of schematic letters 'F', 'G', 'm'.

Form can thus be studied independently of subject-matter, and it is mainly in virtue of their form, as it turns out, rather than their subject-matter that arguments are valid or invalid. Hence it is the forms of argument, rather than actual arguments themselves, that logic investigates.

To sum up the contents of this section, we may define logic as the study, by symbolic means, of the exact conditions under which patterns of argument are valid or invalid: it being understood that validity and invalidity are to be carefully distinguished from the related notions of truth and falsity. But this account is provisional in the sense that it will be better understood in the light of what is to follow.

2 CONDITIONALS AND NEGATION

When we analyse the logical form of arguments (as we did in the last section to obtain (9) from (7) and (8)), words which relate to specific subject-matters disappear but other words remain; this residual vocabulary constitutes the words in which the logician is primarily interested, for it is on their properties that validity hinges.

Of particular importance in this vocabulary are the words ' if . . . then . . .', '. . . and . . .', ' either . . . or . . .', and ' not '. This chapter and the next are in fact devoted to a systematic study of the exact rules for their proper deployment in arguing. We have no single grammatical term for these words in ordinary speech, but in logic they may be called *sentence-forming operators on sentences.* I shall try to explain why they merit this formidable title.

In arguments, as we have already seen, propositions occur; an argument is a certain complex of propositions, among which we may distinguish premisses and conclusion. Propositions are expressed, in natural languages, in *sentences.* However, not all sentences express propositions; some are used to ask questions (such as ' Where is Jack? '), others to give orders (such as ' Open the door '). Where it is desirable to distinguish between sentences expressing propositions and other kinds of sentence, logicians sometimes call the former *declarative* sentences. Always, when I speak of sentences, it is declarative sentences I have in mind, unless there is some explicit denial. Now if we select two English sentences, say ' it is raining ' and ' it is snowing ', then we may suitably place ' if . . . then . . .', '. . . and . . .', and ' either . . . or . . .' to obtain the new English sentences: ' *if* it is raining, *then* it is snowing ', ' it is raining *and* it is snowing ', and ' *either* it is raining *or* it is snowing '. The two original sentences have merely been substituted for the two blanks in ' if . . . then . . .', '. . . and . . .', and ' either . . . or . . .'. Further, if we select one English sentence, say ' it is raining ', then we may suitably place ' not ' to obtain the new English sentence: ' it is *not* raining '. Thus, grammatically speaking, the effect of these words is to form new sentences out of (one or two) given sentences. Hence I call them sentence-forming operators on sentences. Other examples are: ' although . . . nevertheless . . .' (requiring two sentences to complete it), ' because . . ., . . .' (also requiring two), and ' it is said that . . .' (requiring only one).

(This book is written in English, and so mentions *English* sentences and words; but the above account could be applied, by appropriate translation, to all languages I know of. There is nothing parochial about logic, despite this appearance to the contrary.)

In this section we are concerned with the rules for manipulating ' if . . . then . . .' and ' not ', and we begin by introducing special logical symbols for these operators. Suppose that P and Q are any

two propositions; then we shall write the proposition that *if P then Q* as:

$$P \rightarrow Q.$$

Again, let P be any proposition; then we shall write the proposition that it is *not* the case that P as:

$$-P.$$

The proposition $P \rightarrow Q$ will be called a *conditional* proposition, or simply a *conditional,* with the proposition P as its *antecedent* and the proposition Q as its *consequent.* For example, the antecedent of the proposition that if it is raining then it is snowing is the proposition that it is raining, and its consequent is the proposition that it is snowing. The proposition $-P$ will be called the *negation* of P. For example, the proposition that it is not snowing is the negation of the proposition that it is snowing.

The letters ' P ', ' Q ', used here, should be compared with the variables ' x ', ' y ' of algebra; they may be considered as a kind of variable, and are frequently called by logicians *propositional variables.* In introducing the minus sign ' $-$ ', I might say: let x and y be any two numbers; then I shall write the result of subtracting y from x as $x-y$. In an analogous way I introduced ' $\rightarrow$ ' above, using propositional variables in place of numerical variables, since in logic we are concerned with propositions not numbers.

Propositional variables will also help us to express the logical form of complex propositions (compare the use of schematic letters ' F ' and ' G ' in (9) of Section 1). Consider, for example, the complex proposition

(1) If it is raining, then it is not the case that if it is not snowing it is not raining.

Let us use ' P ' for the proposition that it is raining and ' Q ' for the proposition that it is snowing. Then, with the aid of ' $\rightarrow$ ' and ' $-$ ', we may write (1) symbolically as:

(2) $P \rightarrow -(-Q \rightarrow -P)$

(we introduce brackets here in an entirely obvious way). (2), as well as being a kind of shorthand for (1), with the advantages of brevity and clarity—once at least the feeling of strangeness associated with novel symbolism has worn off—succeeds in expressing the logical form of (1). We can see that (2) also gives the logical form of the quite different proposition

(3) If there is a fire, then it is not the case that if there is not smoke there is not a fire:

here P is a stand-in for the proposition that there is a fire, and Q for the proposition that there is smoke.

When we argue, we draw or *deduce* or *derive* a conclusion from given premisses; in logic we formulate rules, called *rules of derivation*, whose object is so to control the activity of deduction as to ensure that the conclusion reached is validly reached. Another feature of ordinary argumentation is that it proceeds *in stages*: the conclusion of one step is used as a premiss for a new step, and so on until a final conclusion is reached. It will be helpful, therefore, if we distinguish at once between *assumptions* and *premisses*. By an *assumption*, we shall understand a proposition which is, in a given stretch of argumentation, the conclusion of *no* step of reasoning, but which is rather taken for granted at the outset of the total argument. By a *premiss*, we shall understand a proposition which is used, at a particular stage in the total argument, to obtain a certain conclusion. An assumption may be—and characteristically will be—*used* as a premiss at a given stage in an argument in order to obtain a certain conclusion. This conclusion may itself then be used as a premiss for a further step in the argument, and so on. Thus a premiss at a certain stage will be *either* an assumption of the argument as a whole *or* a conclusion of an earlier phase in the argument. At any given stage in the total argument, we shall have a conclusion obtained ultimately from a certain assumption or combination of assumptions, and we shall say that this conclusion *rests on* or *depends on* that assumption (those assumptions).

Roughly, our procedure in setting out arguments will be as follows. Each step will be marked by a new line, and each line will be numbered consecutively. On each line will appear *either* an assumption of the argument as a whole *or* a conclusion drawn from propositions at earlier lines and based on these propositions as premisses. To the right of each proposition will be stated the rule of derivation used to justify its appearance at that stage and (where necessary) the numbers of the premisses used. To the left of each

proposition will appear the numbers of the original assumptions on which the argument at that stage depends.

Rule of Assumptions (A)

The first rule of derivation to be introduced is the *rule of assumptions*, which we call A. This rule permits us to introduce at *any* stage of an argument *any* proposition we choose as an assumption of the argument. We simply write the proposition down as a new line, write 'A' to the right of it, and to the left of it we put its own number to show that it depends on itself as an assumption. Thus we might begin an argument

1 (1) $P \rightarrow Q$ A

This means that our first step has been to assume the proposition $P \rightarrow Q$ by the rule of assumptions. Or after nine lines of argument we may proceed

10 (10) $-Q$ A

This means that at the tenth line we assume the proposition $-Q$ by the rule of assumptions.

It may seem dangerously liberal that the rule of assumptions imposes no limits on the kind of assumptions we may make (in particular there is no question of ensuring that assumptions made are true). This is best understood by reminding ourselves that the logician's concern is with the soundness of the argument rather than the truth or falsity of any assumptions made; hence A allows us to make any assumptions we please—the job of the logician is to make sure that any conclusion based on them is validly based, *not* to investigate their credentials.

Modus ponendo ponens (MPP)

The second rule of derivation concerns the operator $\rightarrow$. We name it *modus ponendo ponens*, abbreviated to MPP, which was the medieval term for a closely related principle of reasoning. Given as premisses a conditional proposition and the antecedent of that conditional, MPP permits us to draw the consequent of the conditional as a conclusion. For example, given $P \rightarrow Q$ and P, we can deduce Q. Or, to take a more complicated example, given $-Q \rightarrow (-P \rightarrow Q)$

and $-Q$, we can deduce $-P \rightarrow Q$. Written more formally, these two arguments become:

1 1	(1)	$P \rightarrow Q$	A
2	(2)	P	A
1,2	(3)	Q	1,2 MPP

2 1	(1)	$-Q \rightarrow (-P \rightarrow Q)$	A
2	(2)	$-Q$	A
1,2	(3)	$-P \rightarrow Q$	1,2 MPP

On the first two lines of each of these arguments, we make the required assumptions by the rule A, numbering on the left accordingly. At line (3), we draw the appropriate conclusion by the rule MPP: the *consequent* of the conditional at line (1), given at line (2) the *antecedent* of that conditional. To the right at line (3) in both cases, we note the rule used (MPP) together with the numbers of the premisses used in this application of the rule. To the left at line (3), we mark the assumptions on which the conclusion rests—in this case again (1) and (2), which here are both premisses for the application of MPP and assumptions of the total argument.

