LECTURE NOTES ON ELLIPTIC EQUATIONS

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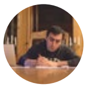

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LECTURE NOTES ON ELLIPTIC EQUATIONS

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Abstract. In this notes we discuss some ideas of De Giorgi and Moser leading to Hölder continuity of the weak bounded solutions. We also give the proof of Krylov-Safonov's theorem following Trudinger's paper.

Contents

1. Outline

The present notes contain the material that I have been covering in the PDE seminar organized for the postdocs and first and second year PhD students. The proof of the De Giorgi oscillation lemma is based on the unpublished notes of Labutin [3]. The discussion on the weak Harnack inequalities for non-divergence form elliptic equations is based on the paper of Trudinger [5]. Other important texts are listed below

- D. Gilbarg, N.S. Trudinger; *Elliptic partial differential equations of second order.* Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001

- Q. Han, F. Lin; *Elliptic partial differential equations.* Second edition. Courant Lecture Notes in Mathematics, 1., New York; AMS, Providence, RI, 2011

- L.A. Caffarelli, *A priori estimates and the geometry of the Monge-Ampère equation.* Nonlinear partial differential equations in differential geometry (Park City, UT, 1992), 5-63, IAS/Park City Math. Ser., 2, Amer. Math. Soc., Providence, RI, 1996.

- L.A. Caffarelli, X. Cabre ; *Fully nonlinear elliptic equations.* American Mathematical Society Colloquium Publications, 43. AMS, Providence, RI, 1995

- E. Landis; *Second order equations of elliptic and parabolic type.* Translated from the 1971 Russian original by Tamara Rozhkovskaya. With a preface by Nina Ural'tseva. Translations of Mathematical Monographs, 171. AMS, Providence, RI, 1998

2. The De Giorgi oscillation lemma

By L we denote the divergence form uniformly elliptic operator with measurable coefficients,

$$Lu = \sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j u).$$

Let $\Lambda > 0$ be the ellipticity of L. Here is the De Giorgi oscillation lemma.

Theorem 1. *Let $u \in H^1(B_1)$ satisfy*

$$Lu \geq 0 \quad in \quad B_1. \tag{2.1}$$

Assume also that u is normalised in the following sense:

$$u \leq 1 \quad in \quad B_1,$$
$$\big|B_{1/2} \cap \{u \leq 0\}\big| \geq \theta |B_{1/2}| \tag{2.2}$$

for some $\theta > 0$. Then

$$\sup_{B_{1/4}} u \leq 1 - \varepsilon, \tag{2.3}$$

for some $\varepsilon = \varepsilon(n, L, \theta) > 0$.

The meaning of this lemma is simple. A function u satisfying (2.1) is more convex than concave. Condition (2.2) means that u is below the level 1 near ∂B_1, and below the level 0 on a set of big measure in $B_{1/2}$. The lemma claims that such u is below the level $1 - \varepsilon$ in $B_{1/2}$. Actually, the name *growth lemma* reflects the nature of estimate (2.3) better.

Utilising translations and stretches of the graph of u we derive the De Giorgi oscillation (growth) lemma at any level $A \in \mathbf{R}^1$:

> *Let $u \in H^1(B_1)$ satisfy*
>
> $$Lu \geq 0 \quad in \quad B_1.$$
>
> *Assume that u is normalised at the level A as follows: for some $\delta, \theta > 0$*
>
> $$u \leq A \quad in \quad B_1,$$
> $$\big|\{u \leq A - \delta\} \cap B_{1/2}\big| \geq \theta |B_{1/2}|.$$
>
> *Then there exists $\varepsilon = \varepsilon(n, L, \theta) > 0$ such that*
>
> $$\sup_{B_{1/4}} u \leq A - \varepsilon\delta. \tag{2.4}$$

Indeed, just consider

$$\tilde{u} = \frac{u}{a} + b$$

and choose a, b such that (2.1)–(2.2) hold for $\tilde{u}$: $a > 0$ and

$$\frac{u}{a} + b \leq 1 \quad \Leftrightarrow \quad u \leq A$$
$$\frac{u}{a} + b \leq 0 \quad \Leftrightarrow \quad u \leq A - \delta.$$

This gives

$$a = \delta, \quad b = 1 - \frac{A}{\delta}.$$

Then (2.3) gives

$$\sup_{B_{1/2}} \left(\frac{u}{\delta} + 1 - \frac{A}{\delta} \right) \le 1 - \varepsilon \Leftrightarrow \sup_{B_{1/2}} u \le A - \varepsilon\delta.$$

Similar estimate holds for supersolutions:

Let $u \in H^1(B_1)$ satisfy

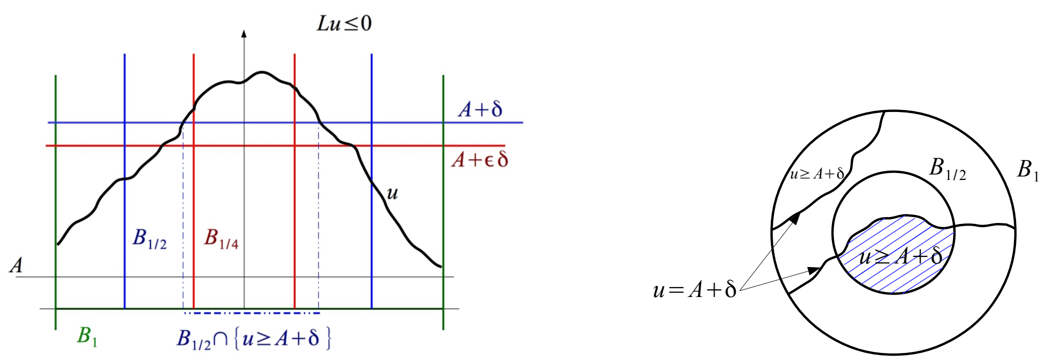
$$Lu \le 0 \quad in \quad B_1.$$

Assume that u is normalised at a level A as follows: for some $\delta, \theta > 0$

$$u \ge A \quad in \quad B_1,$$
$$\left|\{u \ge A + \delta\} \cap B_{1/2}\right| \ge \theta |B_{1/2}|.$$

Then there exists $\varepsilon = \varepsilon(n, L, \theta) > 0$ such that

$$\inf_{B_{1/4}} u \ge A + \varepsilon\delta. \tag{2.5}$$



(A) (B)

FIGURE 1

We also formulate a variation of the previous result.

Theorem 1′. *Let $u \in H^1(B_1)$ satisfy*

$$Lu \ge 0 \quad in \quad B_1. \tag{2.6}$$

Assume also that u is normalised in the following sense:

$$u \le 1 \quad in \quad B_1,$$
$$\left|B_{1/4} \cap \{u \le 0\}\right| \ge \theta |B_{1/4}| \tag{2.7}$$

for some $\theta > 0$. Then

$$\sup_{B_{1/4}} u \le 1 - \varepsilon, \tag{2.8}$$

for some $\varepsilon = \varepsilon(n, L, \theta) > 0$.

Proof.

$$\left|B_{1/2} \cap \{u \leq 0\}\right| \geq \left|B_{1/4} \cap \{u \leq 0\}\right| \geq \frac{\theta}{2^n}$$

then apply Theorem 1 to infer the result. □

Notations: for a function u and $t \in \mathbf{R}$ set

$$E_t = \{u > t\}.$$

Clearly

$$(u-t)^+ = |u-t| = u-t \quad \text{on} \quad E_t.$$

We also have

$$E_k \setminus E_h = \{k < u \leq h\}$$

for $k < h$.

2.1. Main steps in the proof of the De Giorgi oscillation lemma.

Proof. **Step 1: Reversed Poincarè inequality.** The starting point is the Caccioppoli inequality. In order to obtain it, test the equation

$$Lu \geq 0$$

with

$$(u-t)^+ \, \eta_{r,R}^2.$$

Here $\eta_{r,R}$ is the cut-off localised in B_R. The result is

$$\int_{E_t \cap B_r} |Du|^2 \lesssim \frac{1}{(R-r)^2} \int_{E_t \cap B_R} \left|u-t\right|^2. \tag{2.9}$$

In other words,

> for any height t the gradient of the subsolution u is controlled on the set E_t by the *relative* elevation of u on a larger scale.

The rest of the proof consists of a nontrivial bootstrapping of (2.9) at different scales and levels.

Take

$$v(x) = \eta^2(x)(u-t)^+, Dv = 2\eta D\eta\,(u-t)^+ + \eta^2 Du(x)\,1_{E_t}$$

where $\eta(x) = \eta(|x|)$ is radially symmetric continuous functions such that $\eta(s) = 1$ in $(-r, r)$ zero outside of the interval $(-R, R), R > r$ and linear otherwise. From the weak formulation formulation the equation we have

$$\int a(x)\,Du(x)\,Dv(x) \leq 0.$$

Subtracting the expression of Dv into this inequality yields

$$\int \eta^2 1_{E_t}\left(a(x)\,DuDv\right) \leq -\int_{(B_R \setminus B_r) \cap E_t} 2\eta D\eta\,(aDu)\,(u-t)^+.$$

Using the ellipticity of a we obtain

$$\begin{aligned}\lambda\int_{B_R\cap E_t}\eta^2\,|Du|^2 &\le \int_{Et}\eta^2 aDuDu \le 2\Lambda\int_{(B_R\setminus B_r)\cap E_t}\eta\,|D\eta|\,|Du|\,(u-t)^+\\ &\le 2\Lambda\left(\epsilon\int_{(B_R\setminus B_r)\cap E_t}\eta^2|Du|^2+\frac{1}{\epsilon}\int|D\eta|^2[(u-t)^+]^2\right).\end{aligned}$$

Or equivalently

$$(\lambda-2\epsilon\Lambda)\int\eta^2|D\eta|^2\le\frac{\Lambda}{\epsilon}\int_{(B_R\setminus B_r)\cap E_t}|D\eta|^2[(u-t)^+]^2.$$

Choose $\lambda = 2\epsilon\Lambda = \frac{\lambda}{2}\Rightarrow \epsilon=\frac{\lambda}{4\Lambda}$ to get

$$\int_{B_r\cap E_t}|Du|^2\le\int_{B_R\cap E_t}\eta^2|Du|^2\le 8\left(\frac{\Lambda}{\lambda}\right)^2\frac{C_n}{(R-r)^2}\int_{B_R\cap E_t}[(u-t)^+]^2.$$

Step 2: Local maximum principle. Careful iterations combined with the Sobolev and Hölder inequalities give that

Proposition 2. *for $u\in H^1(B_1)$ satisfying (2.9) for all levels and all scales, the following inequality holds:*

$$\begin{aligned}\sup_{B_{1/4}}u &\le t+C(n,L)\left(\int_{B_{1/2}\cap E_t}|u-t|^2\right)^{1/2}\left(\left|E_t\cap B_{1/2}\right|\right)^{\alpha}\\ &\quad\text{for}\quad\text{any}\quad t.\end{aligned}\tag{2.10}$$

Here

$$\alpha=\alpha(n)=\frac{1}{2\kappa}>0\tag{2.11}$$

and κ is a fixed number (the positive root of $2\kappa^2-n(\kappa+1)=0$). The proof of (2.10) relies on recurrent relations between the integral and the measure in the right-hand side there obtained from (2.9), see next section.

Step 3: Dyadic level sets. From (2.10) the proof of the desired estimate (2.3) is almost immediate. Indeed, set

$$t=1-\frac{1}{2^j},\quad j=1,2\dots,$$

in (2.10). We discover that

$$\begin{aligned}\sup_{B_{1/4}}u &\le t+C\left(\int_{B_{1/2}\cap E_t}|u-t|^2\right)^{1/2}\left|E_t\cap B_{1/2}\right|^{\alpha}\\ &\le t+(1-t)\,C\,|E_t\cap B_{1/2}|^{\alpha+1/2}\\ &= 1+(t-1)\left(1-C|E_t\cap B_{1/2}|\right)^{\alpha+1/2} &(2.12)\\ &\le 1-\frac{1}{2^j}\left(1-C\left|E_{1-\frac{1}{2^j}}\cap B_{1/2}\right|^{\alpha+1/2}\right) &(2.13)\\ &\quad\text{for any } j=1,2\dots.\end{aligned}$$

We see that to prove (2.3) it is left to establish that

$$|E_t \cap B_{1/2}| \to 0 \quad \text{when} \quad t \to 1$$

uniformly over all subsolutions normalised by (2.2).

Step 4: Decay rates. The final step is to obtain from (2.1) and (2.2) that

$$j \left| E_{1-\frac{1}{2^j}} \cap B_{1/2} \right|^{2-\frac{2}{n}} \le C(n, L, \theta), \quad j = 1, 2, \dots . \tag{2.14}$$

Combined with (2.12) it completes the proof of the De Giorgi oscillation lemma.

Estimate (2.14) for our u, $u \le 1$, implies that

$$|E_t| \to 0 \quad \text{when} \quad t \to 1.$$

This is not surprising because u is a subsolution bounded by 1 from above. Intuitively we expect this from the strong maximum principle. Formally, the proof of (2.14) relies on the Cacioppoli inequality and the Sobolev estimate.

2.2. **Proof of the decay estimate in Step 4.** To prove (2.14) we need to approach the level 1 for u. To see what can be done fix two numbers h and k such that

$$0 < h < k < 1.$$

We use the following general fact.

Lemma 3. *If $f \in W^{1,1}(B_1)$ and $h < k$, then*

$$(k-h) \left| \{f > k\} \cap B_1 \right|^{1-\frac{1}{n}} \lesssim \frac{1}{|\{f \le h\} \cap B_1|} \int_{\{h<f<k\}\cap B_1} |Df|.$$

Remark 4. *The meaning of Lemma 3 is simple. The left hand side in the last inequality indicates how f grows above the level h. The lemma asserts that we can bound this growth provided we controll Df and f lies below the level h on a set of positive measure.*

The lemma is proved later.

In the ball $B_{1/2}$ apply Lemma 3 to our function u to discover that

$$\begin{aligned} (k-h) \left| E_k \cap B_{1/2} \right|^{(n-1)/n} &\lesssim \frac{1}{\left|\{u \le h\} \cap B_{1/2}\right|} \int_{(E_h \setminus E_k)\cap B_{1/2}} |Du| \\ &\lesssim \frac{1}{\theta} \int_{(E_h \setminus E_k)\cap B_{1/2}} |Du|. \end{aligned} \tag{2.15}$$

In the last inequality we used (2.2).

