DEFINITION 1.3. Let $C \subset X$ be a closed cone with nonempty interior and $T: X \to X$ a linear operator. $T$ is said to be positive with respect to $C$ if

$$H_3: T(C \setminus \{0\}) \subset \text{int } C.$$

An abstract version of Perron's Theorem where $P$ and $A$ are replaced by any cone $C \subset X$, $\dim X < \infty$ and any operator $T$ satisfying the condition in Definitions 1.2 and 1.3 follows from the observation that the only fact used in the proof besides the compactness of the unit sphere in $\mathbb{R}^N$ is the positivity of $A$ with respect to $P$. In 1948 T. Krein and R. Rutman published a remarkable extension of Perron's Theorem [2] that covers the case where $T$ is compact and $\dim X = \infty$. The celebrated Krein–Rutman Theorem can be stated in the following form.

THEOREM 1.4. *Let $C \subset X$, $X$ a Banach space, be a closed cone with nonempty interior and let $T: X \to X$ be a compact linear operator. Assume $T$ and $C$ satisfy $H_3$; then:*

(i) *the spectral radius $\rho(T)$ of $T$ is positive;*
(ii) *$\rho(T)$ is a simple eigenvalue of $T$;*
(iii) *if $\mu \neq \rho(T)$ is an eigenvalue of $T$, then $|\mu| < \rho(T)$;*
(iv) *the eigenvector $v$ associated to $\rho(T)$ can be taken in* int $C$;
(v) *$v$ is the unique eigenvector of $T$ in $C$ up to a multiplicative constant.*

There exist several proofs of this theorem. The original proof makes substantial use of the spectral theory of compact operators. Other proofs like the one presented in [4] use powerful topological theorems. The aim of this work is to present a new proof of the Krein–Rutman Theorem which is completely elementary and establishes the existence of an eigenvector in $C$ without using any information from the spectral theory of compact operators. The central object in our proof is the discrete dynamical system $S: \Sigma^+ \to \Sigma^+$ induced by the operator $T$ on $\Sigma^+$, the intersection of the unit sphere $\Sigma = \{x \in C, \|x\| = 1\}$ with $C$,

$$Su = Tu/\|Tu\|, \quad u \in \Sigma^+. \tag{1.1}$$

We refer to [1] for the basic theory of Dynamical Systems needed in this paper. The two main points in the proof are:

(I) to show that the compactness of $T$ and the convexity of $C$ imply that the $\omega$-limit set $\omega(u)$ of any $u \in \Sigma^+$ is non-empty;

(For the reader's convenience we recall the definition

$$\omega(u) = \bigcap_{k>1} \left( \overline{\bigcup_{i>k} S^i u} \right).)$$

(II) to show that the linearity of $T$ implies $v \in \omega(u)$ is a fixed point of $S$ and therefore an eigenvector of $T$.

When $X$ is finite dimensional, (I) is a trivial consequence of the compactness of the unit sphere. This and the fact that convexity of $C$ is not used in (II) enable us to show that the conclusion of Perron's Theorem holds true (see Theorem 3.1 below) under weaker conditions on the cone $C$ which do not require convexity. In (I) we use the convexity of $C$ in an essential way and therefore we cannot presently extend Theorem 3.1 to the case where $X$ is infinite dimensional. It is an open question whether one can prove that $u \in \Sigma^+$ implies $\omega(u) \neq \varnothing$ under the weaker hypothesis in Theorem 3.1.

## 2. Proof of Theorem 1.4

LEMMA 2.1. *Let $u \in \Sigma^+$ and $\omega(u)$ be the $\omega$-limit set of $u$ for the dynamical system $S: \Sigma^+ \to \Sigma^+$ defined by* (1.1). *Then $\omega(u)$ is nonempty.*

*Proof.* Let $e_1 = u$, $e_{i+1} = Se_i$, $i \geqq 1$. If $e_1, e_2, \ldots$ are linearly independent, we have obviously $\omega(u) \neq \varnothing$. Therefore we assume $e_1, e_2, \ldots$ are linearly independent. From (1.1) we have $Te_i = \mu_i e_{i+1}$ with $\mu_i = \|Te_i\|$.

(a) The sequence $\mu_i$ does not approach 0 as $i \to \infty$. Let

$$\begin{aligned} \varepsilon_1 &= e_1, \\ \varepsilon_2 &= \mu_1 e_2 - c e_1, \\ \varepsilon_{i+1} &= T\varepsilon_i, \quad i = 2, 3, \ldots. \end{aligned}$$

$H_3$ implies $e_2$ is in int $C$; therefore, for $c$ small, $\varepsilon_2 \in C$ and $H_3$ implies $e_i \in C$, $i = 1, 2, \ldots$. It follows that if $\alpha_i$, $i = 1, 2, \ldots$ is a sequence of positive numbers and the series $\sum_i \alpha_i \varepsilon_i$ is convergent, the sum $v = \sum_i \alpha_i \varepsilon_i$ is in $C$ because $C$ is closed and satisfies $H_2$. Consider in particular the series corresponding to

$$\alpha_i^+ = (2c)^{1-i}, \quad \alpha_i^- = (2^i - 1)(2c)^{1-i}, \quad i = 1, 2, \ldots$$

with $c > 0$ sufficiently small.

