$$(13) \xRightarrow{(14),(15)}$$

$$|u(x)-u(y)| \le C r^{1-\frac{n}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

$$= C|x-y|^{1-\frac{n}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

$$\Longrightarrow$$

$$[u]_{C^{0,\gamma}(\mathbb{R}^n)} \le C \|\nabla u\|_{L^p}$$

$$\gamma = 1 - \frac{n}{p}$$

AND $(12) \Rightarrow (2)$.

$\square$

Holder

$$\leq C\left(\int_{B(x,1)} |\nabla u|^p dy\right)^{1/p}\left(\int_{B(x,1)} \frac{1\, dy}{|x-y|^{(n-1)p/p-1}}\right)^{\frac{p-1}{p}} + C\|u\|_{L^p(\mathbb{R}^n)}$$

(10)

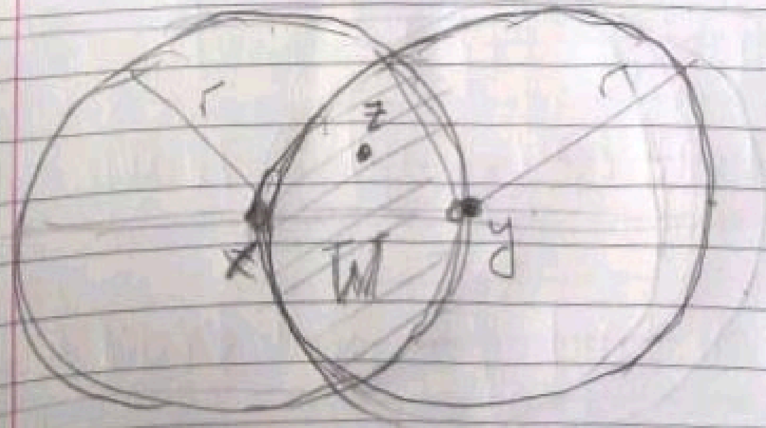
$$\leq C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

$\therefore$

(12) $\boxed{W^{1,p}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n)}$ $\boxed{p>n}$

C. (Coda)

Επιλέγουμε $x, y \in \mathbb{R}^n$, $r = |x-y|$



$W = B(x,r) \cap B(y,r)$

(13) $$|u(x)-u(y)| \leq \fint_W |u(x)-u(z)|\,dz + \fint_W |u(y)-u(z)|\,dz$$

<u>τεριέχουμε</u>?

(14) $$\fint_W |u(x)-u(z)|\,dz \leq \frac{1}{|W|}\int_{B(x,r)} |u(x)-u(z)|\,dz$$

$$\leq \frac{|B(x,r)|}{|W|}\int_{B(x,r)} |u(x)-u(z)|\,dz$$

$$\leq C_1 \fint_{B(x,r)} |u(x)-u(z)|\,dz$$

$$\overset{(*)}{\leq} C\int_{B(x,r)} \frac{|\nabla u(z)|}{|z-x|^{n-1}}\,dz$$

όπως (11)

$$\underset{\boxed{\text{Hölder}}}{\leq} C\left(\int_{B(x,r)} |\nabla u(z)|^p\,dz\right)^{1/p}\left(\int_{B(x,r)} \frac{dz}{|x-z|^{(n-1)\frac{p}{p-1}}}\right)^{\frac{p-1}{p}}$$

$$\leq C\left(r^{n-(n-1)\frac{p}{p-1}}\right)^{\frac{p-1}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

$$= C r^{1-\frac{n}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}$$

Παρόμοια

(15) $$\fint_W |u(y)-u(z)|\,dz \leq C r^{1-\frac{n}{p}} \|\nabla u\|_{L^p(\mathbb{R}^n)}.$$

$$dS_r = r^{n-1}\, dS_1$$

$$= \int_0^1 \left(\int_{\partial B(0,1)} \frac{1}{|r|^{\sigma(n-1)}} r^{n-1}\, dS_1 \right) dr$$

$$= \int_{\partial B(0,1)} \left(\int_0^1 r^{(n-1)-\sigma(n-1)}\, dr \right) dS_1 < \infty \Leftarrow$$

$$\sigma = \frac{p}{p-1}$$

$$(n-1) - \sigma(n-1) > -1 \iff$$

$$(n-1) - \frac{p}{p-1}(n-1) > -1 \iff$$

$$(n-1)\left[1 - \frac{p}{p-1}\right] > -1 \iff \frac{n-1}{p-1} < 1$$

$$\iff \boxed{p > n}$$

Απο την (*)

$$|u(x)| \leq |u(x) - u(y)| + |u(y)|$$

$$\overset{r=1}{\Longrightarrow} (1) \int_{B(x,1)} |u(x)|\, dy \leq \int_{B(x,1)} |u(x)-u(y)|\, dy + \int_{B(x,1)} |u(y)|$$

$$\overset{(*)}{\leq} c \int \frac{|\nabla u(y)|}{|y-x|^{n-1}}\, dy + c\, \|u\|_{p}$$