Now we are going to use Caccioppoli inequality to bound the last integral. Applying Holder inequality and (2.9) derive that

$$\begin{aligned}\int_{(E_h\setminus E_k)\cap B_{1/2}}|Du| &\leq (|E_h\cap B_{1/2}|-|E_k\cap B_{1/2}|)^{1/2}\left(\int_{(E_h\setminus E_k)\cap B_{1/2}}|Du|^2\right)^{1/2}\\ &\leq (|E_h\cap B_{1/2}|-|E_k\cap B_{1/2}|)^{1/2}\left(\int_{E_h\cap B_{1/2}}|Du|^2\right)^{1/2}\\ &\lesssim (|E_h\cap B_{1/2}|-|E_k\cap B_{1/2}|)^{1/2}\left(\int_{E_h\cap B_{1/2}}|u-h|^2\right)^{1/2}\\ &\lesssim (|E_h\cap B_{1/2}|-|E_k\cap B_{1/2}|)^{1/2}(1-h).\end{aligned}$$

Continuing (2.15) deduce that

$$|E_k\cap B_{1/2}|^{2-\frac{2}{n}} \lesssim \left(\frac{(1-h)}{\theta(k-h)}\right)^2\Big(|E_h\cap B_{1/2}|-|E_k\cap B_{1/2}|\Big). \tag{2.16}$$

It is left to iterate (2.16) trying to approach the level 1. For that it is clear that we need to choose h and k such that

$$\frac{1-h}{k-h}\lesssim 1.$$

The simplest choice is

$$h=1-\frac{1}{2^j},\quad k=1-\frac{1}{2^{j+1}},\quad \frac{1-h}{k-h}=2,$$

where $j=1,2,\dots$. Sum the first m inequalities

$$\left|E_{1-\frac{1}{2^{j+1}}}\cap B_{1/2}\right|^{2-\frac{2}{n}}\leq C(n,L,\theta)\left(\left|E_{1-\frac{1}{2^j}}\cap B_{1/2}\right|-\left|E_{1-\frac{1}{2^{j+1}}}\cap B_{1/2}\right|\right),$$

$j=0,\dots,m-1$, and keep in mind that

$$\left|E_{1-\frac{1}{2^m}}\cap B_{1/2}\right|\leq\left|E_{1-\frac{1}{2^j}}\cap B_{1/2}\right|,\quad j\leq m.$$

Discover that

$$m\left|E_{1-\frac{1}{2^m}}\cap B_{1/2}\right|^{2-\frac{2}{n}}\leq C(n,L,\theta)|B_{1/2}|,$$

which proves (2.14). □

3. Proof of Lemma 3 and Proposition 2

Recall the statement of Lemma 3:

If $f\in W^{1,1}(B_1)$ and $h<k$, then

$$(k-h)\left|\{f>k\}\cap B_1\right|^{1-\frac{1}{n}}\lesssim\frac{1}{|\{f\leq h\}\cap B_1|}\int_{\{h<f<k\}\cap B_1}|Df|.$$

Proof. **Step 1.** The proof is simlar to the derivation of (2.15). Truncate f at the levels h and k, and consider the relative value ϕ, $\phi \geq 0$. The formula for ϕ is

$$\phi = ((f-k)^- + h)^+,$$

or

$$\phi(x) = \begin{cases} 0 & \text{if} \quad f(x) \leq h \\ f(x)-h & \text{if} \quad h \leq f(x) \leq k \\ k-h & \text{if} \quad k \leq f(x). \end{cases}$$

The Sobolev inequality gives that

$$\left(\int_{B_1} |\phi - \overline{\phi}_{B_1}|^{n/(n-1)} \right)^{(n-1)/n} \lesssim \int_{B_1} |D\phi|.$$

Let us express this in terms of f and $E_t = \{f > t\}$.

Step 2. First notice that

$$0 \leq \overline{\phi}_{B_1} = \frac{1}{|B_1|} \int_{E_h \cap B_1} \phi \leq \frac{(k-h)|E_h \cap B_1|}{|B_1|}.$$

Consequently

$$\begin{aligned} \int_{B_1} |\phi - \overline{\phi}_{B_1}|^{n/(n-1)} &\geq \int_{E_k \cap B_1} |\phi - \overline{\phi}_{B_1}|^{n/(n-1)} \\ &= \int_{E_k \cap B_1} |(k-h) - \overline{\phi}_{B_1}|^{n/(n-1)} \\ &\geq \left((k-h) - (k-h) \frac{|E_h \cap B_1|}{|B_1|} \right)^{n/(n-1)} |E_k \cap B_1| \\ &= \left((k-h) \frac{|\{f \leq h\} \cap B_1|}{|B_1|} \right)^{n/(n-1)} |E_k \cap B_1|. \end{aligned}$$

At the same time

$$\int_{B_1} |D\phi| = \int_{\{h<f<k\} \cap B_1} |Df|.$$

Substitution in the Sobolev inequality for ϕ now gives

$$\frac{(k-h)|\{f \leq h\} \cap B_1|}{|B_1|} |\{f > k\} \cap B_1|^{1-\frac{1}{n}} \lesssim \int_{\{h<f<k\} \cap B_1} |Df|.$$

The lemma follows. □

Proof of Proposition 2 and (2.10.)

We want to show that

> for $u \in H^1(B_1)$ satisfying (2.9) on all levels and at all scales, the following inequality holds:
>
> $$\sup_{B_{1/2}} u \leq t + C(n,L) \left(\int_{B_1 \cap E_t} |u-t|^2 \right)^{1/2} \Big(|E_t \cap B_1| \Big)^{\alpha} \quad \text{for any } t,$$
>
> where $\alpha = \alpha(n) > 0$, see (2.11).

We will prove a more general statement. We show the bound for functions u from De Giorgi class:

Proposition 2^p. *Let $n > p \geq 1$, and let $u \in W^{1,p}(B_1)$ satisfy*

$$\int_{E_t \cap B_r} |Du|^p \lesssim \frac{1}{(R-r)^p} \int_{E_t \cap B_R} (u-t)^p \tag{3.17}$$

on all levels t and at all scales $0 < r < R$. Then the following inequality holds:

$$\sup_{B_{1/2}} u \leq t + C(n, L, p) \left(\int_{B_1 \cap E_t} (u-t)^p \right)^{1/p} \Big(|E_t \cap B_1| \Big)^{\alpha} \quad \text{for any} \quad t, \tag{3.18}$$

where $\alpha = \alpha(n,p) = \frac{1}{p\kappa} > 0$ and κ is the positive root of $p\kappa^2 - n(\kappa+1) = 0$.

The proof of (3.18) is a careful iteration of (3.17) changing levels and scales.

Proof. **Step 1.** To move closer to the proof of (3.18) let us try to understand the behaviour of two quantities, namely

$$I(r,t) = I_p(r,t) = \int_{E_t \cap B_r} (u-t)^p$$

and

$$\mu(r,t) = |E_t \cap B_r|,$$

when $t \to +\infty$. For any real t we have $(u-t)^+ \in W^{1,p}(B_1)$, and in localised version

$$\eta(u-t)^+ \in H_0^{1,p}(B_1) \quad \text{for all} \quad t \in \mathbf{R}^1, \quad \eta \in C_0^\infty(B_1).$$

Now fix

$$0 < r < \rho < R < 1, \quad \eta \in C_0^\infty(B_\rho), \quad \eta|_{B_r} = 1.$$

Estimate the $W_0^{1,p}$-norm using (3.17):

$$\begin{aligned}
\int_{B_1} \left| D\left(\eta(u-t)^+\right) \right|^p &\lesssim \int_{B_1} |D\eta|^p \left|(u-t)^+\right|^p \\
&\quad + \int_{B_1} \eta^p \left| D\left((u-t)^+\right) \right|^p \\
&\lesssim \int_{E_t \cap B_\rho} |D\eta|^p (u-t)^p \\
&\quad + \int_{E_t \cap B_\rho} |Du|^p \\
&\lesssim \frac{1}{(\rho-r)^p} \int_{E_t \cap B_\rho} (u-t)^p \\
&\quad + \frac{1}{(R-\rho)^p} \int_{E_t \cap B_R} (u-t)^p .
\end{aligned}$$

Now set

$$\rho = \frac{R+r}{2}.$$

Then the last inequality gives that

$$\int_{B_1} \left|D\left(\eta(u-t)^+\right)\right|^p \lesssim \frac{1}{(R-r)^p}\int_{E_t\cap B_R}(u-t)^p.$$

Now, the Poincaré-Sobolev estimate for $H_0^{1,p}, 1\le p<n$ gives[1]

$$\left(\int_{B_1}\left|\eta(u-t)^+\right|^{p^*}\right)^{1/p^*} \lesssim \left(\int_{B_1}\left|D\left(\eta(u-t)^+\right)\right|^p\right)^{1/p}, \quad \frac{1}{p}=\frac{1}{n}+\frac{1}{p^*},$$

and we obtain

$$\left(\int_{E_t\cap B_r}(u-t)^{p^*}\right)^{p/p^*} \lesssim \frac{1}{(R-r)^p}\int_{E_t\cap B_R}(u-t)^p.$$

The Hölder inequality implies that

$$|E_t\cap B_r|^{\frac{p}{p^*}-1}\int_{E_t\cap B_r}(u-t)^p \le \left(\int_{E_t\cap B_r}(u-t)^{p^*}\right)^{p/p^*}.$$

Consequently

$$\int_{E_t\cap B_r}(u-t)^p \lesssim \frac{|E_t\cap B_r|^{p/n}}{(R-r)^p}\int_{E_t\cap B_R}(u-t)^p.$$

Let us record this result in the following form:

$$I(r,t)\lesssim \frac{\mu(r,t)^{p/n}}{(R-r)^p} I(R,t). \tag{3.19}$$

Estimate (3.19) shows what happens when we go to the smaller scale. What happens when we go the higher level?

Step 2. To pass from E_t to E_s with $s>t$ we will use only a very simple fact, namely the Chebyshev inequality:

$$|E_s\cap B_r| \le \frac{1}{(s-t)^p}\int_{E_t\cap B_r}(u-t)^p.$$

In other words,

$$\mu(r,s)\le \frac{1}{(s-t)^p} I(r,t), \quad s>t. \tag{3.20}$$

Step 3. Combining (3.19) and (3.20) we discover that

$$I(r,s)\mu(r,s)\lesssim \frac{\mu(r,t)^{p/n}}{(R-r)^p(s-t)^p} I(R,t)I(r,t), \quad r<R,\ s>t.$$

[1]We tacitly assume that $n\ge 3$ because for $n=2$ Ch. B. Morrey gave the proof of Hölder continuity of u using the regularity theory of quasiconformal mappings. Thus we can assume that $n\ge 3$ and $p=2$ is allowed.

Actually, it is better to take a power $\kappa > 0$ of (3.19) and obtain

$$\begin{aligned} I(r,s)^\kappa \mu(r,s) &\lesssim \frac{\mu(r,t)^{p\kappa/n}}{(R-r)^{p\kappa}(s-t)^p} I(R,t)^\kappa I(r,t), \\ &\lesssim \frac{\mu(R,t)^{p\kappa/n}}{(R-r)^{p\kappa}(s-t)^p} I(R,t)^{1+\kappa}, \\ & \quad r < R, \\ & \quad s > t. \end{aligned} \tag{3.21}$$

The reason for introducing this $\kappa > 0$ is that now we can use

$$\Phi(r,s) = I(r,s)^\kappa \mu(r,s) = \left[\left(\int_{B_r \cap E_s} (u-s)^p \right)^{\frac{1}{p}} |B_r \cap E_s|^{\frac{1}{p\kappa}} \right]^{\kappa p}$$

to express (3.21) simply as

$$\begin{aligned} \Phi(r,s) &\lesssim \frac{\Phi(R,t)^\beta}{(R-r)^{p\kappa}(s-t)^p}, \\ & \quad r < R, \\ & \quad s > t, \end{aligned} \tag{3.22}$$

provided $\kappa = \kappa(n,p)$ and $\beta = \beta(n,p)$ are given by

$$\beta = \frac{p\kappa}{n}, \quad \beta = \frac{\kappa+1}{\kappa} = 1 + \frac{1}{\kappa} \Rightarrow p\kappa^2 - n(\kappa+1) = 0 \text{ see (2.11)}.$$

Notice that

$$\beta > 1.$$

The crucial observation is that (3.22) can be iterated to produce (3.18).

Step 4. We are going to establish (3.18). First notice that

$$\|u\|_{L^\infty(B_{1/2})} \le M \Longleftrightarrow \Phi(1/2, M) = 0.$$

We will establish the last equality. For that we want to iterate (3.22). Therefore we necessarily have to introduce a sequence of radii $\{R_j\}$. Set

$$R_j = \frac{1}{2} + \frac{1}{2^j}, \quad j = 1, 2, \dots.$$

Fix a level

$$M = t + T$$

with some $T = T(n,p,t) > 0$ to be chosen below. Let us try to approach it with the sequence $\{t_j\}$,

$$t_j = t + T - \frac{T}{2^j}, \quad j = 1, 2, \dots.$$

Thus $R_j \to 1/2$, $t_j \to t + T$, and

$$|R_{j+1} - R_j| \asymp \frac{1}{2^j}, \quad |t_{j+1} - t_j| \asymp \frac{T}{2^j}.$$

Utilising (3.22) we discover that

$$\begin{aligned}\Phi(R_{j+1},t_{j+1}) &\le \frac{C2^{p\kappa j}2^{pj}}{T^p}\Phi(R_j,t_j)^\beta \\ &\le \frac{\tilde{C}^j}{T^p}\Phi(R_j,t_j)^\beta \\ &\le \frac{\tilde{C}^j}{T^p}\left(\frac{\tilde{C}^{j-1}}{T^p}\Phi(R_{j-1},t_{j-1})^\beta\right)^\beta \dots, \quad j=1,2\dots,\end{aligned}$$

with some $\widetilde{C}(n,L,p)=\max(1,C)2^{p(\kappa+1)}>0$.