From the definition of $\varepsilon_i$, if we also assume $\mu_0 = 1$, it follows that

$$\varepsilon_{i+1} = T^{i-1}(\mu_1 e_2 - c e_1) = T^i e_1 - cT^{i-1} e_1 = \Big(\prod_0^i \mu_j\Big) e_{i+1} - c\Big(\prod_0^{i-1} \mu_j\Big) e_i, \quad i = 1, 2, \ldots$$

$$\sum_1^n \alpha_i \varepsilon_i = \sum_1^{n-1} (\alpha_i - c\alpha_{i+1})\Big(\prod_0^{i-1} \mu_j\Big) e_i + \alpha_n\Big(\prod_0^{n-1} \mu_j\Big) e_n$$

and therefore

$$\begin{aligned} \sum_1^n \alpha_i^+ \varepsilon_i &= \tfrac{1}{2} \sum_1^{n-1} \frac{\prod_0^{i-1} \mu_j}{(2c)^{i-1}} e_i + \frac{\prod_0^{n-1} \mu_j}{(2c)^{n-1}} e_n, \\ \sum_1^n \alpha_i^- \varepsilon_i &= -\tfrac{1}{2} \sum_1^{n-1} \frac{\prod_0^{i-1} \mu_j}{(2c)^{i-1}} e_i + \frac{(2^n - 1)\prod_0^{n-1} \mu_j}{(2c)^{n-1}} e_n. \end{aligned}$$

From these expressions we see that if $\lim_{i \to \infty} \mu_i = 0$, the series $\sum \alpha_i^{\pm} \varepsilon_i$ are convergent and

$$\sum_1^\infty \alpha_i^- \varepsilon_i = -\sum_1^\infty \alpha_i^+ \varepsilon_i \neq 0$$

in contradiction to $H_1$.

(b) From (a) there is a subsequence $\mu_{i_k}$ and a positive number $\mu$ such that $\mu_{i_k} \geqq \mu$. Since $T$ is compact, by taking a subsequence if necessary, we can assume $Te_{i_k}$ is convergent to some vector $z$. In particular

$$\lim_{k \to \infty} \|Te_{i_k}\| = \lim_{k \to \infty} \mu_{i_k} = \|z\| \geqq \mu,$$

and therefore

$$\lim_{k\to\infty} S^{i_k}u = \lim_{k\to\infty} e_{i_k+1} = \lim_{k\to\infty} \frac{Te_{i_k}}{\|Te_{i_k}\|} = \frac{z}{\|z\|}. \quad \square$$

Lᴇᴍᴍᴀ 2.2. *Let $u \in \Sigma^+$ and $v \in \omega(u)$, then $v$ is a fixed point of $S$ and therefore an eigenvector of $T$.*

*Proof.* By definition, $v \in \omega(u)$ implies the existence of a subsequence $i_k$, $k = 1, 2, \ldots$ such that

$$\lim_{k\to\infty} S^{i_k}u = v. \tag{2.1}$$

If $v \neq Sv$, then $u \neq Su$ and $H_1$ implies that $u$, $Su$ and $v$, $Sv$ are pairs each of which consists of linearly independent vectors. Taking into account this fact, (2.1) and the computation

$$\lim_{k\to\infty} T^{i_k}\left(\frac{Su}{\|T^{i_k}u\|}\right) = \|Tu\|^{-1}\, T\left(\lim_{k\to\infty} \frac{T^{i_k}u}{\|T^{i_k}u\|}\right) = \frac{Tv}{\|Tu\|} = \frac{\|Tv\|}{\|Tu\|} Sv,$$

we obtain

$$\lim_{k\to\infty} S^{i_k}u_\gamma = v_\gamma, \quad \gamma \in \mathbb{R}, \tag{2.2}$$

where

$$u_\gamma = \frac{\cos\gamma u + \sin\gamma Su}{\|\cos\gamma u + \sin\gamma Su\|}; \quad v_\gamma = \frac{\cos\gamma v + \dfrac{\|Tv\|}{\|Tu\|}\sin\gamma Sv}{\left\|\cos\gamma v + \dfrac{\|Tv\|}{\|Tu\|}\sin\gamma Sv\right\|}.$$

Let $\Gamma$ be the set of $\gamma$ such that $v_\gamma \in C$. $\Gamma$ is nonempty because $v_0 = v \in C$ and it is closed because $C$ is closed. Take $\sigma \in \Gamma$; then $H_3$ implies $Sv_\sigma \in \operatorname{int} C$ and therefore from (2.2) it follows that $S^{i_k+1}u_\sigma \in \operatorname{int} C$ for $k \geqq K$ for some integer $K$. But then $H_3$ implies $S^{i_K+1}u_\sigma \in \operatorname{int} C$ and therefore there exists $\varepsilon > 0$ such that $S^{i_K+1}u_\gamma \in \operatorname{int} C$ for all $\gamma \in (\sigma-\varepsilon, \sigma+\varepsilon)$. From this, (2.2) and $H_3$ it follows that $v_\gamma \in C$ for $\gamma \in (\sigma-\varepsilon, \sigma+\varepsilon)$. This shows $\Gamma$ is open and in turn that $\Gamma = \mathbb{R}$. This implies $v_\gamma$ and $v_{\gamma+\pi} = -v_\gamma$ are both in $C$ in contradiction with $H_1$. $\square$