Here are more complicated examples, using only the two rules A and MPP. I shall show first that, given the assumptions $P \rightarrow Q$, $Q \rightarrow R$, and P, we may validly conclude R.

ie YA : $\{P \rightarrow Q, Q \rightarrow R\} \vdash P \rightarrow R$

3 1	(1)	$P \rightarrow Q$	A
2	(2)	$Q \rightarrow R$	A
3	(3)	P	A
1,3	(4)	Q	1,3 MPP
1,2,3	(5)	R	2,4 MPP

The first three lines here merely make the necessary assumptions. At line (4), we draw by MPP the conclusion Q, given at line (1) the conditional $P \rightarrow Q$ and at line (3) its antecedent P. Hence (1) and (3) are mentioned to the right as premisses for the application of the rule and to the left as the assumptions used at that stage. At line (5), we use Q, the conclusion at line (4), as a premiss for a new application of MPP, noting that Q is the antecedent of the conditional $Q \rightarrow R$ assumed at line (2). So we obtain the desired

conclusion R from (2) and (4) as premisses. The numbers 2 and 4 appear on the right accordingly. In deciding what assumptions to cite on the left, we note that (4) rests on (1) and (3), whilst (2) rests only on itself: we 'pool' these assumptions to obtain (1), (2), and (3).

Secondly, I show that, given $P \rightarrow (Q \rightarrow R)$, $P \rightarrow Q$, and P, we may validly conclude R.

ie YA : $\{P \rightarrow (Q \rightarrow R), P \rightarrow Q\} \vdash P \rightarrow R$

4 1	(1)	$P \rightarrow (Q \rightarrow R)$	A
2	(2)	$P \rightarrow Q$	A
3	(3)	P	A
1,3	(4)	$Q \rightarrow R$	1,3 MPP
2,3	(5)	Q	2,3 MPP
1,2,3	(6)	R	4,5 MPP

At lines (4) and (5), the premisses used for the applications of MPP are also assumptions, so that the same pair of numbers appears on the right and on the left. But at line (6), the premisses are the conditional (4), $Q \rightarrow R$, and its antecedent (5), Q, neither of which are assumptions of the argument as a whole: in determining the numbers on the left, therefore, we 'pool' the assumptions on which (4) and (5) rest—(1), (3) and (2), (3) respectively—to obtain (1), (2), and (3).

It should be obvious that MPP is a reliable principle of reasoning. It can never lead us, at least, from true premisses to a false conclusion. For it is a basic feature of our use of 'if . . . then . . .' that if a conditional is true and if also its antecedent is true then its consequent must be true too, and MPP precisely allows us to affirm as a conclusion the consequent of a conditional, given as premisses the conditional itself and its antecedent.

It will be a help to have an abbreviation for the cumbersome expression 'given as assumptions . . ., we may validly conclude . . .'. To this end, I introduce the symbol

$$\vdash,$$

called often but misleadingly in the literature of logic the *assertion-sign*. It may conveniently be read as 'therefore'. Before it, we list (in any order) our assumptions, and after it we write the conclusion drawn. Using this notation, we may conveniently sum up the four pieces of reasoning above (from now on to be called *proofs*) thus:

1 $P \rightarrow Q, P \vdash Q$;
2 $-Q \rightarrow (-P \rightarrow Q), -Q \vdash -P \rightarrow Q$;
3 $P \rightarrow Q, Q \rightarrow R, P \vdash R$;
4 $P \rightarrow (Q \rightarrow R), P \rightarrow Q, P \vdash R$.

Results obtained in this form we shall call *sequents*. Thus a sequent is an argument-frame containing a set of assumptions and a conclusion which is claimed to follow from them. Effectively, sequents which we can prove embody valid patterns of argument in the sense that, if we take the P, Q, R, . . . in a proved sequent to be actual propositions, then, reading '$\vdash$' as 'therefore', we obtain a valid argument. The propositions to the left of '$\vdash$' become assumptions of the argument, and the proposition to the right becomes a conclusion validly drawn from those assumptions. From this point of view, in constructing proofs we are demonstrating the validity of patterns of argument, which is one of the logician's chief concerns.

The sequent proved can be written down immediately from the last line of the proof.[1] In place of the numbers on the left, we write the propositions appearing on the corresponding lines; then we place the assertion sign; finally, we add as conclusion the proposition on the last line itself. To see this, the four sequents above should be compared with the last lines of the corresponding proofs.

Modus tollendo tollens (MTT)

The third rule of derivation concerns both $\rightarrow$ and $-$. Again we use a medieval term for it, *modus tollendo tollens*, abbreviated to MTT. Given as premisses a conditional proposition and *the negation* of its consequent, MTT permits us to draw *the negation* of the antecedent of the conditional as a conclusion.

Here are two simple examples of the use of MTT. I set the precedent of citing the sequent proved before the proof.

5 $P \rightarrow Q, -Q \vdash -P$

1	(1) $P \rightarrow Q$	A
2	(2) $-Q$	A
1,2	(3) $-P$	1,2 MTT

[1] Thus we take a proof as a proof of a *sequent*; but it is also natural to say, in a different sense, that in a proof a *conclusion* is proved *from* certain assumptions. This resultant ambiguity in the word 'prove' is fairly harmless.

6 $P \rightarrow (Q \rightarrow R), P, -R \vdash -Q$

1	(1) $P \rightarrow (Q \rightarrow R)$	A
2	(2) P	A
3	(3) $-R$	A
1,2	(4) $Q \rightarrow R$	1,2 MPP
1,2,3	(5) $-Q$	3,4 MTT

For line (5), we notice that (3), $-R$, is the negation of the consequent of the conditional (4), $Q \rightarrow R$, so that by MTT we may conclude the negation $-Q$ of the antecedent of (4): to the right, we cite (3) and (4), and to the left (1) and (2)—the assumptions on which (4) rests—and (3)—the assumption, namely itself, on which (3) rests.

We may see the soundness of the rule MTT by ordinary examples. The following are evidently sound arguments:

(4) If Napoleon was Chinese, then he was Asiatic; Napoleon was not Asiatic; therefore he was not Chinese.

(5) If Napoleon was French, then he was European; Napoleon was not European; therefore he was not French.

In both cases, given a conditional and the negation of its consequent, we deduce validly the negation of its antecedent: in (4) the conclusion is true, and so are both premisses; in (5) the conclusion is false, but so is one premiss. It should be clear that this pattern of reasoning will never lead from premisses which are *all* true to a *false* conclusion.

Double negation (DN)

The fourth rule of derivation purely concerns negation. By the *double negation* of a proposition P we understand the proposition $--P$. Intuitively, to affirm that it is not the case that it is not the case that it is raining is the same as to affirm that it is raining, and this holds for any proposition whatsoever: the double negation of a proposition is identical with the proposition itself. Hence from the double negation of a proposition we can derive validly the proposition, and vice versa. This principle lies behind the *rule of double negation* (DN): given as premiss the double negation of a proposition, DN permits us to draw the proposition itself as conclusion; and given as premiss any proposition, DN permits us to draw its double negation as conclusion. Unlike MPP and MTT, DN requires only one premiss for its application, not two. Its use is exemplified in the following proofs.

7 $P \rightarrow -Q, Q \vdash -P$

1	(1) $P \rightarrow -Q$	A
2	(2) Q	A
2	(3) $--Q$	2 DN
1,2	(4) $-P$	1,3 MTT

Note especially that, since the consequent of (1) $P \rightarrow -Q$ is $-Q$, we need to obtain the *negation* of this, i.e. $--Q$, before we can apply the rule MTT: hence we require the step of DN from (2) to (3) before the use of MTT at line (4).