Note that for $\beta>1$ we have that

$$\begin{aligned}\sum_{j=1}^\infty \frac{j}{\beta^j} &\lesssim \int_1^\infty \frac{x}{\beta^x}dx \\ &= \int_1^\infty e^{-x\log\beta}xdx = \\ &= \frac{1}{(\log\beta)^2}\int_1^\infty e^{-z}zdz \\ &\le \frac{\Gamma(2)}{(\log\beta)^2}.\end{aligned}$$

Having in mind this and that $\beta>1$ we straightforwardly iterate this estimate to derive that

$$\begin{aligned}\Phi(R_{j+1},t_{j+1}) &\le \frac{\widetilde{C}^{\left(j+(j-1)\beta+(j-2)\beta^2+\dots+1\beta^{j-1}\right)}}{T^{p(1+\beta+\beta^2+\dots+\beta^{j-1})}}\Phi(1,t)^{\beta^j} \\ &= \left(\frac{\widetilde{C}^{\left(\frac{j}{\beta^j}+\frac{(j-1)}{\beta^{j-1}}+\frac{(j-2)}{\beta^{j-2}}+\dots+\frac{1}{\beta}\right)}}{T^{p\left(\frac{1}{\beta^j}+\frac{1}{\beta^{j-1}}+\frac{1}{\beta^{j-2}}+\dots+\frac{1}{\beta}\right)}}\right)^{\beta^j}\Phi(1,t)^{\beta^j} \\ &\le \left(\frac{\widehat{C}\Phi(1,t)}{T^{\frac{p}{\beta}\left(\frac{1-(1/\beta)^j}{1-(1/\beta)}\right)}}\right)^{\beta^j} \qquad \widehat{C}=\widetilde{C}\frac{\Gamma(2)}{(\log\beta)^2} \\ &= \left(\frac{\widehat{C}\Phi(1,t)}{T^{\frac{p}{\beta-1}\left(1-\frac{1}{\beta^j}\right)}}\right)^{\beta^j} \\ &= \left(\frac{\widehat{C}\Phi(1,t)}{T^{\frac{p}{\beta-1}}}\right)^{\beta^j} T^{\frac{p\beta^j}{(\beta-1)\beta^j}} \\ &= \left(\frac{\widehat{C}\Phi(1,t)}{T^{\frac{p}{\beta-1}}}\right)^{\beta^j} T^{p/(\beta-1)}.\end{aligned}$$

Recall that $\beta>1$. Hence we see that

$$\Phi(1/2,t+T)=\lim_{j\to\infty}\Phi(R_j,t_j)=0$$

provided
$$\frac{\widehat{C}\Phi(1,t)}{T^{\frac{p}{\beta-1}}} < 1.$$
For example, we can take
$$T = (\widehat{C}+1)\Phi(1,t)^{(\beta-1)/p}.$$
Then
$$\begin{aligned} T &= \tilde{C}I(1,t)^{\kappa(\beta-1)/p}\mu(1,t)^{(\beta-1)/p} \\ &= \tilde{C}I(1,t)^{1/p}\mu(1,t)^{\alpha} \quad \text{with} \quad \alpha = (\beta-1)/p = \frac{1}{p\kappa} > 0. \end{aligned}$$
We conclude that
$$\begin{aligned} \|u\|_{L^\infty(B_{1/2})} &\le t + T \\ &\le t + CI(1,t)^{1/p}\mu(1,t)^{\alpha} \end{aligned}$$
which establishes (3.18). □

4. Holder continuity from De Giorgi oscillation lemma

The De Giorgi oscillation lemma implies the Hölder continuity of $u \in H^1$ solving
$$Lu = 0.$$
Namely,

for $u \in H^1(B_2)$ solving
$$Lu = 0 \quad \text{in} \quad B_2$$
the estimate
$$[u]_{C^{0,\alpha}(\overline{B}_{1/2})} \lesssim \text{osc}_{B_1} u \tag{4.1}$$
holds.

Note that
$$\text{osc}_{B_1} u \le 2\|u\|_{L^\infty(B_1)},$$
and that by (2.10)
$$\|u\|_{L^\infty(B_1)} \lesssim \|u\|_{L^2(B_{3/2})}.$$
We set
$$\begin{aligned} M(R) &= \sup_{B_R} u, \\ m(R) &= \inf_{B_R} u, \\ o(R) &= \text{osc}_{B_R} u = M(R) - m(R). \end{aligned}$$
To prove (4.1) it is enough to show that there exists $\xi < 1$ such that
$$o(R/4) \le \xi o(R) \quad \text{for} \quad \text{all} \quad R > 0. \tag{4.2}$$
To prove this fix $R > 0$. Take the level $l = (M(R) + m(R))/2$. Either
$$|\{u \ge l\} \cap B_{R/4}| \ge |B_{R/4}|/2,$$

or

$$|\{u \le l\} \cap B_{R/4}| \ge |B_{R/4}|/2.$$

In the first case we utilise that u is a supersolution $Lu \le 0$ and use (2.5). In the second case we utilise that u is a subsolution $Lu \ge 0$ and use (2.4). Assume, for example, that the first possibility is realised. We have

$$Lu \le 0 \quad \text{in} \quad B_R,$$

together with the pointwise bounds

$$\begin{aligned} &u \ge m(R) \quad \text{in} \quad B_R, \\ &\left| \left\{ u \ge m(R) + \frac{o(R)}{2} \right\} \cap B_{R/4} \right| \ge |B_{R/4}|/2. \end{aligned}$$

Estimate (2.5) is scale invariant. Applying it to our u deduce that there exists $\varepsilon = \varepsilon(n, L) > 0$ such that

$$m(R/4) \ge m(R) + \frac{\varepsilon}{2} o(R).$$

At the same time

$$M(R/4) \le M(R).$$

Hence

$$\begin{aligned} o(R/2) &= M(R/4) - m(R/4) \\ &\le o(R) - \frac{\varepsilon}{2} o(R) \\ &= \frac{2-\varepsilon}{2} o(R). \end{aligned}$$

We conclude that (4.2) holds. Thus (4.1) is proved.

Lemma 5. *Let $o(r) \ge 0$ be non-decreasing and $o(r) \le \eta o(4r)$ for some $\eta \in (0,1)$. Then*

$$o(\rho) \le \frac{1}{\eta} \left(\frac{\rho}{R} \right)^{\alpha} o(R), \alpha = \frac{\log \frac{1}{\eta}}{\log 4}.$$

Proof.

$$4^m \le \frac{R}{\rho} \le 4^{m+1} \quad \Rightarrow \quad m \log 4 \le \log \frac{R}{\rho} \le (m+1) \log 4$$

$$m \le \frac{\log \frac{R}{\rho}}{\log 4} \le m + 1 \quad \Rightarrow \quad \eta^m \ge \eta^{\frac{\log \frac{R}{\rho}}{\log 4}} \ge \eta^{m+1}.$$

Iterating $\frac{R}{4^{m+1}} \le \rho \le \frac{R}{4^m}$ we have

$$\begin{aligned} o(\rho) &\le o(\frac{R}{4^m}) \le \eta^m \le \frac{1}{\eta} \eta^{m+1} o(R) \\ &\le \frac{o(R)}{\eta} \eta^{\frac{\log \frac{R}{\rho}}{\log 4}} \\ &= \frac{1}{\eta} \left(\frac{\rho}{R} \right)^{\alpha} o(R). \end{aligned}$$

□

5. The mean curvature equation and a Bernstein type theorem

Let u be a $W^{1,1}(\Omega)$ satisfying

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=0.$$

This PDE $\mathcal{M}u=\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)$ is called the mean curvature equation and arises in the area minimization

$$Area=\int_\Omega\sqrt{1+|Du|^2}\to\min.$$

Theorem 6.

$$\sup_\Omega|Du|\le M\Rightarrow u\in C^{1,\alpha}_{loc}(\Omega).$$

Proof. Let e be a fixed unit vector. We have

$$\partial_e\left(\frac{Du}{\sqrt{1+|Du|^2}}\right)=\frac{Du_e}{\sqrt{1+|Du|^2}}-\frac{DuD^2uDu_e}{\left(\sqrt{1+|Du|^2}\right)^3}=a\left(x\right)Du_e$$

where

$$a\left(x\right)=\frac{1}{\sqrt{1+|Du|^2}}\left(id-\frac{Du\otimes Du}{1+|Du|^2}\right).$$

It follows that u_e solves the equation

$$\operatorname{div}\left(a\left(x\right)Du_e\right)=0.$$

Denote $w=Du_e$, then $w\in L^\infty(\Omega)$. Furthermore,

$$\begin{aligned}a_{ij}\xi^i\xi^j &= \frac{1}{\sqrt{1+|Du|^2}}\left(|\xi|^2-\frac{(Du\cdot\xi)^2}{1+|Du|^2}\right)\\ &\ge \frac{1}{\sqrt{1+|Du|^2}}\left(|\xi|^2-\frac{(|Du||\xi|)^2}{1+|Du|^2}\right)\\ &\ge \frac{1}{\left(\sqrt{1+|Du|^2}\right)^{\frac{3}{2}}}|\xi|^2.\end{aligned}$$

On the other hand

$$a_{ij}\xi^i\xi^j\le\frac{1}{\sqrt{1+|Du|^2}}|\xi|^2,$$

consequently, a is a uniformly elliptic matrix such that

$$\lambda = \frac{1}{(1+M^2)^{3/2}}, \Lambda = \frac{1}{\sqrt{1+M^2}}.$$

Applying De Giorgi's theorem the result follows. □

Theorem 7. *(Bernstein type theorem) Suppose that* $\mathcal{M}u = 0$ *in* $\mathbb{R}^n$ *such that* $\sup_{\mathbb{R}^n} |Du| \leq M$*. Then* u *is a linear function.*

Proof. Suppose $u(0) = 0$ and introduce the scaled functions

$$u \to \frac{u(Rx)}{R} = u_R(x), \quad x \in B_\rho, R > 0.$$

It is easy to check that $\mathcal{M}u_R = 0$. Observe that

$$x = \frac{z}{R} \ \Rightarrow \left|\frac{z}{R}\right| \leq 1.$$

We have from the $C^{1,\alpha}$ result of the previous theorem

$$|Du_R(x) - Du_R(y)| \leq C|x-y|^\alpha.$$

Scaling to the original variables we get that

$$|Du(z) - Du(\xi)| \leq C\frac{|z-\xi|^\alpha}{R^\alpha}.$$

□

6. Weak Harnack inequality from De Giorgi oscillation lemma

The De Giorgi oscillation lemma can be formulated in a seemingly stronger form. For example for supersolutions we have:

Let $u \in H^1(B_1)$ *satisfy*

$$Lu \leq 0 \quad in \quad B_1.$$

Assume that u *is normalised at a level* A *as follows: for some* $\delta, \theta > 0$

$$u \geq A \quad in \quad B_1,$$
$$\left|\{u \geq A + \delta\} \cap B_{1/100}\right| \geq \theta |B_{1/100}|.$$

Then there exists $\varepsilon = \varepsilon(n, L, \theta) > 0$ *such that*

$$\inf_{B_{1/2}} u \geq A + \varepsilon\delta. \tag{6.1}$$

In other words (having in mind the comparison principle), the oscillation lemma says that if a supersolution u lies above the level A everywhere in B_1 (and in particular around ∂B_1) and if in $B_{1/100}$ it lies above the level C, $C > A$, on a set of big measure, then in the ball $B_{1/2}$ it lies above the level

$$A + \varepsilon(C - A),$$

with $\varepsilon > 0$ independent of u, A, and C.

It is convenient to reformulate the De Giorgi oscillation lemma for dyadic cubes instead of balls. Let $w \in H^1(B_1)$ be a positive supersolution:

$$\begin{aligned} Lw \le 0 \quad &\text{in} \quad B_1, \\ w \ge 0 \quad &\text{in} \quad B_1. \end{aligned}$$

We will assume that dyadic cubes stay far away from ∂B_1. Let Q be a dyadic cube, and let $\widetilde{Q}$ be any of $\chi(n)$ dyadic cubes touching Q and belonging to the previous generation.

Proposition 8. *There exists $M > 0$, $M = M(n, L)$, such that*

$$|\{w \ge 1 + M\} \cap Q| > \frac{1}{2}|Q| \quad \Longrightarrow \quad \inf_{\widetilde{Q}} w > 1 \text{ for any } \widetilde{Q}. \tag{6.2}$$

Proof. Indeed, (6.1) implies at once (with $A = 0, \delta = 1$) that

$$\left.\begin{array}{l} u \ge 0 \quad \text{in } B_1, \\ |\{u \ge 1\} \cap Q| > \frac{1}{2}|Q| \end{array}\right\} \Longrightarrow \inf_{\widetilde{Q}} u \ge \varepsilon.$$

Now set $u = \frac{w}{1+M}$. Then

$$\{w \ge 1 + M\} \cap Q = \{u \ge 1\} \cap Q.$$

Hence the normalisation

$$\begin{aligned} &w \ge 0, \\ &|\{w \ge 1 + M\} \cap Q| > \frac{1}{2}|Q| \end{aligned}$$

implies

$$\inf_{\widetilde{Q}} \frac{w}{1 + M} = \inf_{\widetilde{Q}} u \ge \varepsilon.$$

Consequently,

$$\inf_{\widetilde{Q}} w \ge (1 + M)\varepsilon > 1$$

provided we take, say, $M = \dfrac{1}{\varepsilon}$. □

Another way to say (6.2) is:

There exists $M > 0$, $M = M(n, L)$, such that

$$\inf_{\widetilde{Q}} w \le 1 \text{ for some } \widetilde{Q} \Longrightarrow |\{w \ge 1 + M\} \cap Q| \le \frac{1}{2}|Q| \text{ for all dyadic subcubes } Q. \tag{6.3}$$

This is just the trivial negation of both sides in (6.2).

The oscilation lemma toether with the Calderon-Zygmund decomposition implies the *weak Harnack inequality*. It is convinient to have in mind both forms (6.2) and (6.3) of the oscilation lemma. We set

$$Q_0 = (0,1)^n, \quad Q_1 = (-3,4)^n, \quad Q_2 = (-6,8)^n, \quad Q_0 \subset Q_1 \subset Q_2.$$

Theorem 9. *Let $u \geq 0$ satisfy the oscilation conditions (6.2), (6.3) with some constant $M > 0$. Then there exists $\kappa > 0$, $\kappa = \kappa(n, M)$, such that*

$$|E_t \cap Q_0| \lesssim t^{-\kappa} \inf_{Q_1} u. \tag{6.4}$$

In particular, for some $p > 0$, $p = p(n, L)$,

$$\|u\|_{L^p(Q_0)} \lesssim \inf_{Q_1} u.$$

Proof. **Step 1.** Without loss of generality we assume that

$$\inf_{Q_1} u = 1.$$

Then oscilation property (6.3) implies that

$$|\{u \geq 1 + M\} \cap Q_0| \leq \frac{1}{2}|Q_0|.$$

We are going to combine the Calderòn-Zygmund decompoistion with the oscilation property at any scale to derive that

$$\left|\{u \geq (1+M)^k\} \cap Q_0\right| \leq \frac{1}{2^k}|Q_0|, \quad \text{for all } k = 1, 2 \dots. \tag{6.5}$$

Estimate (6.4) follows immediately from (6.5). The rest of the proof is devoted entirely to veryfying (6.5).

Step 2. Set $E_t = \{u \geq t\}$. We have from (6.3)

$$|E_{1+M} \cap Q_0| \leq \frac{1}{2}|Q_0|.$$

Of course we also have

$$E_{1+M} \supset E_{(1+M)^2} \supset \cdots \supset E_{(1+M)^k} \supset E_{(1+M)^{k+1}} \supset \cdots.$$

Fix an element in this chain. The crucial fact here is that its measure is less than the measure of the previous element by a *fixed factor*. Namely, we claim that

$$|\{u \geq (1+M)^2\} \cap Q_0| = |E_{(1+M)^2} \cap Q_0| \leq \frac{1}{2^2}|Q_0|. \tag{6.6}$$

To see this we will use a Calderòn-Zygmund type argument. Assume that there is dyadic cube $Q \subset Q_0$ such that the inequality

$$\left|E_{(1+M)^2} \cap Q\right| > \frac{1}{2}|Q|$$

holds. (If no such Q exists then we have $\sum E_{(1+M)^2} \cap Q \leq \frac{1}{2}\sum |Q| \leq \frac{1}{2^{n+1}}|Q_0| \leq \frac{1}{2^2}|Q_0|$). For Q we have

$$\left|\left\{\frac{u}{1+M} \geq 1 + M\right\}\right| > \frac{1}{2}|Q|.$$

Oscilation property (6.2) applied to $\frac{u}{1+M}$ gives

$$\inf_{\widetilde{Q}} \frac{u}{1+M} > 1,$$

and hence

$$\widetilde{Q} \subset E_{1+M}.$$

Thus we have derived the following implication:

$$\left|E_{(1+M)^2} \cap Q\right| > \frac{1}{2}|Q| \Rightarrow \widetilde{Q} \subset E_{1+M}.$$

In particular, we derived that

> *if a dyadic cube $Q \subset Q_0$ is filled with the set $E_{(1+M)^2}$ by more than $1/2$, then its dyadic parent Q' belongs to E_{1+M}.*

Now apply a Calderon-Zygmund type lemma described in the last step below with $\mathcal{A} = E_{(1+M)^2}, \mathcal{B} = E_{1+M}$ at the level $\delta = 1/2$. We deduce at once that

$$|E_{(1+M)^2} \cap Q_0| \leq \frac{1}{2}|E_{1+M} \cap Q_0|.$$

Recalling that

$$|E_{1+M} \cap Q_0| \leq \frac{1}{2}|Q_0|$$

we see that (6.6) holds.