$1^{ον}$ μέρος της (4) :

$$(7) \quad \int_{|w|=1} |u(x+sw) - u(x)| \, dS_w$$

$$z = x + sw \Rightarrow dS_z = s^{n-1} dS_w$$

$$= \frac{1}{s^{n-1}} \int_{\partial B(x,s)} |u(z) - u(x)| \, dS_z$$

(6), (7) μαζί με την (4) δίνουν :

$$(8) \quad \int_{\partial B(x,s)} |u(z) - u(x)| \, dS_z \leq s^{n-1} \int_{B(x,r)} \frac{|\nabla u(y)|}{|x-y|^{n-1}} \, dy$$

$$\Rightarrow (9) \quad \boxed{\int_0^r \int_{\partial B(x,s)} |u(z) - u(x)| \, \underbrace{dS_z \, ds}_{dy} \leq \boxed{\frac{r^n}{n}} \int_{B(x,r)} \frac{|\nabla u(y)|}{|x-y|^{n-1}} \, dy}$$

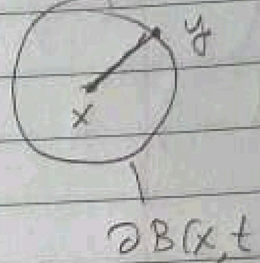
$$\Leftrightarrow (*).$$

B) Σχετικά με την ιδιομορφία $\dfrac{1}{|x|^{(n-1)\frac{p}{p-1}}}$

$$(10) \quad \int_{B(0,1)} \frac{1}{|x|^{q(n-1)}} \, dx = \int_0^1 \left(\int_{\partial B(0,r)} \frac{1}{|r|^{q(n-1)}} \, dS_r \right) dr$$

$$\le \int_0^s |\nabla(x+tw)|\,|w|\,dt = \int_0^s |\nabla(x+tw)|\,dt.$$

(4) $$\int_{|w|=1} |u(x+sw)-u(x)|\,dS_w \le \int_0^s\left(\int_{|w|=1} |\nabla u(x+tw)|\,dS_w\right)dt$$

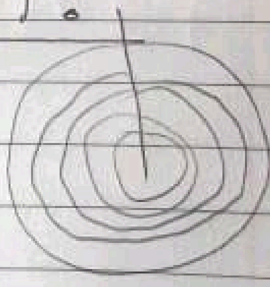


Αντικατάσταση $y = x+tw$, $|y-x|=t$

(5) $$\int_{|w|=1} |\nabla u(x+tw)|\,dS_w = \int_{\partial B_t(x,t)} |\nabla u(y)|\,\frac{dS_w}{dS_y}\,dS_y$$

$$= \int_{\partial B_t(x,t)} \frac{|\nabla u(y)|}{t^{n-1}}\,dS_y$$

Κατά συνέπεια το 2ο μέλος της (4):



(6) $$\int_0^s \int_{|w|=1} |\nabla u(x+tw)|\,dS_w\,dt$$

$$= \int_0^s \int_{\partial B_t(x,t)} \frac{|\nabla u(y)|}{t^{n-1}}\,dS_y\,dt$$

$$= \int_0^s \int_{\partial B_t(x,t)} \frac{|\nabla u(y)|}{t^{n-1}}\,dS_y\,dt = \int_0^s \int_{\partial B_{|x-y|}(x)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dS_y$$

σφαιρικές

$$= \int_{B_s(x,t)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy \le \int_{B_r(x,t)} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy.$$

Διάλεξη 13 : Το Θεώρημα Εμφύτευσης του Morrey

$1 < n < p < \infty$

Θεώρημα 1

Έστω U φραγμένο, ανοικτό, $\subset \mathbb{R}^n$, $\partial U \in C^1$.
Δοθέντος $u \in W^{1,p}(U)$ $\exists\, \hat{u} : U \to \mathbb{R}$, $\hat{u} \in C^{0,\gamma}(\bar{U})$
και $\hat{u} = u$ a.e. U, και ισχύει

(1) $$\|\hat{u}\|_{C^{0,\gamma}(\bar{U})} \leq C \|u\|_{W^{1,p}(U)}$$

$C^{0,\gamma} \equiv C^\gamma$ $\quad C = C(p, n, U)$, $\boxed{\gamma = 1 - \frac{n}{p}}$

Θα αποδείξουμε πρώτα για $C^1(\mathbb{R}^n)$ συναρτήσεις την ανισότητα

(2) $$\|u\|_{C^{0,\gamma}} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

Απόδειξη

A) $$\boxed{⨍_{B(x,r)} |u(x) - u(y)|\, dy \leq C \int_{B(x,r)} \frac{|\nabla u(y)|}{|y-x|^{n-1}}\, dy \quad \forall B(x,r) \subset \mathbb{R}^n} \quad (*)$$

Απ.

Έστω $|w| = 1$, σταθεροποιημένο, $0 < s < r$

(3) $$|u(x+sw) - u(x)| = \left|\int_0^s \frac{d}{dt}\big(u(x+tw)\big)\, dt\right| = \left|\int_0^s \nabla u(x+tw) \cdot w\, dt\right|$$