8 $-P \rightarrow Q, -Q \vdash P$

1	(1) $-P \rightarrow Q$	A
2	(2) $-Q$	A
1,2	(3) $--P$	1,2 MTT
1,2	(4) P	3 DN

Note especially that from (1) and (2) by MTT we draw as conclusion the *negation* of the antecedent of (1), i.e. $--P$: hence we require the step of DN from (3) to (4) in order to obtain the conclusion P. Note also that the conclusion of an application of DN rests on exactly the same assumptions as its premiss.

Conditional proof (CP)

The rules MPP and MTT enable us to use a conditional *premiss*, together with either its antecedent or the negation of its consequent, in order to obtain a certain conclusion, either its consequent or the negation of its antecedent. But how may we derive a conditional *conclusion*? The most natural device is to take the antecedent of the conditional we wish to prove as an extra assumption, and aim to derive its consequent as a conclusion: if we succeed, we may take this as a proof of the original conditional from the original assumptions (if any). For example, given that all Germans are

Europeans, how might we prove that if Napoleon was German then he was European? We naturally say: suppose Napoleon was German (here we take the antecedent of the conditional to be proved as an extra assumption); now all Germans are Europeans (the given assumption); therefore Napoleon was European (here we derive the consequent as conclusion); so if Napoleon was German he was European (here we treat the previous steps of the argument as a proof of the desired conditional).

The fifth rule of derivation, the *rule of conditional proof* (CP), imitates exactly this natural procedure and is our most general device for obtaining conditional conclusions. Its working is harder to grasp than that of the earlier rules, but familiarity with it is indispensable. I first state it, then exemplify and discuss it.

Suppose some proposition (call it B) depends, as one of its assumptions, on a proposition (call it A); then CP permits us to derive the conclusion A $\rightarrow$ B on the remaining assumptions (if any). In other words, at a certain stage in a proof we have derived the conclusion B from assumption A (and perhaps other assumptions); then CP enables us to take this as a proof of A $\rightarrow$ B from the other assumptions (if any).

For example:

9 $P \rightarrow Q \vdash -Q \rightarrow -P$

1	(1) $P \rightarrow Q$	A
2	(2) $-Q$	A
1,2	(3) $-P$	1,2 MTT
1	(4) $-Q \rightarrow -P$	2,3 CP

In attempting to derive the conditional $-Q \rightarrow -P$ from $P \rightarrow Q$, we first assume its antecedent $-Q$ at line (2), and derive its consequent $-P$ at line (3); CP at line (4) enables us to treat this as a proof of $-Q \rightarrow -P$ from just assumption (1). On the right, we give first the number of the assumed antecedent and second the number of the concluded consequent. On the left, the assumption (2) at line (3) disappears into the antecedent of the new conditional, and we are left with (1) alone. Always, in an application of CP, the number of assumptions falls by one in this manner, the one omitted being called the *discharged assumption*.

10 $P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)$

1	(1) $P \rightarrow (Q \rightarrow R)$	A
2	(2) Q	A
3	(3) P	A
1,3	(4) $Q \rightarrow R$	1,3 MPP
1,2,3	(5) R	2,4 MPP
1,2	(6) $P \rightarrow R$	3,5 CP
1	(7) $Q \rightarrow (P \rightarrow R)$	2,6 CP

A more complicated example, involving double use of CP: in attempting to derive the conditional $Q \rightarrow (P \rightarrow R)$ from $P \rightarrow (Q \rightarrow R)$, we first assume its antecedent Q at line (2), and aim to derive its consequent $P \rightarrow R$; since this consequent is also conditional, we assume *its* antecedent P at line (3), and aim to derive its consequent R. This is achieved by two steps of MPP (lines (4) and (5)); at line (6), we treat this by CP as a proof of $P \rightarrow R$ from assumptions (1) and (2), and we cite to the right line (3) (the assumption of the antecedent) and line (5) (the derivation of the consequent). In turn, we treat this at line (7) as a proof of $Q \rightarrow (P \rightarrow R)$ from assumption (1) alone, and we cite to the right line (2) (the assumption of its antecedent) and line (6) (the derivation of its consequent). As before, the assumptions on the left decrease by one at each step of CP.

11 $Q \rightarrow R \vdash (-Q \rightarrow -P) \rightarrow (P \rightarrow R)$

1	(1) $Q \rightarrow R$	A
2	(2) $-Q \rightarrow -P$	A
3	(3) P	A
3	(4) $--P$	3 DN
2,3	(5) $--Q$	2,4 MTT
2,3	(6) Q	5 DN
1,2,3	(7) R	1,6 MPP
1,2	(8) $P \rightarrow R$	3,7 CP
1	(9) $(-Q \rightarrow -P) \rightarrow (P \rightarrow R)$	2,8 CP

This proof uses all five rules of derivation introduced so far, and deserves study. Aiming to prove a complex conditional, we assume its antecedent $-Q \rightarrow -P$ at line (2), and try to prove its consequent $P \rightarrow R$. Since this is conditional, we assume its antecedent P at line (3), and after a series of steps using DN, MTT, and MPP we derive its consequent R at line (7). Two steps of CP, paralleling the last two steps in the proof of 10, complete the job by discharging in turn the assumptions (3) and (2).

Proofs 10 and 11 suggest a useful and important general method for discovering the proofs of sequents with complex conditionals as conclusion. After using the rule A for the assumptions given in the sequent, we assume also the antecedent of the desired conditional conclusion, and aim to prove its consequent; if this is also a conditional, we assume its antecedent, and aim to prove its consequent; we repeat this procedure, until our target becomes to prove a non-conditional conclusion. If we can derive this from the assumptions we now have, the right number of CP steps, applied in reverse order, will prove the original sequent.

I end this section with a remark on two common fallacies, so common that they have received names. In accordance with rule MPP, if a conditional is true and also its antecedent, then we can soundly derive its consequent. If a conditional is true and also its *consequent*, is it sound to derive its *antecedent*? The following example shows that it is not sound to do so: it is true that if Napoleon was German then he was European, since all Germans are Europeans; and it is true that Napoleon was European; but it is false, and so cannot soundly be deduced from these true premisses, that Napoleon was German. To suppose that it is sound to derive the antecedent of a conditional from the conditional and its consequent is to commit the fallacy of *affirming the consequent*. Again, in accordance with rule MTT, if a conditional is true and also the negation of its consequent, then we can soundly derive the negation of its antecedent. But it is not sound to derive the negation of a conditional's *consequent* from the conditional itself and the negation of its *antecedent*, and to suppose that it is sound is to commit the fallacy of *denying the antecedent*. The same example may be used: it is true that if Napoleon was German then he was European, and true also that he was *not* German; but it is not true that Napoleon was not European.

Put schematically: the sequents

1 $P \to Q, P \vdash Q$ and

5 $P \to Q, -Q \vdash -P$

are sound patterns of reasoning, as we have proved. But the sequents

6 $P \to Q, Q \vdash P$ and

7 $P \to Q, -P \vdash -Q$

are not sound patterns, as we have shown by finding *examples* of propositions P and Q such that the assumptions of (6) and (7) turn out true whilst their conclusions turn out false; for it is a necessary condition of a sound pattern of argument that it shall *never* lead us from assumptions that are all true to a false conclusion. (6) is in fact the pattern of the fallacy of affirming the consequent, and (7) that of the fallacy of denying the antecedent.

EXERCISES

1 Find proofs for the following sequents, using the rules of derivation introduced so far:

(*a*) $P \to (P \to Q), P \vdash Q$

(*b*) $Q \to (P \to R), -R, Q \vdash -P$

(*c*) $P \to --Q, P \vdash Q$

(*d*) $--Q \to P, -P \vdash -Q$

(*e*) $-P \to -Q, Q \vdash P$

(*f*) $P \to -Q \vdash Q \to -P$

(*g*) $-P \to Q \vdash -Q \to P$

(*h*) $-P \to -Q \vdash Q \to P$

(*i*) $P \to Q, Q \to R \vdash P \to R$

(*j*) $P \to (Q \to R) \vdash (P \to Q) \to (P \to R)$

(*k*) $P \to (Q \to (R \to S)) \vdash R \to (P \to (Q \to S))$

(*l*) $P \to Q \vdash (Q \to R) \to (P \to R)$

(*m*) $P \vdash (P \to Q) \to Q$

(*n*) $P \vdash (-(Q \to R) \to -P) \to (-R \to -Q)$

2 Show that the following sequents are unsound patterns of argument, by finding actual propositions for P and Q such that the assumption(s) are true and the conclusion false:

(*a*) $P \to -Q, -P \vdash Q$

(*b*) $-P \to -Q, -Q \vdash -P$

(*c*) $P \to Q \vdash Q \to P$

3 CONJUNCTION AND DISJUNCTION

Of the four sentence-forming operators on sentences mentioned in the last section as being of importance to the logician, only two have so far been discussed: 'if . . . then . . .' and 'not'. In the present section, we introduce rules for arguments involving '. . . and . . .' and 'either . . . or . . .'.