Further iterations give

$$\left|E_{(1+M)^k} \cap Q_0\right| \leq \frac{1}{2^k}|Q_0|$$

which proves (6.5).

Step 3. We present a Calderón-Zygmund type argument and prove the following statement. By Q we denote a dyadic cube, and by Q' his parent.

Lemma 10. *Let $\delta \in (0,1)$, let $A \subset B \subset Q_0$, be measurable and let $\mathcal{A}$ satisfy*

(i) $|\mathcal{A}| \leq \delta |Q_0|$.
(ii) *For any dyadic cube Q the following implication holds:*

$$\frac{|\mathcal{A} \cap Q|}{|Q|} > \delta \Longrightarrow Q' \subset \mathcal{B}.$$

Then

$$|A| \leq \delta |B|. \tag{6.7}$$

Proof. Recall the Calderòn-Zygmund decomposition: for every $f \in L^1(Q_0)$ and $\delta > 0$ there are disjoint dyadic countably many cubes $\{Q_j\}$ such that

$$|f| \leq \delta \quad \text{a.e. in } Q_0 \setminus \bigcup_j Q_j$$

$$\delta \leq \fint_{Q_j} |f| < 2^n \delta.$$

Apply this to $1_{\mathcal{A}}$ to infer

$$\mathcal{A} \subset \bigcup Q_j \quad \text{modulo a set of measure zero}$$

$$\delta \leq \frac{|\mathcal{A} \cap Q_j|}{|Q_j|} < 2^n \delta \quad \frac{|\mathcal{A} \cap \widetilde{Q}_j|}{|\widetilde{Q}_j|} < \delta$$

where the last inequality follows from the Claderòn-Zygmung argument where those cube which satisfy $\frac{|\mathcal{A} \cap \widetilde{Q}_j|}{|\widetilde{Q}_j|}$ bisected (resulting Q_j) until the opposite inequality is true for Q_j. From our assumption (ii) we see that

$$\mathcal{A} \subset \bigcup_j \widetilde{Q}_j \subset \mathcal{B}.$$

Relabelling $\widetilde{Q}_j$ such that they are non-overlapping we get

$$|\mathcal{A}| \leq \sum_j |\mathcal{A} \cap \widetilde{Q}_j| \leq \delta \sum_j |\widetilde{Q}_j| \leq \delta |\mathcal{B}|.$$

□

The L^p estimate now follows from an elementary integration argument as in Section 10.3. □

7. Harnack inequality

For $u \in H^1(B_1)$ satisfying

$$u \geq 0, \quad Lu \geq 0 \quad \text{in} \quad B_1,$$

estimate (2.10) with $t = 0$ gives

$$\sup_{B_{1/2}} u \lesssim \left(\int_{B_1} |u|^2 \right)^{1/2}.$$

Simple translations and scalings give that for any $B(p,R) \subset B_1$ we have

$$\sup_{B(p,R/2)} u \lesssim \left(\frac{1}{R^n} \int_{B(p,R)} |u|^2 \right)^{1/2}.$$

Moreover, we have

$$\sup_{B(p,\tau R)} u \lesssim \left(\frac{1}{((1-\tau)R)^n} \int_{B(p,R)} |u|^2 \right)^{1/2}.$$

A simple scaling is not enough to prove this estimate. However, we can argue as follows. Choose $x_0 \in B(p, \tau R)$ such that

$$\left(\sup_{B(p,\tau R)} u \right)^2 \leq 2u(x_0)^2.$$

At the same time $B(x_0,(1-\tau)R)\subset B(p,R)$ and hence

$$\begin{aligned} u(x_0)^2 &\le \sup_{B(x_0,(1-\tau)R/4)} u^2 \\ &\lesssim \frac{1}{((1-\tau)R)^n}\int_{B(x_0,(1-\tau)R/2)}|u|^2 \\ &\lesssim \frac{1}{(1-\tau)^n}\left(\frac{1}{R^n}\int_{B(p,R)}|u|^2\right). \end{aligned}$$

8. The Aleksandrov maximum principle

Normal mapping is defined as followsiu

$$\chi_u(y)=\{p\in\mathbb{R}^n:\ \text{s.t. } u(x)\ge u(y)+p(x-y),\ \forall x\in\Omega\}.$$

Lower contact set is defined

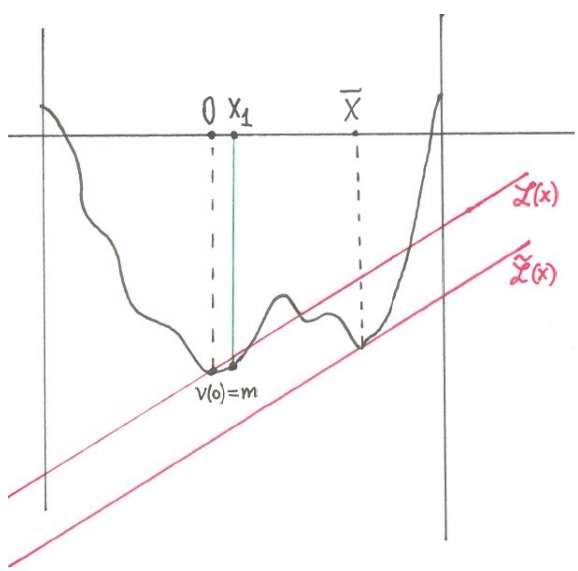
$$\begin{aligned} \Gamma_u^- &= \{y\in\Omega \text{ s.t. } \chi_u(y)\ne\emptyset\} \\ &= \{y\in\Omega \text{ s.t. } u(x)\ge u(y)+p\cdot(x-y)\ \ \forall x\in\Omega \text{ and some } p\in\mathbb{R}^n\} \\ &= \{y\in\Omega \text{ where } u \text{ is convex}\}. \end{aligned}$$

Proposition 11. *Let $u\in C^0(\Omega)$ then*

- *If u is differentiable at $y\in\Gamma^-$ then $\chi_u(y)=Du(y)$.*
- *if u is convex in Ω then $\Gamma^-=\Omega$ and χ is subgradient of u.*
- *If u is twice continuously differentiable at $y\in\Gamma^-$ then $D^2u(y)\ge 0$.*

Lemma 12. *Let $M=\dfrac{\inf_{\partial\Omega}u-\inf_\Omega u}{d}$, Ω bounded $u\in C^2(\Omega)\cap C^0(\overline{\Omega})$. Then*

$$B_M(0)\subset\chi_u(\Gamma^-).$$



Proof. Note that $\partial\Omega \subset \overline{\Omega}$ thus $\inf_{\partial\Omega} u - \inf_\Omega u \geq 0$. If $M = 0$ then $B_M(0) = \emptyset$. Thus assume $M > 0$. Replacing u with $v = u - \inf_{\partial\Omega} v$ we have $\inf_{\partial\Omega} v = 0, v \geq 0$ on $\partial\Omega$ and in terms of the new function v we have that

$$M = -\frac{\inf_\Omega v}{d}.$$

Let $,\Omega^- = \{v < 0\}$. It is enough to show that

$$B_M(0) \subset \chi_u\left(\Gamma^- \cap \Omega^-\right).$$

Given $\xi \in B_M(0)$. Suppose $m = \min_\Omega v = v(0), 0 \in \Omega$ (if the minimum is attained on $\partial\Omega$ then we are done). In our notations

$$M = -\frac{m}{d}, m = v(0) < 0.$$

Consider the linear function $\mathscr{L}(x) = m + \xi \cdot x, |\xi| < M$ then

$$\mathscr{L}(x) < m + \left(\frac{-m}{d}\right) d = 0, \quad \forall x \in \Omega, \mathscr{L}(0) = m < 0.$$

There is x_1 close to 0 such that

$$v(x_1) < \mathscr{L}(x_1) < 0.$$

Thus we have "room" to translate $\mathscr{L}(x)$ downwards before it becomes tangent to the graph of v at some point $\bar{x}$. Note that $\mathscr{L}(x) < 0, x \in \Omega$ thus after translating downwards the resulted functions $\widetilde{\mathscr{L}}(x) < 0, x \in \Omega$ and consequently $\bar{x} \notin \partial\Omega$. Thus we see that for given $\xi \in B_M(0)$ there is $\bar{x} \in \Omega^-$ such that ξ is the slope of the tangent at $\bar{x}$. □

Theorem 13. *Let $u \in C^2(\Omega) \cap C^0\left(\overline{\Omega}\right)$ satisfy*

$$\det D^2 u \leq f(x)\, h(Du) \quad \text{on } \Gamma^-$$

for some $h > 0, h \in C^2(\mathbb{R}^n)$. Assume $\int_{\Gamma^-} f < \int_{\mathbb{R}^n} \dfrac{dp}{h(p)}$

$$\min_\Omega \geq \min_{\partial\Omega} -cd\left(\int_{\Gamma^-} f\right)^{\dfrac{1}{n}}$$

where $C = C(n)$ depends only on the dimension n.

Proof.

$$\frac{M^n}{\sup h} \leq \int_{B_M(0)} \frac{dp}{h(p)} \leq \int_{\chi(\Gamma^-)} \frac{dp}{h(p)} \leq \int_{\Gamma-} \frac{\det D^2 u}{h(Du)} \leq \int_{\Gamma-} f(x)$$

By approximation $\chi_u^\varepsilon = Du + \varepsilon Id$, and then letting $\varepsilon \to 0$ we can justify the change of variable formula.

Consequently

$$M^n = \left(\frac{\inf_{\partial\Omega} u - \inf_\Omega u}{d}\right)^n \leq \int_{\Gamma-} f(x)\, dx,$$

and the result follows. □

Consider

$$Lu = a_{ij}u_{ij} \le g \text{ in } \Omega$$

For $n \times n$ matrices $A, B \ge 0$ we have

$$\det A \det B \le \left(\frac{\mathrm{Tr}(AB)}{n}\right)^n$$

$$\inf_{\partial\Omega} u \le \inf_{\Omega} u - dC(n) \left\| \frac{g}{(\det a)^{\frac{1}{n}}} \right\|_{L^n(\Gamma^-)}, \quad d = \mathrm{diam}\Omega. \tag{8.8}$$

Similarly for $Lu \ge g$ we have

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u + dC(n) \left\| \frac{g}{(\det a)^{\frac{1}{n}}} \right\|_{L^n(\Gamma^+)}, \quad d = \mathrm{diam}\Omega, \tag{8.9}$$

where Γ^+ is the upper contact set defined by

$$\Gamma_u^+ = \{y \in \Omega \text{ s.t. } u(x) \le u(y) + p \cdot (x - y),\ \forall x \in \Omega \text{ and some } p \in \mathbb{R}^n\}.$$

9. Applications of Aleksandrov's maximum principle

9.1. Generalizations for $Lu = a_{ij}u_{ij} + b_iu_i + cu$. [2]

Next proposition contains a generalization of the Aleksandrov maximum principle.

Proposition 14. *Let $\mathscr{D} = \det a, \mathscr{D}^* = \mathscr{D}^{\frac{1}{n}}$ such that*

$$\lambda \le \mathscr{D}^* \le \Lambda.$$

Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfying

$$Lu := a_{ij}u_{ij} + b_iu_i + cu \ge f \quad in \quad \Omega$$

such that the following holds

$$\frac{|b|}{\mathscr{D}^*}, \frac{f}{\mathscr{D}^*} \in L^n(\Omega), \qquad c \le 0 \quad in \quad \Omega$$

Then

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+ + c \left\| \frac{f}{\mathscr{D}^*} \right\|_{L^n(\Gamma^+)}$$

In fact the constant C depends on $n, d = diam\Omega, \left\| \frac{b}{\mathscr{D}^*} \right\|_{L^n(\Gamma^+)}$ and more explicitly

$$C = d\left\{exp\left\{\frac{2^n + n - 2}{\omega_n n^{n+1}}\left(\left\|\frac{b}{\mathscr{D}^*}\right\|^n_{L^n(\Gamma^+)} + 1\right)\right\} - 1\right\}$$

[2]This section covers some of the material from Section 2.5 of the lecture notes of Q. Han and F. Lin.

where $\omega_n = |B_1|$.

Proof. In $\Omega^+ = \{u > 0\}$ with $c \le 0$ we have from $Lu \ge f$

$$
\begin{aligned}
-a_{ij}u_{ij} &\le \overrightarrow{b} \cdot Du - f + c(x)\,u \\
&= \overrightarrow{b} \cdot Du - f + cu^+ - cu^- \\
&= \overrightarrow{b} \cdot Du - f + cu^+ \\
&\le |f|\,|Du| + f^-.
\end{aligned} \tag{9.10}
$$

It is clear from the binomial theorem that

$$\left(|b|\,|Du| + f^-\right)^n = \sum_{k=0}^{n} \left(|b|\,|Du|\right)^k \left(f^-\right)^{n-k} \binom{n}{k}.$$

For $1 \le k \le n-1$ we have from holder inequality $ab \le \frac{a^p}{p} + \frac{b^q}{q}, p = \frac{n}{k}, \frac{1}{p} + \frac{1}{q} = 1$ we have

$$
\begin{aligned}
\left(|b|\,|Du|\right)^k \left(f^-\right)^{n-k} &= |Du|^n\,|b|^k \left(\frac{f^-}{|Du|}\right)^{n-k} \\
&\le |Du|^n \left(\frac{k}{n}\,|b|^n + \frac{n-k}{n}\left(\frac{f^-}{|Du|}\right)^n\right) \\
&\le \max_{1\le k\le n-1}\left(\frac{k}{n}, \frac{n-k}{n}\right)\left(|b|\,|Du|\right)^n + \left(f^-\right)^n.
\end{aligned}
$$

Using this computation we get

$$
\begin{aligned}
\left(|b|\,|Du| + f^-\right)^n &\le \left(|b|\,|Du|\right)^n + \left(f^-\right)^n + \sum_{x=1}^{n-1}\left(|b|\,|Du|\right)^k (f)^{n-k}\binom{n}{k} \\
&\le \left(\left(|b|\,|Du|\right)^n + \left(f^-\right)^n\right)\left[1 + \max_{1\le k\le n-1}\left(\frac{k}{n}, \frac{n-k}{n}\right)\sum_{k=1}^{n-1}\binom{n}{k}\right] \\
&\le \left(\left(|b|\,|Du|\right)^n + \left(f^-\right)^n\right)\left[\max_{1\le k\le n-1}\left(\frac{k}{n}, \frac{n-k}{n}\right)[2^n - 2] + 1\right] \\
&\le \left[\frac{1}{n}[2^n - 2] + 1\right]\left(\left(|b|\,|Du|\right)^n + \left(f^-\right)^n\right).
\end{aligned}
$$