From Lemma 2.1 and Lemma 2.2 it follows that there exists $v \in \Sigma^+$ such that $Sv = v$ or equivalently $Tv = \|Tv\|\, v$. Therefore $v$ is an eigenvector of $T$ corresponding to a positive eigenvalue $\lambda = \|Tv\|$. Also $v \in \operatorname{int} C$ by $H_3$. Assume $w \neq v$, $\|w\| = 1$, is an eigenvector of $T$ corresponding to a real eigenvalue $\mu$. Then, if $\alpha$, $\beta$ are real numbers such that $|\alpha| + |\beta| \neq 0$, it results that:

$$S^n\left(\frac{\alpha v + \beta w}{\|\alpha v + \beta w\|}\right) = \frac{\alpha v + \beta\left(\dfrac{\mu}{\lambda}\right)^n w}{\left\|\alpha v + \beta\left(\dfrac{\mu}{\lambda}\right)^n w\right\|}. \tag{2.3}$$

If $\mu = \lambda$, $\alpha v + \beta w$ is an eigenvector of $T$ and $H_1$ implies $\alpha v + \beta w \in \partial C$ for a suitable choice of $\alpha$, $\beta$. This is in contradiction with $H_3$ which implies $T$ cannot have eigenvectors on $\partial C$. Therefore $\mu \neq \lambda$. If $|\mu| > \lambda$ then, provided $\beta \neq 0$, equation (2.3) implies

$$\lim_{n\to\infty} S^{2n}\left(\frac{\alpha v + \beta w}{\|\alpha v + \beta w\|}\right) = \pm w, \tag{2.4}$$

depending on whether $\beta > 0$ or $\beta < 0$. By $H_1$, either $w$ or $-w$ is not in $C$. By changing $w$ with $-w$, if necessary, we can assume $w \notin C$. Since $C$ is closed there is a neighbourhood $W$ of $w$ such that $W \cap C = \varnothing$. On the other hand, $v \in \operatorname{int} C$ implies we can choose $\alpha, \beta > 0$ such that $(\alpha v + \beta w) \in \operatorname{int} C$. But then (2.4) is in contradiction with $H_3$ and therefore $|\mu|$ cannot be larger than $\lambda$. If $|\mu| < \lambda$, it follows from (2.3) that, provided $\alpha < 0$,

$$\lim_{n\to\infty} S^{n}\left(\frac{\alpha v + \beta w}{\|\alpha v + \beta w\|}\right) = -v. \tag{2.5}$$

Assume $w \in C$; then we can also assume $w \in \operatorname{int} C$ because $w$ is an eigenvector of $T$. Therefore there are $\alpha$, $\beta$, $\alpha < 0$, such that $(\alpha v + \beta w) \in \operatorname{int} C$. This and (2.5) again lead to a contradiction with $H_3$. We conclude $w \notin C$. Similar arguments show that the algebraic multiplicity of $\lambda$ is one and that $|\mu| < \lambda$ also for any complex eigenvalue of $T$. □

## 3. An extension of Perron's Theorem

In the above proof of the Krein–Rutman Theorem, the only place where convexity of $C$ was used was in the proof of Lemma 2.1 to show that $\omega(u)$ is nonempty. If $X$ is finite dimensional, as we have already observed in the Introduction, the fact that $\omega(u)$ is nonempty is just a consequence of the compactness of the unit sphere. Therefore we can state the following theorem which is a generalisation of Perron's Theorem:

THEOREM 3.1. *Let $X$ be a finite dimensional real normed linear space. Let $K \subset X$ be a closed set with nonempty interior such that*

$$x \in K, \quad -x \in K \Rightarrow x = 0,$$
$$\alpha \in [0, \infty), \quad x \in K \Rightarrow \alpha x \in K.$$

*Let $T: X \to X$ be a linear operator such that*

$$T(K\setminus\{0\}) \subset \operatorname{int} K.$$

*Then the conclusions of Theorem 1.4 hold.*

In connection with this theorem and in order to clarify the relationship with the classical Perron Theorem, it is natural to ask what can be said about the geometry of the set $K$. More precisely, we can ask what can be said about a set $K$ which, together with a linear operator $T$, satisfies the assumptions in Theorem 3.1. We have the following proposition:

PROPOSITION 3.2. *Let $X$ be a finite dimensional real normed linear space and let $K$ be a closed set with non-empty interior such that*

(a) $x \in K$, $-x \in K \Rightarrow x = 0$;

(b) $\alpha \in [0, \infty)$, $x \in K \Rightarrow \alpha x \in K$;

(c) *there exist a linear operator $T: X \to X$ with the property $T(K\setminus\{0\}) \subset \operatorname{int} K$.*

*Then the convex hull $\hat{K}$ of $K$ is a cone.*