Let P and Q be any two propositions. Then the proposition that *both P and Q* is called the *conjunction* of P and Q and is written

$$P \,\&\, Q.$$

P and Q are called the *conjuncts* of the conjunction $P \,\&\, Q$. Similarly, the proposition that *either P or Q* is called the *disjunction* of P and Q and is written

$$P \vee Q.$$

P and Q are called the *disjuncts* of the disjunction $P \vee Q$. (The symbol 'v' is intended to remind classicists of the Latin 'vel' as opposed to 'aut': for $P \vee Q$ is understood not to exclude the possibility that both P and Q might be the case.)

There are two rules of derivation concerning &, the *rule of &-introduction* and the *rule of &-elimination*; and there are two rules concerning v, the *rule of* v*-introduction* and the *rule of* v*-elimination*. Introduction-rules serve the purpose of enabling us to derive *conclusions* containing & or v, whilst elimination-rules serve the purpose of enabling us to use *premisses* containing & or v. We discuss and exemplify these rules in turn.

&-introduction (&I)

The rule of &-introduction (&I) is exceptionally easy to master. Given any two propositions as premisses, &I permits us to derive their conjunction as a conclusion. The rule clearly corresponds to a sound principle of reasoning; for if A and B are the case separately, it is obvious that A & B must be the case. The following proofs exemplify the use of &I.

12 $P, Q \vdash P \,\&\, Q$

1	(1) P	A
2	(2) Q	A
1,2	(3) $P \,\&\, Q$	1,2 &I

At line (3), by &I we conclude the conjunction of the assumptions (1) and (2). To the right, we cite (1) and (2) as the premisses for the application of &I; to the left we cite the pool of the assumptions on which these premisses rest—in this case themselves.

13 $(P \,\&\, Q) \rightarrow R \vdash P \rightarrow (Q \rightarrow R)$

1	(1) $(P \,\&\, Q) \rightarrow R$	A
2	(2) P	A
3	(3) Q	A
2,3	(4) $P \,\&\, Q$	2,3 &I
1,2,3	(5) R	1,4 MPP
1,2	(6) $Q \rightarrow R$	3,5 CP
1	(7) $P \rightarrow (Q \rightarrow R)$	2,6 CP

In attempting to prove the conditional $P \rightarrow (Q \rightarrow R)$, we assume first its antecedent P (line (2)) and second the antecedent of its consequent Q (line (3)). A step of &I at line (4) gives us the conjunction of these assumptions, enabling us to apply MPP at line (5) to obtain R. Two steps of CP complete the proof.

&-elimination (&E)

The rule of &-elimination (&E) is just as straightforward. Given any conjunction as premiss, &E permits us to derive either conjunct as a conclusion. Again, the rule is evidently sound; for if A & B is the case, it is obvious that A separately and B separately must be the case. Here are examples.

14 $P \,\&\, Q \vdash P$

1	(1) $P \,\&\, Q$	A
1	(2) P	1 &E

15 $P \,\&\, Q \vdash Q$

1	(1) $P \,\&\, Q$	A
1	(2) Q	1 &E

16 $P \rightarrow (Q \rightarrow R) \vdash (P \,\&\, Q) \rightarrow R$

1	(1) $P \rightarrow (Q \rightarrow R)$	A
2	(2) $P \,\&\, Q$	A
2	(3) P	2 &E
2	(4) Q	2 &E
1,2	(5) $Q \rightarrow R$	1,3 MPP
1,2	(6) R	4,5 MPP
1	(7) $(P \,\&\, Q) \rightarrow R$	2,6 CP

We desire the conditional conclusion $(P \,\&\, Q) \rightarrow R$, and so we assume its antecedent at line (2) and aim for R. &E is used at lines (3) and (4) to obtain the conjuncts P and Q separately, which are required for the MPP steps at lines (5) and (6). To the right, in an application of &E, we cite the conjunction employed as a premiss, and to the left the assumptions on which that conjunction rests.

The rules &I and &E are frequently used together in the same proof. For example:

17 $P \,\&\, Q \vdash Q \,\&\, P$

1	(1) $P \,\&\, Q$	A
1	(2) P	1 &E
1	(3) Q	1 &E
1	(4) $Q \,\&\, P$	3,2 &I

18 $Q \rightarrow R \vdash (P \,\&\, Q) \rightarrow (P \,\&\, R)$

1	(1) $Q \rightarrow R$	A
2	(2) $P \,\&\, Q$	A
2	(3) P	2 &E
2	(4) Q	2 &E

1,2 (5) R 1,4 MPP

1,2 (6) $P \& R$ 3,5 &I

1 (7) $(P \& Q) \rightarrow (P \& R)$ 2,6 CP

We desire the conditional conclusion $(P \& Q) \rightarrow (P \& R)$; hence we assume the antecedent $P \& Q$ and aim for $P \& R$. This aim is translated into the aim for P and R separately, from which $P \& R$ will follow by &I. P follows from $P \& Q$ by &E, and so does Q, which can be used in conjunction with line (1) to obtain R by MPP (line (5)). When &I is used at line (6), the premisses are (3) and (5), and these rest respectively on assumption (2) and assumptions (1) and (2). Hence the pool of these—(1) and (2)—is cited to the left.

v-*introduction* (vI)

The rule of v-introduction we name vI. Given any proposition as premiss, vI permits us to derive the disjunction of that proposition and *any* proposition as a conclusion. Thus from P as premiss, we may derive $P \vee Q$ as a conclusion, or $Q \vee P$ as a conclusion; and here it makes no difference what proposition Q is. Clearly the conclusion will in general be much weaker than the premiss, in an application of vI. It may, that is, be the case that either P or Q even when it is not the case that P. None the less, the rule is acceptable in the sense that when P is the case it must be also the case that either P or Q. For example, it is the case that Charles I was beheaded. It follows that either he was beheaded or he was sent to the electric chair, even though of course he was not sent to the electric chair. A disjunction $P \vee Q$ is true if *at least one* of its disjuncts is true, so that rule vI cannot lead from a true premiss to a false conclusion (though it may lead to a dull one).

v-*elimination* (vE)

The rule of v-elimination (vE) is rather more complex. I first state it, then explain and justify it, and finally exemplify both it and vI. Let A, B, and C be any three propositions, and suppose (*a*) that we are given that A v B, (*b*) that from A as an assumption we can derive C as conclusion, (*c*) that from B as an assumption we can derive C as conclusion; then vE permits us to draw C as a conclusion from any assumptions on which A v B rests, together with any

assumptions (apart from A itself) on which C rests in its derivation from A and any assumptions (apart from B itself) on which C rests in its derivation from B. Thus the typical situation for a step of vE is as follows: we have a disjunction A v B as a premiss, and wish to derive a certain conclusion C; we aim first to derive C from the first disjunct A, and second to derive C from the second disjunct B. When these phases of the argument are completed, we have the situation described in (*a*), (*b*), and (*c*) above, and can apply vE to obtain the conclusion C direct from A v B. On the right, we unfortunately need to cite five lines: (i) the line where the disjunction A v B appears; (ii) the line where A is assumed; (iii) the line where C is derived from A; (iv) the line where B is assumed; (v) the line where C is derived from B. And on the left, the conclusion may rest on rather a complex pool of assumptions, derived from three sources: (i) any assumptions on which A v B rests; (ii) any assumptions on which C rests in its derivation from A, though not A itself; (iii) any assumptions on which C rests in its derivation from B, though not B itself.

Though involved to state exactly, the rule vE corresponds to an entirely natural principle of reasoning. Suppose it is the case that either A or B, i.e. that *one* of A or B is true; and suppose that on the assumption A, we can show C to be the case, i.e. that if A holds C holds; suppose also that on the assumption B, we can still show that C holds, i.e. that if B holds C also holds; then C holds either way. For example: you agree that either it is raining or it is fine (A v B); given that it is raining, then it is not fit to go for a walk (from A we derive C); given that it is fine, then it must be very hot, so that again it is not fit to go for a walk (from B we derive C). Hence either way it is not fit to go for a walk (we conclude C).