For $\mu > 0$ to be fixed below we notice that

$$
\begin{aligned}
\left(|b|^n + \left(\frac{f^-}{\mu}\right)^n\right)\left(|Du|^n + \mu^n\right) &= |b|^n\,|Du|^n + \left(f^-\right)^n + |b|^n\,\mu^n + \left(\frac{f^-}{\mu}\right)^n |Du|^n \\
&\ge |b|^n\,|Du|^n + \left(f^-\right)^n
\end{aligned}
$$

implying the estimate

$$|b|\,|Du|+f^-\le\left(|b|^n+\frac{(f^-)^n}{\mu^n}\right)^{\frac{1}{n}}(|Du|^n+\mu^n)^{\frac{1}{n}}\left(\frac{2}{n}(2^{n-1}-1)+1\right)^{\frac{1}{n}}.$$

Returning to (9.10) we conclude

$$\det a\det(-D_{ij}u)\le\left(-\frac{a_{ij}u_{ij}}{n}\right)^n\le\left(|b|^n+\frac{(f^-)^n}{\mu^n}\right)(|Du|^n+\mu^n)\frac{1}{n}\left(2^n+n-2\right).$$

Define

$$g\left(\xi\right)=\frac{1}{|\xi|^n+\mu^n}$$

and use Lemma 12 (applied to $-u$ and consequently replacing Γ^- with Γ^+ and $M=\frac{\sup_\Omega u^+-\sup_{\partial\Omega}u}{d}$) to obtain

$$\int_{B_M}g(\xi)d\xi\le\frac{2^n+n-2}{n^{n+1}}\int_{\Gamma^+\cap\Omega^+}\frac{|b|^n+\mu^{-n}\left(f^-\right)^n}{\mathscr{D}}.$$

It remains to evaluate the integral

$$\begin{aligned}\int_{B_M}g(\xi)d\xi&=Area(\partial B_1)\int_0^M\frac{r^{n-1}dr}{r^n+\mu^n}\\&=\frac{Area(\partial B_1)}{n}\log\left(\frac{M^n}{\mu^n}+1\right)\\&=\omega_n\log\left(\frac{M^n}{\mu^n}+1\right).\end{aligned}$$

Consequently

$$M^n\le\mu^n\left\{\exp\left\{\frac{2^n+n-2}{\omega_n n^{n+1}}\left[\left\|\frac{b}{\mathscr{D}^*}\right\|^n_{L^n(\Gamma^+\cap\Omega^+)}+\mu^{-n}\left\|\frac{f}{\mathscr{D}^*}\right\|^n_{L^n(\Gamma^+\cap\Omega^+)}\right]\right\}-1\right\}.$$

If $f\not\equiv0$ then we choose $\mu=\left\|\frac{f}{\mathscr{D}^*}\right\|_{L^n(\Gamma^+\cap\Omega^+)}$. If $f=0$ then we let $\mu\to0$. □

We begin with the following simple application of strong maximum principle due to James Serrin. Note that there is no assumption of the sign of c.

Lemma 15. *Let $u\in C^2(\Omega)\cap C\left(\overline{\Omega}\right)$ satisfies $Lu\ge0$. If $u\le0$ in Ω then either $u<0$ in Ω or $u\equiv0$ in Ω .*

Proof. We have $c=c^+-c^-$, consequently

$$a_{ij}u_{ij}+b_iu_i-c^-u\ge-c^+u\ge0.$$

Applying the strong maximum principle the result follows. □

Next lemma is valid for the domains with small volume.

Lemma 16. *Let $Lu = a_{ij}u_{ij} + b_iu_i + cu$ in Ω*

$$\{a_{ij}\} \text{ is positive definite in } \Omega, |b_i| + |c| \leq \Lambda, \quad \det(a_{ij}) \geq \lambda.$$

Suppose $u \in C(\overline{\Omega}) \cap C^2(\Omega)$, $Lu \geq 0$ in Ω, $u \leq 0$ in Ω, diam$\Omega \leq d$. There is a constant $\delta = \delta(n, \lambda, \Lambda, d) > 0$ such that if $|\Omega| < \delta$ then

$$u \leq 0 \quad \text{in } \Omega.$$

Proof. Write $c = c^+ - c-$, then it follows

$$a_{ij}u_{ij} + b_iu_i - c^-u \geq -c^+u = -c^+\left(u^+ - u^-\right) \geq -c^+u^+.$$

We can apply now the Aleksandrov maximum principle

$$\begin{aligned}\sup_\Omega u &\leq C\left\|c^+u^+\right\|_{L^n(\Omega)}\\ &\leq C\left\|c^+\right\|_\infty |\Omega|^{\frac{1}{n}} \sup_\Omega u\\ &\leq \delta^{\frac{1}{n}} C\left\|c^+\right\|_\infty \sup_\Omega u.\end{aligned}$$

This implies that $u \leq 0$ if δ is sufficiently small. □

9.2. **The moving plane method.**

Theorem 17. *Suppose $u \in C^2(B_1) \cap C(\overline{B_1})$ is a positive solution of*

$$\begin{aligned}\Delta u + f(u) = 0 \quad &\text{in} \quad B_1,\\ u = 0 \quad &\text{on} \quad \partial B_1,\end{aligned}$$

where f is locally Lipschitz in $\mathbb{R}$. Then u is radially symmetric in B_1 and $\frac{\partial u}{\partial r} < 0$ for every $x \neq 0$.

The proof follows from the following

Lemma 18. *Suppose Ω is a bounded domain convex in x_1 direction and symmetric with respect to the plane $\{x_n = 0\}$. Let $u \in C^2(B_1) \cap C(\overline{B_1})$ be a* **positive** *solution of*

$$\begin{aligned}\Delta u + f(u) = 0 \quad &\text{in} \quad B_1,\\ u = 0 \quad &\text{on} \quad \partial B_1,\end{aligned}$$

where f is locally Lipschitz in $\mathbb{R}$. Then u is symmetric in x_1 variable and $\frac{\partial u}{\partial x_1} < 0$ for every $x \in \Omega$ with $x_1 > 0$.

Proof. **Step 1:** We use the notation $x = (x_1, y), y \in \mathbb{R}^{n-1}$. We shall prove that

$$u(x_1, y) < u(x_1^*, y) \tag{9.11}$$

for any $x_1 > 0$ and $x_1^* < x_1$ with $x_1^* + x_1 > 0$. Then by letting $x_1^* \to x_1$ we get $u(x_1, y) \leq u(-x_1, y)$ for any x_1. Then changing the direction $x_1 \to -x_1$ we get the symmetry.

Step 2: We introduce now the main notations

$$\begin{aligned} a = \sup_{x\in\Omega} x_1, 0 < \lambda < a, \\ \Sigma = \{x \in \Omega \ : \ x_1 > \lambda\}, \\ T_\lambda = \{x_1 = \lambda\}, \\ \Sigma'_\lambda = \text{reflection of } \Sigma_\lambda \text{ with respect } T_\lambda, \\ x_\lambda = (2\lambda - x_1, x_2, \dots, x_n), \quad x = (x_1, x_2, \dots, x_n). \end{aligned}$$

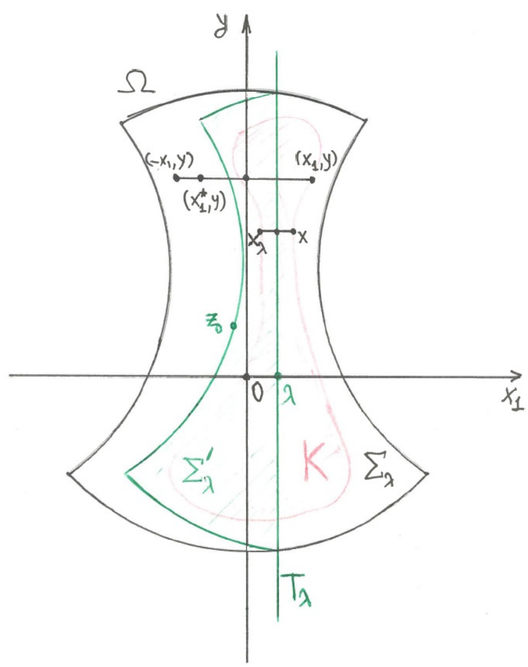
In Σ_λ we define

$$w_\lambda = u(x) - u(x_\lambda), \quad x \in \Sigma_\lambda.$$

Then we have by the mean value theorem

$$\begin{aligned} \Delta w_\lambda + c(x,\lambda) w_\lambda = 0 \qquad & \text{in } \Sigma_\lambda, \\ w_\lambda \le 0, \quad \text{and } w_\lambda \not\equiv 0 \qquad & \text{on } \partial\Sigma_\lambda, \end{aligned}$$

where $c(x,\lambda)$ is a bounded function in Σ_λ.



Claim 19. *$w_\lambda \le 0$ in Σ_λ for any $\lambda \in (0,a)$.*

Proof. For λ close to a we have $w_\lambda < 0$ by Lemma 16.

Let (λ_0, a) be the largest interval of values of λ such that $w_\lambda < 0$ in Σ_λ. We want to show $\lambda_0 = 0$.

If $\lambda_0 > 0$ by continuity, $w_{\lambda_0} \leq 0$ in Σ_{λ_0} and $w_{\lambda_0} \not\equiv 0$ on $\partial\Sigma_{\lambda_0}$(because otherwise we will have an interior point z_0 where u vanishes which is precluded by our assumptions.).

Let us take small $\delta > 0$ to be fixed below. Let K be a closed subset of Σ_{λ_0} such that

$$|\Sigma_{\lambda_0} \setminus K| < \frac{\delta}{2}.$$

The fact that $w_{\lambda_0} < 0$ in Σ_{v_0} implies there is $\eta > 0$ such that

$$w_{\lambda_0}(x) \leq -\eta, \forall x \in K.$$

Choosing ϵ small we get

$$|\Sigma_{\lambda_0-\epsilon} \setminus K| < \delta$$

we choose δ so small that we can apply Lemma 16 to $w_{\lambda_0-\epsilon}$ in $\Sigma_{\lambda_0-\epsilon} \setminus K$. Hence we get that

$$w_{\lambda_0-\epsilon} \leq 0 \text{ in } \quad \Sigma_{\lambda_0-\epsilon} \setminus K$$

and then my Lemma 15 we infer

$$w_{\lambda_0-\epsilon} < 0 \ \text{ in } \quad \Sigma_{\lambda_0-\epsilon} \setminus K.$$

Therefore we obtain that

$$w_{\lambda_0-\epsilon} \ \text{ in } \quad \Sigma_{\lambda_0-\epsilon}$$

which is in contradiction with the definition of λ_0.

□

Step 3: From the claim we see that w_λ assumes along $\partial\Sigma_\lambda \cap \Omega$ its maximum in Σ_λ. By Hopf's lemma we have that for any such $\lambda \in (0, a)$

$$D_{x_1} w_\lambda|_{x_1=\lambda} = 2D_{x_1} u|_{x_1=\lambda} < 0.$$

□

10. Local estimates for subsolutions and supersolution of linear equations $a_{ij}u_{ij} = f$

Let us define

$$u = u_{\epsilon,\alpha}(x) = \begin{cases} 1 - r^\alpha & \text{if } \ r > \epsilon, \\ 1 - \alpha\epsilon^{\alpha-2}r^2 - (1-\alpha)\epsilon^\alpha & \text{if } \ r \leq \epsilon. \end{cases}$$

Let

$$F(D^2u) = \frac{1}{1-\alpha}\sum_{\lambda_j \geq 0} \lambda_j + \frac{1}{n-1}\sum_{\lambda_j \leq 0} \lambda_j.$$

This operator is convex for $\alpha < 1$ and hence one has a priori $C^{2,\alpha}$ estimates of Evans.

For $\alpha < 1$ we have $F(D^2u) = -C(\alpha)\epsilon^{\alpha-2}1_{B_\epsilon}$ hence

$$\|F(D^2u)\|^p_{L^p} = \epsilon^{p(\alpha-2)}\epsilon^n$$

goes to zero as $\epsilon \to 0$ for any

$$p < \frac{n}{2-\alpha}.$$

This shows that $W^{2,p}$ a priori estimates for fully non-linear equations cannot hold for $p < n$.

Note that from Sobolev's embedding theorem we have that $u \in C^\alpha$ if $u \in W^{2,p}, p \geq n$ with

$$\|u\|_{C^\alpha} \leq C(n, \|u\|_{W^{2,p}}).$$

In this section we show that a better estimate holds

$$\|u\|_{C^\alpha} \leq C(n, \|u\|_\infty, L)$$

for the solutions of the elliptic equation

$$Lu := a_{ij}u_{ij} = 0(\text{or } = f).$$

10.1. **Subsolutions.**

Theorem 20. *Let $\Omega \subset \mathbb{R}^n$ be open such that it contains the ball $B_R = B_R(y)\,(y) \subset \Omega, u \in C^2(\Omega), Lu \geq f$ in Ω, with $f \in L^n(\Omega)$. Then $\forall \sigma \in (0,1), n \geq p > 0$*

$$\sup_{B_{R\sigma}} u \leq C\left\{\left(\frac{1}{R^n}\int_{B_R}\left(u^+\right)^p\right)^{\frac{1}{p}} + \frac{R}{\lambda}\|f\|_{L^n(B_R)}\right\}$$

where $C = C\left(n, \sigma, p, \dfrac{\Lambda}{\lambda}\right)$.

Proof. By scale invariance we can take $R = 1$. Define

$$\eta = \begin{cases} \left(1 - |x|^2\right)^\beta, & x \in B_1, \\ 0 & \text{otherwise}, \end{cases} \tag{10.12}$$

where the exponent $\beta > 1$ will be fixed later.

$$|D\eta| \leq c\eta^{1-\frac{1}{\beta}}, \tag{10.13}$$

$$\left|D^2\eta\right| \leq c\eta^{1-\frac{2}{\beta}} \tag{10.14}$$

Define $v = \eta\left(u^+\right)^2$, consequently

$$\begin{aligned} Dv &= D\eta\left(u^+\right)^2 + 2\eta u^+ Du, \\ D^2 v &= D^2\eta\left(u^+\right)^2 + D\eta \otimes D\eta 4u^+ + 2\eta Du^+ \otimes Du^+ + 2\eta u^+ D^2 u. \end{aligned}$$

Combining

$$\begin{aligned} Lv &= \left(u^+\right)^2 a_{ij}\eta_{ij} + 4u^+ a_{ij}\eta_i u_j + 2\eta a_{ij}u_i^+ u_j^+ + 2\eta u^+ Lu \\ &\geq -\Lambda\left(u^+\right)^2 \eta^{1-\dfrac{2}{\beta}} + 4u^+ a_{ij}\eta_i u_j^+ + 2\eta a_{ij} u_i^+ u_j^+ + 2\eta u^+ f. \end{aligned}$$

Clearly we also have

$$\begin{aligned} 4u^+ a_{ij}\eta_i u_j + 2\eta a_{ij}u_i^+ u_j^+ &\geq 2\lambda\eta\left|Du^+ +\right|^2 - 4u^+\Lambda|D\eta|\left|Du^+\right| \\ &\geq 2\lambda\eta\left|Du^+\right|^2 - 4\Lambda\left(\frac{\left(u^+|D\eta|\right)^2}{\eta\varepsilon} + \varepsilon\eta\left|Du^+\right|^2\right). \end{aligned}$$

Returning to Lv we infer

$$\begin{aligned} Lv &\geq -\Lambda \left(u^+\right)^2 \eta^{1-\frac{2}{\beta}} + 2\lambda\eta \left|Du^+\right|^2 - 4\Lambda \left(\frac{\left(u^+ \left|D\eta\right|\right)^2}{\eta\varepsilon} + \varepsilon\eta \left|Du^+\right|^2 \right) + 2\eta u^+ f \\ &\geq -\Lambda \left(u^+\right)^2 \eta^{1-\frac{2}{\beta}} \left(1 + \frac{4}{\varepsilon}\right) + 2\eta \left|Du^+\right|^2 (\lambda - 2\varepsilon\Lambda) + 2\eta u^+ f. \end{aligned}$$

Choose

$$\varepsilon = \frac{\lambda}{2\Lambda}$$

then

$$Lv \geq -\Lambda \left(u^+\right)^2 \eta^{1-\frac{2}{\beta}} \left(1 + \frac{8\Lambda}{\lambda}\right) + 2\eta u^+ f.$$

Now apply Aleksandrov's maximum principle to conclude

$$\begin{aligned} \sup_{B_1} v &\leq \sup_{\partial B_1} v + \frac{C(n)}{\lambda} \left\| -\Lambda \left(1 + \frac{8\Lambda}{\lambda}\right) \left(u^+\right)^2 \eta^{1-\frac{2}{\beta}} + 2\eta u^+ f \right\|_{L^n(B_1)} \\ &= \sup_{\partial B_1} v + C(n) \left\| -\frac{\Lambda}{\lambda} \left(1 + \frac{8\Lambda}{\lambda}\right) v^{1-\frac{2}{\beta}} \left(u^+\right)^{\frac{4}{\beta}} + \frac{2}{\lambda} v^{\frac{1}{2}} \eta^{\frac{1}{2}} f \right\|_{L^n(B_1)} \\ &\leq C(n) \left\{ \frac{\Lambda}{\lambda} \left(1 + \frac{8\Lambda}{\lambda}\right) \left\| v^{1-\frac{2}{\beta}} \left(u^+\right)^{\frac{4}{\beta}} \right\|_{L^n(B_1)} + \frac{2}{\lambda} \parallel (v\eta)^{\frac{1}{2}} f \parallel_{L^n(B_1)} \right\} \\ &\leq C(n) \max\left[\frac{\Lambda}{\lambda}\left(1 + \frac{8\Lambda}{\lambda}\right), 2\right] \left\{ \sup_{B1} v^{1-\frac{2}{\beta}} \left\| \left(u^+\right)^{\frac{4}{\beta}} \right\|_{L^n(B_1)} + \frac{\sup v^{\frac{1}{2}}}{\lambda} \|f\|_{L^n(B_1)} \right\}. \end{aligned}$$

After dividing both sides by $\sup_{B_1} v^{\frac{1}{2}}$ this yields the estimate

$$\sup_{B_1} v^{\frac{1}{2}} \leq C \left\{ \sup_{B_1} v^{1-\frac{2}{\beta}} \left\| \left(u^+\right)^{\frac{4}{p}} \right\|_{L^n(B_1)} + \frac{1}{\lambda} \|f\|_{L^n(B_1)} \right\}.$$

Now choose

$$\beta = \frac{4n}{p}, \quad \Rightarrow \quad \frac{1}{2} - \frac{2}{\beta} = \frac{1}{2}\left(1 - \frac{4}{\beta}\right) = \frac{1}{2}\left(1 - \frac{p}{n}\right).$$

Applying Hölder's inequality we get

$$\sup_{B1} v^{\frac{1}{2}\left(1-\frac{p}{n}\right)} \left\| \left(u^+\right)^{\frac{p}{n}} \right\|_{L^n} \leq \varepsilon^q \frac{\sup v^{\frac{q}{2}\left(1-\frac{p}{n}\right)}}{q} + \frac{1}{\varepsilon^{q'} q} \left(\left\| \left(u^+\right)^{\frac{p}{n}} \right\|_{L^n} \right)^{q'}$$

where q' is the conjugate of q, i.e. $\frac{1}{q} + \frac{1}{q'} = 1$. We have to choose ε and q appropriately. In order to match the powers and get the L^p norm of u^+ on the right hand side we should take $q' = \frac{n}{p}$. Consequently

$$q' = \frac{n}{p} \Rightarrow q = \frac{1}{1 - \dfrac{1}{q'}} = \frac{1}{1 - \dfrac{p}{n}} = \frac{n}{n-p}.$$

Thus returning to $\sup v^{\frac{1}{2}}$ we conclude that

$$\sup_{B_1} v^{\frac{1}{2}} \leq C\left\{\varepsilon^{\frac{n}{n-p}}\frac{n-p}{n}\sup_{B1} v^{\frac{1}{2}} + \frac{1}{\varepsilon^{\frac{n}{p}}}\frac{p}{n}\left\|u^+\right\|_{L^p(\beta_1)} + \frac{1}{\lambda}\|f\|_{L^n}\right\}$$

or equivalently

$$\left[1 - C\frac{n-p}{n}v\varepsilon^{\frac{n}{p}}\right]\sup_{B1} v^{\frac{1}{2}} \leq C\left\{\frac{p}{n\varepsilon^n p}\left\|u^+\right\|_{L^p(B_1)} + \frac{1}{\lambda}\|f\|_{L^n}\right\}.$$

Choose ε so that

$$C\frac{n-p}{n}\varepsilon^{\frac{n}{n-p}} = \frac{1}{2}, \quad \Rightarrow \quad \varepsilon = \left[\frac{n}{2C\,(n-p)}\right]^{\frac{n-p}{n}}$$

and we finally obtain

$$\sup_{B_1} v^{\frac{1}{2}} \leq 2C\left(\frac{p}{n}\left(\frac{2C\,(n-p)}{n}\right)^{\frac{n-p}{p}}\left\|u^+\right\|_{L^p(B_1)} + \frac{1}{\lambda}\|f\|_{L^n(B_1)}\right).$$

It is left to choose $\eta \geq 1$ in B_σ to conclude the proof of the theorem. □

10.2. **Supersolutions.**

Theorem 21. *Let $u \in W^{2,n}(\Omega)$, satisfying $u \geq 0, Lu \leq 0$ in Ω, and let $B_R \subset \Omega$. Then for some p (we will later see that $p = \frac{1}{2C}$) and any $\sigma \in (0,1)$ we have the estimate*

$$\left(\frac{1}{|B_{R\sigma}|}\int_{B_{R\sigma}} u^p\right)^{\dfrac{1}{p}} \leq C \inf_{B_{R\sigma}} u$$

where $\gamma > p > 0$ and C depend only on n, σ and $\frac{\lambda}{\Lambda}$, $\gamma = \frac{\log M}{\log \delta} > 0$.

Proposition 22. *Let $u \geq 0$ be as in Theorem 21.*

(10.15) $$\text{If } |K_R(y) \cap \{u \geq 1\}| \geq \delta |K_R(y)| \text{ then } \inf_{B_{3R}} \geq M.$$

Proof. Set $w = \log\dfrac{1}{u+\varepsilon}, \varepsilon > 0$ and normalize $R = 1, x \in B_1$.

$$\begin{aligned} Dw &= -\frac{Du}{u+\varepsilon}, \\ D^2 w &= -\frac{D^2 u}{u+\varepsilon} + \frac{Du \otimes Du}{(u+\varepsilon)^2} = -\frac{D^2 u}{u+\varepsilon} + Dw \otimes Dw. \end{aligned}$$

From here we see that

$$Lw = -\frac{aD^2u}{u+\varepsilon} + (aDw)\,Dw = -\frac{Lu}{u+\varepsilon} + (aDw)\,Dw \geq (aDw)\,Dw.$$

Take

$$\eta = \left(1 - |x|^2\right)^{\beta}, \beta > 0,$$

By direct computation we have

$$\begin{aligned}
D\eta &= -2\beta\left(1-|x|^2\right)^{\beta-1} x,\\
D^2\eta &= -2\beta\left(1-|x|^2\right)^{\beta-1} id + 4\beta\,(\beta-1)\left(1-|x|^2\right)^{\beta-2} x\otimes x\\
&= 2\beta\left(1-|x|^2\right)^{\beta-2}\left[2\,(\beta-1)\,x\otimes x - id\left(1-|x|^2\right)\right].
\end{aligned}$$

From these computations it follows that

$$\begin{aligned}
L\eta &= 2\beta\left(1-|x|^2\right)^{\beta-2}\left[2\,(\beta-1)\,(ax)\,x - \left(1-|x|^2\right)\mathrm{Tr}a\right]\\
&\geq 2\beta\left(1-|x|^2\right)^{\beta-2}\left[2\,(\beta-1)\,\lambda\,|x|^2 - n\Lambda\left(1-|x|^2\right)\right]
\end{aligned}$$

For $|x| \geq \alpha$ we have

$$\begin{aligned}
L\eta &\geq 2\beta\left(1-|x|^2\right)^{\beta-2}\left[2\,(\beta-1)\,\lambda\,|x|^2 - n\Lambda\left(1-|x|^2\right)\right]\\
&\geq 2\beta\lambda\left(1-|x|^2\right)^{\beta-2}\left[2\,(\beta-1)\,\alpha^2 - n\frac{\Lambda}{\lambda}\left(1-\alpha^2\right)\right]\\
&= 2\beta\lambda\left(1-|x|^2\right)^{\beta-2}\left[\alpha^2\left(2\,(\beta-1) + n\frac{\Lambda}{\lambda}\right) - n\frac{\Lambda}{\lambda}\right] \geq 0
\end{aligned}$$

provided that

$$\alpha^2 \geq \frac{n\dfrac{\Lambda}{\lambda}}{2\,(\beta-1) + n\dfrac{\Lambda}{\lambda}}, \quad \text{or} \quad \beta \geq 1 + \frac{n}{2}\frac{\Lambda}{\lambda}\left(1-\alpha^2\right). \tag{10.16}$$

This computation shows that if $\alpha > 0$ is a given small number $\alpha < 1$ then we can choose $\beta \geq 2$ large enough so that

$$L\eta \geq 0 \quad \text{in} \quad B_1 \setminus B_\alpha. \tag{10.17}$$

Now we turn to

$$\begin{aligned}
L(\eta w) &= a\left(\eta D^2 w\right) + 2aD\eta Dw + waD^2\eta \\
&= \eta Lw + 2aD\eta Dw + wL\eta \\
&\geq \eta\,(aDw)\,Dw + 2aD\eta Dw + wL\eta \\
&\geq \lambda\eta\,|Dw|^2 - 2\Lambda\,|D\eta|\,|Dw| + wL\eta \\
&\geq \lambda\eta\,|Dw|^2 - 2\Lambda\left(\varepsilon\,|Dw|^2\,\eta + \frac{1}{\varepsilon}\frac{|D\eta|^2}{\eta}\right) + wL\eta \\
&= \eta\,|Dw|^2\,(\lambda - \varepsilon 2\Lambda) - \frac{2\Lambda}{\varepsilon\eta}\,|D\eta|^2 + wL\eta \\
&= \lambda\eta\,|Dw|^2\left(1 - 2\varepsilon\frac{\Lambda}{\lambda}\right) - \frac{2\Lambda}{\varepsilon\eta}\,|D\eta|^2 + wL\eta.
\end{aligned}$$

Choose

$$\varepsilon = \frac{\lambda}{2\Lambda} \tag{10.18}$$

in the last inequality to have

$$L(\eta w) \geq -4\Lambda\frac{\Lambda}{\lambda}\frac{|D\eta|^2}{\eta} + wL\eta.$$

We want to rewrite the last inequality in terms of $v := \eta w$ and η only. To do so recall that

$$|D\eta| = 2\beta\,|x|\left(1 - |x|^2\right)^{\beta-1}.$$

Using this we obtain

$$\frac{|D\eta|^2}{\eta} = \frac{4\beta^2\,|x|^2\left(1-|x|^2\right)^{2\beta-2}}{\left(1-|x|^2\right)^{\beta}} = 4\beta^2\,|x|^2\,(1-|x|^2)^{\beta-2} \leq 4\beta^2 \quad \text{if } \beta \geq 2. \tag{10.19}$$

Finally, writing $w = \dfrac{\eta w}{\eta} = \dfrac{v}{\eta}$, with $v = \eta w$ we obtain the inequality

$$Lv \geq -16\Lambda\frac{\Lambda}{\lambda}\beta^2 + \frac{v}{\eta}L\eta \tag{10.20}$$

We need to consider this inequality in $B^+ = B_1 \cap \{w > 0\}$. Utilizing the fact that

$$L\eta \geq 0 \quad \text{in} \quad B_1 \backslash B_\alpha \ \ \forall \alpha \in (0,1)$$

provided that

$$\beta \geq 1 + \frac{n}{2}\frac{\Lambda}{\lambda}\left(1 - \alpha^2\right) \quad \text{and} \quad \beta \geq 2$$

we can further estimate

$$Lv \geq -16\Lambda\beta^2\frac{\Lambda}{\lambda} - v1_{B_\alpha}\sup_{B_\alpha}\left(-\frac{L\eta}{\eta}\right) \quad \text{in} \quad B^+ \tag{10.21}$$

$$= -16\Lambda\beta^2\frac{\Lambda}{\lambda} - v1_{B_\alpha}\sup_{B_\alpha}\frac{(L\eta)^-}{\eta} \tag{10.22}$$

where 1_{B_α} is the characteristic function of the ball B_α and $(L\eta)^-$ is the negative part of $L\eta$. Now recall Aleksandrov's maximum principle

> If $u \leq 0$ on $\partial\Omega$ and Γ^+ is the upper contact set of $u \in W^{2,n}(\Omega) \cap C^0\left(\overline{\Omega}\right)$ then we have
>
> $$\sup_\Omega u \leq C \mathrm{diam}\Omega \left[\int_{\Gamma^+} \frac{(Lu)^n}{\det a}\right]^{\frac{1}{n}}$$

One may replace Γ^+ in the former estimate by $\Omega^+ = \{u > 0\}$ which yields the estimate

$$\sup_\Omega u \leq C\frac{\mathrm{diam}\Omega}{\lambda}\left[\int_{\Omega+}(Lu)^n\right]^{\frac{1}{n}} \tag{10.23}$$

$$= C\frac{\mathrm{diam}\Omega}{\lambda}\|Lu\|_{L^n(\Omega^+)}. \tag{10.24}$$

We apply this estimate to $v = \eta w$

$$\sup_{B_1} v \leq C\frac{1}{\lambda}\left\|16\beta^2\Lambda\frac{1}{\lambda} + v1_{B_\alpha}\sup_{B_\alpha}\frac{(L\eta)^-}{\eta}\right\|_{L^n(B^+)}. \tag{10.25}$$