19 $P \vee Q \vdash Q \vee P$

1 (1) $P \vee Q$ A

2 (2) P A

2 (3) $Q \vee P$ 2 vI

4 (4) Q A

4 (5) $Q \vee P$ 4 vI

1 (6) $Q \vee P$ 1,2,3,4,5 vE

On line (1), we assume $P \vee Q$; since this is a disjunction, we aim to derive the conclusion $Q \vee P$ from the first disjunct P, assumed at line (2), and also from the second disjunct Q, assumed at line (4). This is achieved on lines (3) and (5) by steps of vI which should be obvious. At line (6), we conclude $Q \vee P$ from assumption (1) directly, since it follows from each disjunct separately. On the right, we cite line (1) (the disjunction), line (2) (assumption of first disjunct), line (3) (derivation of conclusion from that disjunct), line (4) (assumption of second disjunct), and line (5) (derivation of conclusion from that disjunct). To the left, we cite any assumptions on which the disjunction rests (here (1) rests on itself, which is therefore cited), together with any assumptions used to derive the conclusion from the disjuncts apart from the disjuncts themselves (inspection of the citations to the left of line (3) and (5) shows that there are none such). This proof should reveal the importance of keeping accurate assumption-records on the left of proofs: lines (3) and (5) here give indeed the right conclusion $Q \vee P$, but not from the right assumption, which is (1); this is achieved only at line (6), which differs from lines (3) and (5) in the annotation on the left.

20 $Q \rightarrow R \vdash (P \vee Q) \rightarrow (P \vee R)$

1	(1) $Q \rightarrow R$	A
2	(2) $P \vee Q$	A
3	(3) P	A
3	(4) $P \vee R$	3 vI
5	(5) Q	A
1,5	(6) R	1,5 MPP
1,5	(7) $P \vee R$	6 vI
1,2	(8) $P \vee R$	2,3,4,5,7 vE
1	(9) $(P \vee Q) \rightarrow (P \vee R)$	2,8 CP

The desired conclusion here is conditional; so we assume its antecedent $P \vee Q$ (line (2)), and aim to derive $P \vee R$; this assumption is a disjunction, so we assume each disjunct in turn (lines (3) and (5)) and derive the conclusion $P \vee R$ from each (lines (4) and (7)). Hence the citation on the right at line (8) is 2,3,4,5,7. The assumptions at

line (8) are those on which the disjunction $P \vee Q$ rests (itself, (2)), together with any used to obtain $P \vee R$ from (3) apart from (3) itself (none, as line (4) reveals) and any used to obtain $P \vee R$ from (5) apart from (5) itself (namely (1), as line (7) reveals). A step of CP completes the proof from (1) of the desired conditional.

21 $P \vee (Q \vee R) \vdash Q \vee (P \vee R)$

1	(1) $P \vee (Q \vee R)$	A
2	(2) P	A
2	(3) $P \vee R$	2 vI
2	(4) $Q \vee (P \vee R)$	3 vI
5	(5) $Q \vee R$	A
6	(6) Q	A
6	(7) $Q \vee (P \vee R)$	6 vI
8	(8) R	A
8	(9) $P \vee R$	8 vI
8	(10) $Q \vee (P \vee R)$	9 vI
5	(11) $Q \vee (P \vee R)$	5,6,7,8,10 vE
1	(12) $Q \vee (P \vee R)$	1,2,4,5,11 vE

This proof deserves detailed study, in the use both of vI and of vE. Careful attention to bracketing is required. The assumption is a disjunction, the second of whose disjuncts is a disjunction itself. The proof falls into two distinct parts, lines (2)–(4) and lines (5)–(11): the first part establishes the desired conclusion from the first disjunct of the original disjunction (line (4)), and the second part establishes the same conclusion from the second disjunct (line (11)). This should explain the final step of vE at line (12). The second part (lines (5)–(11)), which begins with a disjunctive assumption, also falls into two sub-parts and involves a subsidiary step of vE at line (11). Lines (6)–(7) obtain the conclusion from the first disjunct Q of (5), and lines (8)–(10) obtain the conclusion from its second disjunct R. Hence the final conclusion is obtained no less than five times in the proof, from different assumptions each time.

Reductio ad absurdum (RAA)

The last rule to be introduced at this stage is in many ways the most powerful and the most useful; it is easy to understand, though a little difficult to state precisely. We shall call it the *rule of reductio ad absurdum* (RAA). First, we define a *contradiction*. A *contradiction* is a conjunction the second conjunct of which is the negation of the first conjunct: thus $P \mathbin{\&} -P$, $R \mathbin{\&} -R$, $(P \rightarrow Q) \mathbin{\&} -(P \rightarrow Q)$ are all contradictions. Now suppose that from an assumption A, together perhaps with other assumptions, we can derive a contradiction as a conclusion; then RAA permits us to derive $-$A as a conclusion from those other assumptions (if any). This rule rests on the natural principle that, if a contradiction can be deduced from a proposition A, A cannot be true, so that we are entitled to affirm its negation $-$A.

Here are examples.

22 $P \rightarrow Q, P \rightarrow -Q \vdash -P$

1	(1) $P \rightarrow Q$	A
2	(2) $P \rightarrow -Q$	A
3	(3) P	A
1,3	(4) Q	1,3 MPP
2,3	(5) $-Q$	2,3 MPP
1,2,3	(6) $Q \mathbin{\&} -Q$	4,5 &I
1,2	(7) $-P$	3,6 RAA

This is a typical example of the use of RAA. Aiming at the conclusion $-P$, we assume (line (3)) P and hope to derive from it a contradiction; for, if P leads to a contradiction, we can conclude $-P$ by RAA. We obtain the contradiction $Q \mathbin{\&} -Q$ at line (6), and so conclude $-P$ at line (7). On the right, we cite the assumption which we are blaming for the contradiction—the one whose negation we conclude in the RAA step, here (3)—and the contradiction itself, here (6). On the left, as in a CP step, the number of assumptions naturally falls by one, there being omitted the one which we blame for the contradiction.

23 $P \rightarrow -P \vdash -P$

1	(1) $P \rightarrow -P$	A
2	(2) P	A
1,2	(3) $-P$	1,2 MPP
1,2	(4) $P \mathbin{\&} -P$	2,3 &I
1	(5) $-P$	2,4 RAA

Again desiring $-P$, we assume P (line (2)) and obtain a contradiction (line (4)). Given (1), therefore, we conclude $-P$ by RAA. The sequent proved is striking, and perhaps unexpected—given that if a proposition is the case then so is its negation, we can conclude that its negation is true. This is the first surprising result to be established by our rules, but there will be more.

The rule RAA is particularly useful when we wish to derive *negative* conclusions. It suggests that, instead of attempting a direct proof, we should assume the corresponding *affirmative* proposition and aim to derive a contradiction, thus indirectly establishing the negative. It can also be used, however, to establish affirmatives themselves, *via* DN. If we want to derive A, we may assume $-$A and obtain a contradiction. Hence by RAA we can conclude $--$A (the negation of what we assumed) and so by DN we obtain A. It is a good general tip for proof-discovery that, when direct attempts fail, often an RAA proof will succeed.

So far, ten rules of derivation have been introduced: we shall need no new ones until Chapter 3.

EXERCISES

1 Find proofs for the following sequents:

(*a*) $P \vdash Q \rightarrow (P \mathbin{\&} Q)$

(*b*) $P \mathbin{\&} (Q \mathbin{\&} R) \vdash Q \mathbin{\&} (P \mathbin{\&} R)$

(*c*) $(P \rightarrow Q) \mathbin{\&} (P \rightarrow R) \vdash P \rightarrow (Q \mathbin{\&} R)$

(*d*) $Q \vdash P \vee Q$

(*e*) $P \mathbin{\&} Q \vdash P \vee Q$

(*f*) $(P \rightarrow R) \mathbin{\&} (Q \rightarrow R) \vdash (P \vee Q) \rightarrow R$

(*g*) $P \rightarrow Q, R \rightarrow S \vdash (P \mathbin{\&} R) \rightarrow (Q \mathbin{\&} S)$

(*h*) $P \rightarrow Q, R \rightarrow S \vdash (P \vee R) \rightarrow (Q \vee S)$

(*i*) $P \rightarrow (Q \,\&\, R) \vdash (P \rightarrow Q) \,\&\, (P \rightarrow R)$

(*j*) $-P \rightarrow P \vdash P$

2 Show that the following sequents are unsound, by finding actual propositions for P and Q such that the assumption is true and the conclusion false:

(*a*) $P \vdash P \,\&\, Q$

(*b*) $P \vee Q \vdash P$

(*c*) $P \vee Q \vdash P \,\&\, Q$

(*d*) $P \rightarrow Q \vdash P \,\&\, Q$

4 THE BICONDITIONAL

There is a sentence-forming operator on sentences, of considerable importance to the logician though of rare occurrence in ordinary speech, which we have not so far introduced. This is '. . . if and only if . . .'. We study it in the present section.