Recall that

$$\frac{L\eta}{\eta} = \frac{2\beta\left(1-|x|^2\right)^{\beta-2}\left[2(\beta-1)(ax)x - \left(1-|x|^2\right)\mathrm{Tr}u\right]}{\eta} \tag{10.26}$$

$$\leq \frac{2\beta(2\beta-1)\Lambda}{1-\alpha^2} \tag{10.27}$$

provided

$$\beta \geq 2, \quad \text{and} \quad \beta \geq 1 + \frac{n\Lambda}{\lambda}\left(1-\alpha^2\right). \tag{10.28}$$

Therefore

$$\begin{aligned}
\sup_{B1} v &\leq C\left\|\frac{\Lambda}{\lambda}16\beta^2\frac{\Lambda}{\lambda} + v1_{B_\alpha}4\beta^2\right\|_{L^n\left(B_1^+\right)} \\
&\leq C4\beta^2\frac{\Lambda}{\lambda}\max\left(4\frac{\Lambda}{\lambda}, 1\right)\|1 + v1_{B_\alpha}\|_{L^n\left(B_1^+\right)} \\
&\leq C_0\left(1 + \left\|v^+\right\|_{L^n(B_\alpha)}\right).
\end{aligned}$$

Next we switch to cubes in order to employ cube decomposition argument. For cubes the last estimates takes the form

$$\begin{aligned}
\sup_{B_1} v &\leq C\left(1+\left|B_\alpha^+\right|^{\frac{1}{n}}\sup_{B_\alpha} v+\right)\\
&\leq C\left(1+\left|K_\alpha^+\right|^{\frac{1}{n}}\sup_{B_\alpha} v^+\right)\\
&\leq c\left(1+\left(\frac{|K_\alpha^+|}{|K_\alpha|}\right)^{\frac{1}{n}}|K|^{\frac{1}{n}}\sup_{B_1} v^+\right)\\
&\leq C_A\left(1+\theta^{\frac{1}{n}}2\alpha\sup_{B_1} v\right)
\end{aligned}$$

Implying

$$\left[1-2C_A\alpha\theta^{\frac{1}{n}}\right]\sup_{B_1} v\leq C_A \quad\Rightarrow\quad \sup_{B_1} v\leq\frac{C_A}{1-2C_A\alpha\theta^{\frac{1}{n}}},$$

where we assumed that

$$\frac{|K_\alpha^+|}{|K_\alpha|}\leq\theta, \tag{10.29}$$

Recall that here C depends on $n, \frac{\Lambda}{\lambda}$. Choose

$$\theta=\frac{1}{(4\alpha C_A)^n},$$

then

$$\frac{C_A}{1-2C_A\alpha\theta^{\frac{1}{n}}}=2C_A.$$

Summarizing we have proved the following statement (here and henceforth $\theta=\frac{1}{(4\alpha C_A)}$):

If

$$\frac{|K_\alpha^+|}{|K_\alpha|}\leq\theta \quad \text{then} \quad \sup_{B_1} v\leq 2C_A. \tag{10.30}$$

Here $0<\alpha<1$ is arbitrary and $\beta\geq 2$ must be fixed with respect to α such that all constraints imposed on β are satisfied.

Our choice is

$$\alpha=\frac{1}{3n} \tag{10.31}$$

then $\frac{|K_\alpha^+|}{|K_\alpha|}\leq\theta$ becomes $\frac{|K^+_{\frac{1}{3n}}|}{|K_{\frac{1}{3n}}|}\leq\theta$. Now upon rescaling we get

If

$$\frac{|K_R^+|}{|K_R|} \le \theta \quad \text{then} \quad \sup_{3R} w \le \sup_{B_{3nR}} v \le 2C_A. \tag{10.32}$$

in terms of u this reads

$$\text{If } |K_R(y) \cap \{u \ge 1\}| \ge \delta |K_R(y)| \quad \text{then} \quad \inf_{B_{3R}} \ge M. \tag{10.33}$$

$M = e^{-2C_A}$.

Note that $M = e^{-2CA} < 1$, because $w = \log \frac{1}{u}$ and hence

$$\{w > 0\} \Leftrightarrow \left\{\frac{1}{u} > 1\right\}, K_R^+ = K_R \cap \{u < 1\}$$

and consequently

$$|K_R \cap \{u \ge 1\}| \ge (1-\theta) |K_R(y)|$$

□

10.3. **Proof of Theorem 21.** We can reformulate our results obtained so far as

Claim 23. *If for some $\delta > 0$*

$$|K_R \cap \{u \ge 1\}| \ge \delta |K_R(y)|$$

then

$$\inf_{B_{3R}(y)} u \ge M \quad \forall K_{3R}(y) \text{ such that } K_{3nR}(y) \subset \Omega.$$

From Claim 23 we deduce

Lemma 24. *If*

$$\left|\{u \ge 1\} \cap K_{\frac{1}{3n}}\right| \ge \delta^m \left|K_{\frac{1}{3n}}\right|$$

then

$$\inf_{K_{\frac{1}{3n}}} u \ge M^m.$$

Proof. Demonstration is by induction. The base step is easy: if $m = 1$ then by Claim 23 we have

$$\left|\{u \ge 1\} \cap K_{\frac{1}{3n}}\right| \ge \delta \left|K_{\frac{1}{3n}}\right| \quad \Rightarrow \inf_{K_{\frac{1}{3n}}} \ge M$$

Inductive step $m \ge 2$. Let us prove that

$$\text{if} \quad \left|\{u \ge 1\} \cap K_{\frac{1}{3n}}\right| \ge \delta^{m+1} \left|K_{\frac{1}{3n}}\right| \quad \text{then} \quad \inf_{K_{\frac{1}{3n}}} u \ge M^{m+1}.$$

Let $Q \in \mathscr{B}$ then Claim 23 implies that $\inf_{\widetilde{Q}} u \ge M$. In particular

$$\inf_{\widetilde{\Gamma}_\delta} u = \inf_{\cup_{Q \in \mathscr{B}} \widetilde{Q}} u \ge M$$

For $\overline{u} = \dfrac{u}{M}$ we have $\left\{\overline{u} = \dfrac{u}{M} \geq 1\right\} \supset \widetilde{\Gamma_\delta}$. Moreover, from the covering Lemma 25 it follows that

$$|\widetilde{\Gamma}_\delta| \geq \frac{1}{\delta}|\Gamma| \geq \frac{1}{\delta}\delta^{m+1}\left|K_{\frac{1}{3n}}\right| = \delta^{m+1}\left|K_{\frac{1}{3n}}\right|.$$

Thus for the supersolution $\overline{u} = \frac{u}{M}$ we have that

$$\left|\{\overline{u} \geq 1\} \cap K_{\frac{1}{3n}}\right| \geq \delta^m \left|K_{\frac{1}{3n}}\right|$$

Applying inductive hypothesis we infer

$$\inf_{K_{\frac{1}{3n}}} \overline{u} \geq M^m \quad \Rightarrow \quad \inf_{K_{\frac{1}{3n}}} u \geq M^{m+1}$$

and this finishes the proof of the claim. □

From

$$\left|\Gamma \cap K_{\frac{1}{3n}}\right| \geq \delta^m \left|K_{\frac{1}{3n}}\right|$$

after taking the logarithm we get

$$\log\left|\Gamma \cap K_{\frac{1}{3n}}\right| \geq m\log\delta + \log\left|K_{\frac{1}{3n}}\right| \quad \Rightarrow \quad \log\frac{\left|\Gamma \cap K_{\frac{1}{3n}}\right|}{\left|K_{\frac{1}{3n}}\right|} \geq m\log\delta$$

Since $M < 1$ it follows

$$M^{\log\frac{\left|\Gamma\cap K_{\frac{1}{3n}}\right|}{\left|K_{\frac{1}{3n}}\right|}} \leq (M^m)^{\log\delta}$$

implying

$$\inf_{K_{\frac{1}{3n}}} u \geq M^{\frac{\log\frac{\left|\Gamma\cap K_{\frac{1}{3n}}\right|}{\left|K_{\frac{1}{3n}}\right|}}{\log\delta}} = \left(\log\frac{\left|\Gamma \cap K_{\frac{1}{3n}}\right|}{\left|K_{\frac{1}{3n}}\right|}\right)^{\frac{\log M}{\log\delta}}.$$

Note that $\gamma := \frac{\log M}{\log\delta} > 0$, hence this yields

$$\inf_{K_{\frac{1}{3n}}} u \geq \left(\frac{\left|\Gamma \cap K_{\frac{1}{3n}}\right|}{\left|K_{\frac{1}{3n}}\right|}\right)^\gamma.$$

Returning to balls

$$\inf_{B_{\frac{1}{3n}}} u \geq \inf_{K_{\frac{1}{3n}}} u \geq \left(\frac{\left|\Gamma \cap K_{\frac{1}{3n}}\right|}{\left|K_{\frac{1}{3n}}\right|}\right)^\gamma \geq c(n)\left(\frac{\left|\Gamma \cap B_{\frac{1}{3n}}\right|}{\left|B_{\frac{1}{3n}}\right|}\right)^\gamma.$$

Finally setting $\{u \geq t\} = \Gamma_t$

$$\inf_{B_{\frac{1}{3n}}} u \geq C_0 t \left(\frac{\left|\Gamma_t \cap B_{\frac{1}{3n}}\right|}{\left|B_{\frac{1}{3n}}\right|} \right)^{\gamma}$$

Set $\overline{u} = \dfrac{u}{\inf_{B_\alpha} u}$ then

$$1 \geq \frac{t}{\inf_{B_\alpha} u} c_0 \left(\frac{\left|\left\{\overline{u} \geq \dfrac{t}{\inf_{B_\alpha} u}\right\} \cap B_\alpha\right|}{|B_\alpha|} \right)^{\gamma}$$

substituting $s = \dfrac{t}{\inf_{B_\alpha} u}$ we infer

$$\left(\frac{1}{sc_0}\right)^{\dfrac{1}{\gamma}} |B_\alpha| \geq |\{\overline{u} \geq s\} \cap B_\alpha| \Rightarrow$$

$$\begin{aligned}\int_{B_\alpha} \overline{u}^p &= \int_0^{+\infty} ps^{p-1} \left|\{\overline{u} \geq s\} \cap B_\alpha\right| ds \\ &\leq \int_1^{+\infty} ps^{p-1-\dfrac{1}{\gamma}} ds \\ &= \frac{p}{p - \dfrac{1}{\gamma}}, \quad \text{if } \ p < \frac{1}{\gamma}.\end{aligned}$$

10.4. **Proof of Lemma 25.**

Lemma 25. *Let K_0 be cube in $\mathbb{R}^n$ and $\Gamma \subset K_0$ a measurable subset. Let $0 < \delta < 1$ and set*

$$\Gamma_\delta = \bigcup \{K_{3R}(y) \cap K_0 \ : \ K_R(y) \subset K_0, |K_R(y) \cap \Gamma| \geq \delta |K_R(y)|\}.$$

If $\Gamma_\delta \neq K_0$ then $|\Gamma| \leq \delta |\Gamma_\delta|$.

Proof is by Calderòn-Zygmund type argument similar to one we did in the proof of the Harnack inequality.

Proof. First suppose that

$$|K_0 \cap \Gamma| > \delta \, |K_0|$$

then it implies

$$\Gamma_\delta = K_0.$$

Therefore, we assume $\Gamma_\delta \neq K_0$ and consequently $|K_0 \cap \Gamma| \leq \delta \, |K_0|$.

Let us divide K_0 into 2^n subcubes $\{K(j_1)\}_{j_1=1}^{2^n}$. Consider two cases:

(a) $|\Gamma \cap K(j_1)| \leq \delta \, |K(j_1)|$,
(b) $|\Gamma \cap K(j_1)| > \delta \, |K(j_1)|$.

Denote

$$\mathscr{B}_1 = \{K(j_1) : \quad \text{for which case (b) is true}\}.$$

If $K(j_1) \notin \mathscr{B}$ then subdivide $K(j_1)$ into 2^n smaller congruent cubes $\{K(j_1, j_2)\}_{j_2=1}^{2^n}$. Denote by $\mathscr{B}_2$ the collection of those cubes $K(j_1, j_2)$ for which the case (b) is true. By repeating this procedure we obtain a sequence $\mathscr{B}_1, \mathscr{B}_2, \dots$ containing cubes for which (b) is true. Let

$$\mathscr{B} = \left\{K\left(j_1, j_2, \dots, j_{m-1}\right) : K\left(j_1, j_2, j_{m-1}), j_m\right) \in \mathscr{B}_m\right\}.$$

Observe that for

$$K\left(j_1, j_2, \dots, j_m\right) \in \mathscr{F}_m$$

we have

$$\left|K\left(j_1, \dots, j_m\right) \cap \Gamma\right| > \delta \left|K\left(j_1, \dots, j_m\right)\right|$$

while

$$\left|K\left(j_1, \dots, j_{m-1}\right) \cap \Gamma\right| \leq \delta \left|K\left(j_1, \dots, j_{m-1}\right)\right|. \tag{10.34}$$

Note that $K(j_1, \dots, j_{m-1}) \in \Gamma_\delta$ be definition, and, moreover

$$\bigcup_{K \in \mathscr{B}} K \subset \Gamma_\delta.$$

It follows from (10.34)

$$\begin{aligned} |\widetilde{\Gamma}_\delta \cap \Gamma| &= \left|\bigcup_{K \in \mathscr{B}} \Gamma_\delta \cap \Gamma\right| \\ &= \sum_{K \in \mathscr{B}} |K \cap \Gamma| \leq \delta \sum_{K \in \mathscr{B}} |K| \\ &\leq \delta \left|\Gamma_\delta\right|. \end{aligned}$$

It remains to apply Lebesgue's theorem to finish the proof. □

11. Pogorelov's estimate for the Monge-Ampère equation

Theorem 26. (à la Pogorelov [4, Theorem 5.1]) *Let $D \subset \mathbb{R}^d$ be a bounded convex domain, and $z \in C^4(D) \cap C^1(\overline{D})$ be a convex solution to*

$$\begin{cases} \det D^2 z = \varphi(x, z(x), Dz(x)), & \text{in } D, \\ z = 0, & \text{on } \partial D, \\ z < 0, & \text{in } D, \end{cases} \tag{11.35}$$

and $0 < \varphi \in C^3(\mathbb{R}^d \times \mathbb{R}_- \times \mathbb{R}^d)$. Then, there exists a constant $C > 0$ depending only on d, $||\varphi||_{C^3}$, $||Dz||_{L^\infty}$ such that

$$-zz_{ij} \leq C \text{ on } D.$$

Remark 27. *Since $\varphi \in C^4(D)$ it follows that $D^2 z > 0$ as $\varphi > 0$ and z is convex.*

Proof of Theorem 26. On $\overline{D} \times \mathbb{S}^{d-1}$ consider the function auxiliary function

$$w(x,\alpha) = w = -ze^{\mu\left(\frac{|Dz|^2}{2}\right)} z_{\alpha\alpha} = -z(x)e^{\mu\left(\frac{|Dz|^2}{2}\right)}(z_{ij}\alpha_i\alpha_j),$$

where μ is a smooth function to be defined below. Since $w = 0$ on ∂D and $w > 0$ in D, then there is $x_0 \in D$ and $\alpha_0 \in \mathbb{S}^{d-1}$ where w attains its absolute positive maximum. Without loss of generality we will assume that $x_0 = 0$ as otherwise we may translate the origin onto x_0 by considering the function $\tilde{z}(x) = z(x + x_0)$ in $D - x_0$. Next, by means of a rotation we may assume further that $D^2z(0)$ is diagonal with the property that $z_{11}(0) \geq z_{ij}(0)$ for all $1 \leq i, j \leq d$. Hence, we get that $\alpha_0 = e_1 = (1, 0, ..., 0) \in \mathbb{R}^d$.