To begin with, let us consider the differences between 'if . . . then . . .' and 'only if . . . then . . .'. Compare the following two propositions:

(1) if it snows it turns colder;

(2) only if it snows it turns colder.

(1) affirms that its snowing is *sufficient* for it to turn colder, whilst (2) affirms that its snowing is *necessary* for it to turn colder, that if it is to turn colder it *must* snow. Hence we shall say that, whenever it is the case that if P then Q, P is a *sufficient condition* for Q, and, whenever it is the case that only if P then Q, P is a *necessary condition* for Q. To make this fundamental distinction clearer, let us compare

(3) if you hit the glass with a hammer, you will break it;

(4) only if you hit the glass with a hammer will you break it.

(3) is very likely true; (4) is very likely false, since there are other ways of breaking the glass than by wielding a hammer. On the other hand, of the two propositions

(5) if you use a screwdriver, you will unscrew that very tight screw;

(6) only if you use a screwdriver will you unscrew that very tight screw,

(5) may very well be false (you may use the screwdriver and still not unscrew the screw), and (6) true, since there may be no other way of turning the screw than by wielding a screwdriver. Hitting with a hammer is (probably) a sufficient but not a necessary condition for breaking a glass; using a screwdriver is (possibly) a necessary but not a sufficient condition for turning a tight screw.

In scientific and in mathematical reasoning, and consequently in logic, we are often interested in a condition being *both sufficient and necessary*. P will be a sufficient and necessary condition for Q in just the case that Q holds *if and only if* P holds. Hence our interest in '. . . if and only if . . .'. It may seem, therefore, that we require a special symbol for 'only if . . . then . . .'; but that this is not so may be seen as follows.

Suppose that only if P then Q; then P is a necessary condition for Q, that is, for Q to be the case P must be the case; hence if Q is the case, so is P. For example, suppose, as before, that using a screwdriver is a necessary condition for turning the screw; then if the screw is turned, a screwdriver has been used. In short, given that only if P then Q, we can infer that if Q then P. Conversely, suppose that if Q then P; then for Q to be the case P must be the case, for if Q is the case and P not the case it cannot hold that if Q then P; hence P is a necessary condition for Q, that is, only if P then Q. These two arguments suggest that to affirm only if P then Q is to affirm if Q then P. Hence to express symbolically 'only if P then Q' we may use '$\rightarrow$' and simply write

$$Q \rightarrow P.$$

To affirm, therefore, that Q *if and only if* P is to affirm that if P then Q and only if P then Q, which is to affirm that if P then Q and if Q then P; or, in symbols,

$$(P \rightarrow Q) \,\&\, (Q \rightarrow P).$$

But, rather than use this complex expression, we may conveniently adopt a double arrow and write as an abbreviation

$$P \leftrightarrow Q.$$

(This symbol helps to emphasize the mutuality of the relationship between P and Q.) We call the proposition $P \leftrightarrow Q$ the *biconditional* of P and Q.

What are the properties of the biconditional in argument? We could lay down rules of derivation for this operator, as we have done for the four operators of the previous two sections. But in fact the properties of '$\leftrightarrow$' follow readily from those of '&' and '$\rightarrow$', in terms of which we have just defined the biconditional. For example:

24 $P \leftrightarrow Q \vdash Q \leftrightarrow P$

1	(1) $P \leftrightarrow Q$	A
1	(2) $(P \rightarrow Q) \,\&\, (Q \rightarrow P)$	1
1	(3) $P \rightarrow Q$	2 &E
1	(4) $Q \rightarrow P$	2 &E
1	(5) $(Q \rightarrow P) \,\&\, (P \rightarrow Q)$	4,3 &I
1	(6) $Q \leftrightarrow P$	5

Here the step from (1) to (2) is justified by our taking '$P \leftrightarrow Q$' as an abbreviation for '$(P \rightarrow Q) \,\&\, (Q \rightarrow P)$': at (2) we merely expand what we have assumed at (1). Similarly, but in reverse, the step from (5) to (6) is justified: for (6) is merely an abbreviation for the conclusion (5). However, we need to ratify such steps more precisely, and to this end we introduce the following formal definition of the biconditional:

$$Df. \leftrightarrow: \quad A \leftrightarrow B = (A \rightarrow B) \,\&\, (B \rightarrow A).$$

This definition is to be understood as a very condensed way of saying: given any two sentences A and B, we may replace in a proof the sentence $A \leftrightarrow B$ by the sentence $(A \rightarrow B) \,\&\, (B \rightarrow A)$, and vice versa. When this definition is applied, we shall cite '$Df. \leftrightarrow$' on the right. Lines (2) and (6) of the last proof should in fact be so marked.

The next few proofs exemplify the use of this definition.

25 $P, P \leftrightarrow Q \vdash Q$

1	(1) P	A
2	(2) $P \leftrightarrow Q$	A
2	(3) $(P \rightarrow Q) \,\&\, (Q \rightarrow P)$	2 $Df. \leftrightarrow$
2	(4) $P \rightarrow Q$	3 &E
1,2	(5) Q	1,4 MPP

26 $P \leftrightarrow Q, Q \leftrightarrow R \vdash P \leftrightarrow R$

1	(1) $P \leftrightarrow Q$	A
2	(2) $Q \leftrightarrow R$	A
1	(3) $(P \rightarrow Q) \,\&\, (Q \rightarrow P)$	1 $Df. \leftrightarrow$
1	(4) $P \rightarrow Q$	3 &E
1	(5) $Q \rightarrow P$	3 &E
2	(6) $(Q \rightarrow R) \,\&\, (R \rightarrow Q)$	2 $Df. \leftrightarrow$
2	(7) $Q \rightarrow R$	6 &E
2	(8) $R \rightarrow Q$	6 &E
9	(9) P	A
1,9	(10) Q	4,9 MPP
1,2,9	(11) R	7,10 MPP
1,2	(12) $P \rightarrow R$	9,11 CP
13	(13) R	A
2,13	(14) Q	8,13 MPP
1,2,13	(15) P	5,14 MPP
1,2	(16) $R \rightarrow P$	13,15 CP
1,2	(17) $(P \rightarrow R) \,\&\, (R \rightarrow P)$	12,16 &I
1,2	(18) $P \leftrightarrow R$	17 $Df. \leftrightarrow$

To derive $P \leftrightarrow R$ by $Df. \leftrightarrow$ we need to derive $(P \rightarrow R)$ & $(R \rightarrow P)$, and we aim at each conjunct separately. The first eight lines of the proof merely itemize the information in the assumptions, by applying $Df. \leftrightarrow$ and &E. This information (lines (4), (5), (7), (8)) is then used in a straightforward manner to derive the two required conjuncts (lines (9)–(12) and (13)–(16)).

27 $(P \,\&\, Q) \leftrightarrow P \vdash P \rightarrow Q$

1	(1) $(P \,\&\, Q) \leftrightarrow P$	A
1	(2) $((P \,\&\, Q) \rightarrow P) \,\&\, (P \rightarrow (P \,\&\, Q))$	1 $Df. \leftrightarrow$
1	(3) $P \rightarrow (P \,\&\, Q)$	2 &E
4	(4) P	A
1,4	(5) $P \,\&\, Q$	3,4 MPP

1,4	(6) Q	5 &E
1	(7) $P \rightarrow Q$	4,6 CP

By $Df \leftrightarrow$, (1) is an abbreviation for (2), if we take A to be the sentence 'P & Q' and B to be the sentence 'P'.

28 $P \,\&\, (P \leftrightarrow Q) \vdash P \,\&\, Q$

1	(1) $P \,\&\, (P \leftrightarrow Q)$	A
1	(2) $P \,\&\, ((P \rightarrow Q) \,\&\, (Q \rightarrow P))$	1 $Df. \leftrightarrow$
1	(3) P	2 &E
1	(4) $(P \rightarrow Q) \,\&\, (Q \rightarrow P)$	2 &E
1	(5) $P \rightarrow Q$	4 &E
1	(6) Q	3,5 MPP
1	(7) $P \,\&\, Q$	3,6 &I

In the proofs preceding this, when $Df. \leftrightarrow$ was applied, it was applied to a sentence as a whole, i.e. the sentence to which it was applied was of the form A $\leftrightarrow$ B; but this is not essential: here in fact at line (2) it is applied to the second conjunct of the proposition at line (1).