We have

$$\log w = \log(-z) + \mu\left(\frac{|Dz|^2}{2}\right) + \log z_{\alpha\alpha}.$$

Differentiating this expression we obtain

$$(\log w)_i = \frac{z_i}{z} + \dot{\mu}\sum_{k=1}^{d} z_{ki}z_k + \frac{z_{\alpha\alpha i}}{z_{\alpha\alpha}}$$

and

$$(\log w)_{ii} = \frac{z_{ii}}{z} - \left(\frac{z_i}{z}\right)^2 + \ddot{\mu}\left(\sum_{k=1}^{d} z_{ki}z_k\right)^2 + \dot{\mu}\sum_{k=1}^{d} z_{kii}z_k + \dot{\mu}\sum_{k=1}^{d} z_{ki}z_{ki} + \frac{z_{\alpha\alpha ii}}{z_{\alpha\alpha}} - \left(\frac{z_{\alpha\alpha i}}{z_{\alpha\alpha}}\right)^2.$$

Since $D^2z(0)$ is diagonal and w has maximum at 0 from the last computation we get

$$0 = (\log w)_i(0) = \frac{z_i}{z} + \dot{\mu}z_{ii}z_i + \frac{z_{\alpha\alpha i}}{z_{\alpha\alpha}} = 0, \tag{11.36}$$

and

$$0 \geq (\log w)_{ii}(0) = \frac{z_{ii}}{z} - \frac{z_i^2}{z^2} + \ddot{\mu}(z_i z_{ii})^2 + \dot{\mu}\sum_{k=1}^{d} z_{kii}z_k + \dot{\mu}(z_{ii})^2 + \frac{z_{\alpha\alpha ii}}{z_{\alpha\alpha}} - \frac{z_{\alpha\alpha i}^2}{z_{\alpha\alpha}^2}. \tag{11.37}$$

Multiplying both sides of (11.37) by $\frac{z_{\alpha\alpha}}{z_{ii}}$ and summing up over $1 \leq i \leq d$ yields

$$\underbrace{\sum_{i=1}^{d} \frac{z_{\alpha\alpha ii}}{z_{ii}}}_{\text{4th order}} + \underbrace{\dot{\mu}z_{\alpha\alpha}\sum_{i=1}^{d}\sum_{k=1}^{d}\frac{z_{kii}z_k}{z_{ii}} - \sum_{i=1}^{d}\frac{z_{\alpha\alpha i}^2}{z_{ii}z_{\alpha\alpha}}}_{\text{3rd order}} + \ddot{\mu}z_{\alpha\alpha}\sum_{i=1}^{d} z_i^2 z_{ii} + \dot{\mu}z_{\alpha\alpha}\sum_{i=1}^{d} z_{ii} + \frac{z_{\alpha\alpha}}{z}d - \sum_{i=1}^{d}\frac{z_i^2}{z^2}\frac{z_{\alpha\alpha}}{z_{ii}} \leq 0. \tag{11.38}$$

We now want to eliminate the terms in (11.38) containing 3rd and 4th order derivatives of z. Differentiating the equation $\log\det D^2z = \log\varphi$ at 0 in the direction γ and using the fact that $D^2z(0)$ is diagonal, by a simple computation we obtain

$$\sum_{i=1}^{d}\frac{z_{ii\gamma\gamma}}{z_{ii}} - \sum_{i,j=1}^{d}\frac{z_{ij\gamma}^2}{z_{ii}z_{jj}} = \partial^2_{\gamma\gamma}\log\varphi, \tag{11.39}$$

$$\sum_{i=1}^{d}\frac{z_{ii\gamma}}{z_{ii}} = \partial_\gamma \log\varphi, \tag{11.40}$$

where on the right-hand sides of (11.39) and (11.40) are full derivatives of $\log\varphi$. Using (11.39) with $\gamma=\alpha$ we replace the term containing 4th order derivatives of z in (11.38), by so inferring that the remaining 3rd order term will be as follows

$$\begin{aligned}\text{3rd order} = \sum_{i,j=1}^{d}\frac{z_{ij\alpha}^2}{z_{ii}z_{jj}} - \sum_{i=1}^{d}\frac{z_{\alpha\alpha i}^2}{z_{ii}z_{\alpha\alpha}} + \dot\mu z_{\alpha\alpha}\sum_{i=1}^{d}\sum_{k=1}^{d}\frac{z_{kii}z_k}{z_{ii}} = \\ \sum_{i=1}^{d}\sum_{j\neq\alpha}\frac{z_{ij\alpha}^2}{z_{ii}z_{jj}} + \dot\mu z_{\alpha\alpha}\sum_{i=1}^{d}\sum_{k=1}^{d}\frac{z_{kii}z_k}{z_{ii}} = \\ \sum_{j\neq\alpha}\frac{z_{\alpha\alpha j}^2}{z_{\alpha\alpha}z_{jj}} + \sum_{i\neq\alpha}\sum_{j\neq\alpha}\frac{z_{ij\alpha}^2}{z_{ii}z_{jj}} + \dot\mu z_{\alpha\alpha}\sum_{i=1}^{d}\sum_{k=1}^{d}\frac{z_{kii}z_k}{z_{ii}} \overset{(11.36)}{=} \\ \sum_{j\neq\alpha}\frac{z_{\alpha\alpha}}{z_{jj}}\left(\frac{z_j}{z}+\dot\mu z_{jj}z_j\right)^2 + \sum_{i\neq\alpha}\sum_{j\neq\alpha}\frac{z_{ij\alpha}^2}{z_{ii}z_{jj}} + \dot\mu z_{\alpha\alpha}\sum_{i=1}^{d}\sum_{k=1}^{d}\frac{z_{kii}z_k}{z_{ii}} \overset{(11.40)}{=:} \\ \Sigma_1+\Sigma_2+\dot\mu z_{\alpha\alpha}\sum_{k=1}^{d}z_k\partial_k\log\varphi,\end{aligned}$$

where $\Sigma_{1,2}$ are respectively the first and the second sums of the second to the last row. Substituting these computations in (11.38) we get

$$\begin{aligned}\partial^2_{\alpha\alpha}\log\varphi + \sum_{j\neq\alpha}\frac{z_{\alpha\alpha}}{z_{jj}}\left(\frac{z_j}{z}+\dot\mu z_{jj}z_j\right)^2 + \sum_{i\neq\alpha}\sum_{j\neq\alpha}\frac{z_{ij\alpha}^2}{z_{ii}z_{jj}} + \\ \dot\mu z_{\alpha\alpha}\sum_{k=1}^{d}z_k\partial_k\log\varphi + \ddot\mu z_{\alpha\alpha}\sum_{i=1}^{d}z_i^2z_{ii} + \dot\mu z_{\alpha\alpha}\sum_{i=1}^{d}z_{ii} + \\ d\frac{z_{\alpha\alpha}}{z} - \sum_{i=1}^{d}\frac{z_i^2}{z^2}\frac{z_{\alpha\alpha}}{z_{ii}} =: I \le 0.\end{aligned}$$

A further simplification of I gives

$$\begin{aligned}0\ge I = \sum_{i\neq\alpha}\sum_{j\neq\alpha}\frac{z_{ij\alpha}^2}{z_{ii}z_{jj}} + (\dot\mu)^2\sum_{j\neq\alpha}\frac{z_{\alpha\alpha}}{z_{jj}}(z_{jj}^2)(z_j)^2 + \partial^2_{\alpha\alpha}\log\varphi + \\ 2\dot\mu z_{\alpha\alpha}\sum_{j\neq\alpha}\frac{z_j^2}{z} + \dot\mu z_{\alpha\alpha}\sum_{k=1}^{d}z_k\partial_k\log\varphi + \\ \ddot\mu z_{\alpha\alpha}\sum_{i=1}^{d}z_i^2z_{ii} + \dot\mu z_{\alpha\alpha}\sum_{i=1}^{d}z_{ii} + d\frac{z_{\alpha\alpha}}{z} - \frac{z_\alpha^2}{z^2}.\end{aligned}$$

Finally, we deal with the derivatives of $\log\varphi$. Computing the derivatives at 0 and taking into account that D^2z is diagonal at 0, we obtain

$$\partial_j\log\varphi = \partial_{x_j}\log\varphi + (\partial_z\log\varphi)z_j + \sum_{k=1}^{d}(\partial_{z_k}\log\varphi)z_{kj}, \tag{11.41}$$

$$\partial^2_{\alpha\alpha}\log\varphi = \sum_{k=1}^{d}(\partial_{z_k}\log\varphi)z_{k\alpha\alpha} + z^2_{\alpha\alpha}(\partial^2_{z_\alpha}\log\varphi) + \\ z_{\alpha\alpha}\big(\underbrace{\partial_z\log\varphi + \partial_{x_\alpha}\partial_{z_\alpha}\log\varphi + \partial_z\partial_{z_\alpha}\log\varphi}_{Q}\big) + R,$$

where in R we collect all terms containing lower order derivatives of z. We next eliminate the term containing the 3rd order derivative of z using (11.36). By doing so, we arrive at (11.42)

$$\partial^2_{\alpha\alpha}\log\varphi = z^2_{\alpha\alpha}(\partial^2_{z_\alpha}\log\varphi) - \dot{\mu}z_{\alpha\alpha}\sum_{k=1}^{d} z_k z_{kk}\partial_{z_k}\log\varphi - \frac{z_{\alpha\alpha}}{z}\sum_{k=1}^{d} z_k\partial_{z_k}\log\varphi + z_{\alpha\alpha}Q + R$$

Plugging (11.41)-(11.42) into I, after some simplifications we get

$$0 \ge I \ge -\dot{\mu}z_{\alpha\alpha}\sum_{k=1}^{d}\frac{\partial\log\varphi}{\partial z_k}z_k z_{kk} + \frac{\partial^2\log\varphi}{\partial z^2_\alpha}z^2_{\alpha\alpha} + \frac{z_{\alpha\alpha}}{z}Q' + R + 2\dot{\mu}z_{\alpha\alpha}\sum_{j\neq\alpha}\frac{z_j^2}{z} + \\ \dot{\mu}z_{\alpha\alpha}\sum_{k=1}^{d} z_k\frac{\partial\log\varphi}{\partial z_k}z_{kk} + \ddot{\mu}z_{\alpha\alpha}\sum_{i=1}^{d} z_i^2 z_{ii} + \dot{\mu}z_{\alpha\alpha}\sum_{i=1}^{d} z_{ii} + d\frac{z_{\alpha\alpha}}{z} - \frac{z_\alpha^2}{z^2} = \\ z_{\alpha\alpha}\sum_{i=1}^{d}\ddot{\mu}z_i^2 z_{ii} + \dot{\mu}z_{\alpha\alpha}\sum_{i=1}^{d} z_{ii} + \frac{\partial^2\log\varphi}{\partial z^2_\alpha}z^2_{\alpha\alpha} + \\ \frac{z_{\alpha\alpha}}{z}\underbrace{(Q'+d)}_{Q''} + 2\dot{\mu}z_{\alpha\alpha}\sum_{j\neq\alpha}\frac{z_j^2}{z} + \underbrace{\left(R - \frac{z_\alpha}{z^2}\right)}_{\frac{s_0}{z^2}}.$$

Now choose $\mu(t) = e^{ct}$ with $c > 0$. Then $\dot{\mu} = c\mu$ and $\ddot{\mu} = c^2\mu$. Therefore

$$0 \ge z_{\alpha\alpha}c^2\mu\sum_{i=1}^{d} z_i^2 z_{ii} + c\mu z_{\alpha\alpha}\sum_{i=1}^{d} z_{ii} + \frac{\partial^2\log\varphi}{z^2_\alpha}z^2_{\alpha\alpha} + z_{\alpha\alpha}zQ'' + 2c\mu z_{\alpha\alpha}\sum_{j\neq\alpha}\frac{z_j^2}{z} + \frac{s_0}{z^2} = \\ \mu c\sum_{i=1}^{d} z_{\alpha\alpha}(cz_i^2 z_{ii} + z_{ii}) + \frac{\partial^2\log\varphi}{\partial z^2_\alpha}z^2_{\alpha\alpha} + \frac{z_{\alpha\alpha}}{z}Q'' + 2c\mu z_{\alpha\alpha}\sum_{j\neq\alpha}\frac{z_j^2}{z} + \frac{S_0}{z^2} = \\ \mu c z_{\alpha\alpha}\underbrace{\sum_{i=1}^{d} z_{ii}(cz_i^2+1)}_{\ge z_{\alpha\alpha}} + \frac{\partial^2\log\varphi}{\partial z^2_\alpha}(z_{\alpha\alpha})^2 + \frac{z_{\alpha\alpha}}{z}Q'' + 2c\mu z_{\alpha\alpha}\sum_{j\neq\alpha}\frac{z_j^2}{z} + \frac{S_0}{z^2} \ge \\ \left(\mu c + \frac{\partial^2\log\varphi}{\partial z^2_\alpha}\right)z^2_{\alpha\alpha} + \frac{z_{\alpha\alpha}}{z}\underbrace{\left(Q'' + 2ce^{c\frac{|Dz|^2}{2}}\sum_{j\neq\alpha} z_j^2\right)}_{S} + \frac{S_0}{z^2} \ge \\ \left(c - \sup\left|\frac{\partial^2\log\varphi}{\partial z^2_\alpha}\right|\right)z^2_{\alpha\alpha} + S\frac{z_{\alpha\alpha}}{z} + \frac{S_0}{z^2}.$$

Choosing $c \gg \sup \left|\frac{\partial^2 \log \varphi}{\partial z_\alpha^2}\right|$ and multiplying both sides by z^2 we get

$$cz^2 z_{\alpha\alpha}^2 + Szz_{\alpha\alpha} + S_0 \leq 0,$$

hence

$$zz_{\alpha\alpha} \leq \frac{-s + \sqrt{S^2 - 4cS_0}}{2c} := C_1,$$

where the left-hand side is evaluated at 0, and $C_1 = C_1(d, ||z||_{L^\infty}, ||Dz||_{L^\infty}, ||\varphi||_{C^3})$. Since w attains it maximum at 0, from the last estimate we get $w(x) \leq C_1$ for all $x \in D$, hence

$$-z(x)z_{\alpha\alpha}(x) \leq C_1, \qquad x \in D,\ \alpha \in \mathbb{S}^{d-1}.$$

The proof is complete.

□

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