Although $Df. \leftrightarrow$ is like the ten rules of derivation introduced so far, in that it justifies transitions in a proof, it should not be thought of as another rule on a par with the rest. Its role in proofs is to enable us to take advantage of a piece of symbolic shorthand, rather than to enable us genuinely to derive conclusions from premisses. It happens that, for certain ends, we are interested in complex propositions such as $(P \rightarrow Q) \,\&\, (Q \rightarrow P)$, and to facilitate our study of them we agree to abbreviate our expressions for them to sentences such as '$P \leftrightarrow Q$'. This is a guide to the eye, a sop thrown to human weakness: were we brave enough, in place of 27 above, for example, we might merely prove

$$(7)\ ((P \,\&\, Q) \rightarrow P) \,\&\, (P \rightarrow (P \,\&\, Q)) \vdash P \rightarrow Q;$$

but the expression we have used discloses a pattern which we might miss in the expression of (7). Given, therefore, that we wish to take advantage of this abbreviation in proofs, we need a device for transforming sentences containing '$\leftrightarrow$' into sentences lacking it, and a reverse device for transforming sentences of the right form

lacking '$\leftrightarrow$' into sentences containing it: that is exactly what $Df. \leftrightarrow$ provides. To put the point in a slightly different way, any logical properties which '$\leftrightarrow$' may seem to have are merely properties of '&' and '$\rightarrow$' in symbolic disguise.

A definition such as $Df. \leftrightarrow$ may be called a *stipulative* definition, in that it stipulates or lays down the meaning of the symbol '$\leftrightarrow$' in terms of symbols '$\rightarrow$' and '&' whose meaning is known from the rules governing their deployment in proofs. To say that a definition is stipulative is not to say that it is *arbitrary* (though the actual symbol '$\leftrightarrow$' chosen is in a sense arbitrarily chosen). Indeed, I carefully prepared the ground for the definition by arguing that what we in fact understood by the proposition that Q if and only if P was that if P then Q and if Q then P. But formally the definition is stipulative in that it announces that a sign *is to be taken* in a certain way.

EXERCISES

1 Using $Df. \leftrightarrow$ in conjunction with the rules of derivation of sections 2 and 3, find proofs for the following sequents:

(*a*) $Q, P \leftrightarrow Q \vdash P$
(*b*) $P \rightarrow Q, Q \rightarrow P \vdash P \leftrightarrow Q$
(*c*) $P \leftrightarrow Q \vdash -P \leftrightarrow -Q$
(*d*) $-P \leftrightarrow -Q \vdash P \leftrightarrow Q$
28.4 (*e*) $(P \vee Q) \leftrightarrow P \vdash Q \rightarrow P$
(*f*) $P \leftrightarrow -Q, Q \leftrightarrow -R \vdash P \leftrightarrow R$

2 Just as '. . . if and only if . . .' can be defined in terms of 'if . . . then . . .' and '. . . and . . .', so 'unless . . ., then . . .' can be defined in terms of 'if . . . then . . .' and 'not'. For to affirm that unless P then Q is to affirm that if not P then Q (justify this by taking cases). Let us, therefore, stipulate

$$Df. *: A * B = -A \rightarrow B\ .$$

Using $Df. *$ in a way parallel to $Df. \leftrightarrow$, find proofs for the following sequents:

(*a*) $P * Q \vdash Q * P$
(*b*) $P * Q, P * R \vdash P * (Q \,\&\, R)$
(*c*) $P * Q, R * -Q \vdash P * R$
(*d*) $P * P \vdash P$
(*e*) $-P * R, -Q * R, P \vee Q \vdash R$

5 FURTHER PROOFS: RÉSUMÉ OF RULES

We now exemplify the use of our rules of derivation by some more advanced proofs. The sequents proved are themselves worth studying, as exhibiting some of the more basic formal properties of the operators concerned; the formal work deserves attention too, since it frequently illustrates a technique which the student should master as an aid to his own discovery of proofs. After many proofs I add notes which pick out interesting features and try to indicate how the proofs are discovered. There are not, it should be remembered, precise rules for proof-discovery; hints can be given, but actual practice is all-important. (To this end, the student might try to rediscover proofs of sequents in preceding sections.) At the end of the section, I add for reference purposes a statement of the rules introduced so far.

29 $P \vdash P$

1	(1) P	A

No shorter sequent than this can be proved, and its proof is the shortest possible proof: yet it is worth close attention. Line (1) affirms that, given (1), P follows; what is (1)?—the proposition P itself. That is, given P, we may conclude that P, which is the sequent to be proved. Is this really sound? It is often thought that to infer P from P is unsound, on the grounds that the argument is circular, but this is a misunderstanding; certainly the argument is circular (in the popular sense), but a circular argument is entirely sound (though extremely dull). Given that it is raining, the *safest possible* conclusion is that it is raining. If I infer a proposition from itself, I do not err in reasoning, though I do not advance in information either. From this standpoint, the rule of assumptions is precisely based on the principle of the soundness of a circular argument; for the rule of assumptions affirms that, given a certain proposition, we can at least infer that proposition.

Let A and B be two propositions such that we can prove both the sequent A $\vdash$ B and the sequent B $\vdash$ A; then we say that A and B are *interderivable*, and we write the fact thus:

$$A \dashv\vdash B,$$

using a suggestive symbol. For example, a comparison of sequents 13 and 16 (Section 3) reveals that $(P \,\&\, Q) \rightarrow R$ and $P \rightarrow (Q \rightarrow R)$ are interderivable, so that we may write in summary:

30 $(P \,\&\, Q) \rightarrow R \dashv\vdash P \rightarrow (Q \rightarrow R)$

In establishing an interderivability result, the work naturally falls into two halves. Thus:

31 $P \,\&\, (P \vee Q) \dashv\vdash P$

(*a*) $P \,\&\, (P \vee Q) \vdash P$

1	(1) $P \,\&\, (P \vee Q)$	A
1	(2) P	1 &E

(*b*) $P \vdash P \,\&\, (P \vee Q)$

1	(1) P	A
1	(2) $P \vee Q$	1 vI
1	(3) $P \,\&\, (P \vee Q)$	1,2 &I

In proving (31(*b*)) that $P \,\&\, (P \vee Q)$ follows from P, we prove that each conjunct follows separately: that P follows from P is in fact given at line (1) (compare 29 above and the note following).

32 $P \vee (P \,\&\, Q) \dashv\vdash P$

(*a*) $P \vee (P \,\&\, Q) \vdash P$

1	(1) $P \vee (P \,\&\, Q)$	A
2	(2) P	A
3	(3) $P \,\&\, Q$	A
3	(4) P	3 &E
1	(5) P	1,2,2,3,4 vE

(*b*) $P \vdash P \vee (P \,\&\, Q)$

1	(1) P	A
1	(2) $P \vee (P \,\&\, Q)$	1 vI

In 32(*a*), to show that P follows from the disjunction $P \vee (P \,\&\, Q)$, we need to show that it follows from each disjunct in turn in order to apply vE. That P follows from P is given by line (2) (compare again 29), and that it follows from P & Q is proved by &E at line (4). This should explain the double citation of '2' at line (5) on the right: the first '2' signifies the assumption of the first disjunct P at that line, and the second '2' signifies that the conclusion P is derived from that assumption at the same line.

33 $P \vee P \dashv\vdash P$

(*a*) $P \vee P \vdash P$

1	(1) $P \vee P$	A
2	(2) P	A
1	(3) P	1,2,2,2,2 vE

(*b*) $P \vdash P \vee P$

1	(1) P	A
1	(2) $P \vee P$	1 vI

33(*a*) line (3) reveals a limiting case of the use of vE. To derive P from $P \vee P$, by vE we need to show that P follows from each disjunct in turn; but the disjuncts are the same, P itself, so that the whole work is done by line (2): hence the four citations of '2' to the right at line (3).

34 $P, -(P \,\&\, Q) \vdash -Q$

1	(1) P	A
2	(2) $-(P \,\&\, Q)$	A
3	(3) Q	A
1,3	(4) $P \,\&\, Q$	1,3 &I
1,2,3	(5) $(P \,\&\, Q) \,\&\, -(P \,\&\, Q)$	2,4 &I
1,2	(6) $-Q$	3,5 RAA

To derive $-Q$, we proceed indirectly and assume Q, hoping to obtain a contradiction; this is achieved at line (5), whence RAA yields the desired sequent. The principle of reasoning associated with 34 has the medieval name *modus ponendo tollens*: if P is the case, and it is not the case that both P and Q, then it is not the case that Q.

35 $P \rightarrow Q \dashv\vdash -(P \,\&\, -Q)$

(*a*) $P \rightarrow Q \vdash -(P \,\&\, -Q)$

1	(1) $P \rightarrow Q$	A
2	(2) $P \,\&\, -Q$	A
2	(3) P	2 &E
2	(4) $-Q$	2 &E
1,2	(5) Q	1,3 MPP
1,2	(6) $Q \,\&\, -Q$	4,5 &I
1	(7) $-(P \,\&\, -Q)$	2,6 RAA

(*b*) $-(P \,\&\, -Q) \vdash P \rightarrow Q$

1	(1) $-(P \,\&\, -Q)$	A
2	(2) P	A
3	(3) $-Q$	A
2,3	(4) $P \,\&\, -Q$	2,3 &I
1,2,3	(5) $(P \,\&\, -Q) \,\&\, -(P \,\&\, -Q)$	1,4 &I
1,2	(6) $--Q$	3,5 RAA
1,2	(7) Q	6 DN
1	(8) $P \rightarrow Q$	2,7 CP

35(*a*): another indirect proof—we assume (line (2)) $P \,\&\, -Q$ and aim for a contradiction. Lines (3) and (4) unpack by &E the information of line (2), and the desired contradiction is almost immediate (line (6)). 36(*b*) is a little more complex. Aiming to prove $P \rightarrow Q$, we assume P (line (2)) and take Q as a subsidiary target, relying on CP to redress the balance at the last step. There seems to be no direct way of deriving Q from (1) and (2), so we assume $-Q$ (line (3)) and aim for a contradiction. By &I, assumptions (2) and (3) contradict assumption (1), as we establish at line (5). By RAA, this yields $--Q$ from (1) and (2), and hence Q (line (8)) by DN.

36 $P \vee Q \dashv\vdash -(-P \,\&\, -Q)$

(*a*) $P \vee Q \vdash -(-P \,\&\, -Q)$

1	(1) $P \vee Q$	A

2	(2) $-P \& -Q$	A
3	(3) P	A
2	(4) $-P$	2 &E
2,3	(5) $P \& -P$	3,4 &I
3	(6) $-(-P \& -Q)$	2,5 RAA
7	(7) Q	A
2	(8) $-Q$	2 &E
2,7	(9) $Q \& -Q$	7,8 &I
7	(10) $-(-P \& -Q)$	2,9 RAA
1	(11) $-(-P \& -Q)$	1,3,6,7,10 vE

(*b*) $-(-P \& -Q) \vdash P \vee Q$

1	(1) $-(-P \& -Q)$	A
2	(2) $-(P \vee Q)$	A
3	(3) P	A
3	(4) $P \vee Q$	3 vI
2,3	(5) $(P \vee Q) \& -(P \vee Q)$	2,4 &I
2	(6) $-P$	3,5 RAA
7	(7) Q	A
7	(8) $P \vee Q$	7 vI
2,7	(9) $(P \vee Q) \& -(P \vee Q)$	2,8 &I
2	(10) $-Q$	7,9 RAA
2	(11) $-P \& -Q$	6,10 &I
1,2	(12) $(-P \& -Q) \& -(-P \& -Q)$	1,11 &I
1	(13) $--(P \vee Q)$	2,12 RAA
1	(14) $P \vee Q$	13 DN

Both 36(*a*) and 36(*b*) are instructive proofs, and merit close scrutiny. The basic idea of 36(*a*) is proof by vE. Given a disjunctive assumption, we assume (line (3)) the first disjunct and aim for the conclusion, and assume (line (7)) the second disjunct and aim for the same conclusion. In each case, the conclusion is obtained by RAA, so that we assume once and for all (line (2)) $-P \& -Q$ whose negation we wish to derive. Lines (3)–(6) achieve the first objective, lines

(7)–(10) the second. The basic idea of 36(*b*) is proof by RAA. We assume (line (2)) the negation of the desired conclusion, and aim for a contradiction. Clearly what contradicts assumption (1) is $-P \& -Q$, so that the objective becomes to derive $-P$ and $-Q$ separately from (2). To derive $-P$, we assume P (line (3)), and obtain a contradiction (line (5)); hence $-P$ follows from (2) (line (6)). In a parallel way, $-Q$ also follows from (2) (line (10)). We thus achieve the desired contradiction at line (12). It is worth noting that at line (11) of this proof we have actually proved the sequent $-(P \vee Q) \vdash -P \& -Q$ (compare Exercise 1 (*f*) at the end of the section).

The ten rules we have used hitherto enable us to prove interesting, and in certain cases unobvious, results concerning the interrelations of our sentence-forming operators on sentences. Yet they are all rules which after reflection we are inclined to accept as corresponding to sound and obvious principles of reasoning: at least, from true premisses we shall not be led by them to false conclusions. It should be clear by now that any insights we have so far obtained into the proper codification of arguments are mainly due to the adoption of a special logical notation and of rules the application of which can be *mechanically checked*. Indeed, if someone queries our conclusions, we can present him with the proofs and ask him to state exactly which step he regards as invalid and why. In this respect, the situation is like that in arithmetic: it is idle merely to disagree with a certain calculation; you should say *where* the mistake has been made, and why you consider it to be such. There is a difference, however: calculations can be *performed*, as well as checked, mechanically, whilst we so far know of no mechanical way of generating proofs—though, once discovered, a machine could certify them as valid.

SUMMARY OF RULES OF DERIVATION

1 *Rule of Assumptions* (A)

Any proposition may be introduced at any stage of a proof. We write to the left the number of the line itself.

2 *Modus Ponendo Ponens* (MPP)

Given A and A → B, we may derive B as conclusion. B depends on any assumptions on which either A or A → B depends.

3 *Modus Tollendo Tollens* (MTT)

Given $-B$ and $A \rightarrow B$, we may derive $-A$ as conclusion. $-A$ depends on any assumptions on which either $-B$ or $A \rightarrow B$ depends.

4 *Double Negation* (DN)

Given A, we may derive $--A$ as conclusion, and vice versa. In either case, the conclusion depends on the same assumptions as the premiss.

5 *Conditional Proof* (CP)

Given a proof of B from A as assumption, we may derive $A \rightarrow B$ as conclusion on the remaining assumptions (if any).

6 *&-Introduction* (&I)

Given A and B, we may derive A & B as conclusion. A & B depends on any assumptions on which either A or B depends.

7 *&-Elimination* (&E)

Given A & B, we may derive either A or B separately. In either case, the conclusion depends on the same assumptions as the premiss.

8 v-*Introduction* (vI)

Given either A or B separately, we may derive A v B as conclusion. In either case, the conclusion depends on the same assumptions as the premiss.

9 v-*Elimination* (vE)

Given A v B, together with a proof of C from A as assumption and a proof of C from B as assumption, we may derive C as conclusion. C depends on any assumptions on which A v B depends or on which C depends in its derivation from A (apart from A) or on which C depends in its derivation from B (apart from B).

10 *Reductio ad Absurdum* (RAA)

Given a proof of B & $-B$ from A as assumption, we may derive $-A$ as conclusion on the remaining assumptions (if any).

Note: The biconditional-sign '$\leftrightarrow$' is introduced by the following definition:

$$Df. \leftrightarrow: \quad A \leftrightarrow B = (A \rightarrow B) \,\&\, (B \rightarrow A)$$

This definition permits the replacement of $A \leftrightarrow B$ appearing in a conclusion by $(A \rightarrow B) \,\&\, (B \rightarrow A)$, and vice versa.

EXERCISE

1 Find proofs for the following sequents:

(a) $P \vee Q \dashv\vdash P \vee Q$

(b) $P \,\&\, P \dashv\vdash P$

(c) $P \,\&\, (Q \vee R) \dashv\vdash (P \,\&\, Q) \vee (P \,\&\, R)$

(d) $P \vee (Q \,\&\, R) \dashv\vdash (P \vee Q) \,\&\, (P \vee R)$

(e) $P \,\&\, Q \dashv\vdash -(P \rightarrow -Q)$

(f) $-(P \vee Q) \dashv\vdash -P \,\&\, -Q$

(g) $-(P \,\&\, Q) \dashv\vdash -P \vee -Q$

(h) $P \,\&\, Q \dashv\vdash -(-P \vee -Q)$

(i) $P \rightarrow Q \vdash -P \vee Q$

(j) $-P \rightarrow Q \vdash P \vee Q$