Estimates for Second Order Elliptic Partial Differential Equations

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Course Outline

I. Introduction
II. Preliminaries
III. Maximum principle methods
IV. L^2 estimates
V. Bounds & Hölder continuity for equations in divergence form (de Giorgi – Moser – Stampacchia)
VI. L^p estimates (Calderon – Zygmund)
VII. $C^{2,\alpha}$ estimates (Schauder)

Main references

[L-U] O.A. Ladyženskaja & N.N. Ural'ceva, Linear & Quasilinear Elliptic Equations, Academic Press

[S] G. Stampacchia, Equations Elliptiques du Second Ordre à Coefficients Discontinus, U. of Montreal Press

[G-T] D. Gilbarg & N. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer

[ST] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton U. Press

Inspiring Quotes

"Estimating is the central [illegible] differential equations: it serves to justify both abstract existence proofs and numerical computations. Estimates, as often presented in a string of lemmas, may look singularly unattractive, lacking the elegance of giving the best constants and merely concerned with orders of magnitude. They do, however, express deep truths and lead to results not easily obtained by algebraic manipulations of the differential operators. The most complete estimates exist for differential operators.. of the type called elliptic..." – Fritz John, (AMS Bulletin, Nov. 1972)

"..in the right hands, Schwarz's inequality and integration by parts are still among the most powerful tools of analysis." – Mark Kac, (Quart. Appl. Math., 1972)

"God is in the details" – Walter Gropius (?)

"Τείνω να πιστεύω ότι δεν υπάρχουν ρηχοί [illegible] μαθηματικά και έχουν κατά κάποιο τρόπο [illegible]
Ανακάλυψα όπως υπάρχουν ρηχοί [illegible] και έχουν κατά κάποιο τρόπο [illegible]"

I. Introduction

1.1 Statement of the basic problem

Ω = bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, with smooth boundary $\partial\Omega$

Assume $u : \overline{\Omega} \to \mathbb{R}$ solves

$$(\star)\begin{cases} -\displaystyle\sum_{i,j}^{n} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i}(x) + c(x)u(x) = f(x) & x \in \Omega \\ u(x) = \phi(x) & x \in \partial\Omega, \end{cases}$$

where $a_{ij}, b_i, c, f,$ and ϕ are given.

$\boxed{\text{[illegible]: Given various assumptions as to the properties of } a_{ij}, b_i, c, f, \text{ and } \phi, \text{ what can be said about the boundedness and smoothness of } u \text{ and its derivatives? More precisely, what } \underline{a\ priori} \text{ estimates on } u \text{ can be obtained in terms of known properties of } a_{ij}, b_i, \text{ etc.??}}$

A priori = "from before" (Latin)

Notation (a) $u_{x_i} \equiv \dfrac{\partial u}{\partial x_i}$, $u_{x_i x_j} \equiv \dfrac{\partial^2 u}{\partial x_i \partial x_j}$

(b) Summation Convention Any subscript occurring twice in a term is assumed to be summed from 1 to n (although the "Σ" sign is not written)

(c) Arguments of functions are often omitted.

By (a)-(c), (*) is rewritten:

$$\boxed{(*)\ \begin{cases} -a_{ij}u_{x_i x_j} + b_i u_{x_i} + cu = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}}$$

Ellipticity assumption

$$\boxed{(E)\ \begin{cases} \exists \text{ real numbers } \Theta \geq \theta > 0 \text{ such that} \\ \theta|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Theta|\xi|^2 \quad \text{for all } x \in \bar{\Omega} \\ \text{and all } \xi = (\xi_1, \cdots \xi_n) \in \mathbb{R}^n \end{cases}}$$

θ = constant of ellipticity

$A(x) = ((a_{ij}(x)))$ = matrix whose $(i,j)^{th}$ component is the function $a_{ij}(x)$

Unless otherwise noted, we henceforth assume

$$\boxed{a_{ij}(x) = a_{ji}(x)} \quad \forall x \in \bar{\Omega},\ \forall i,j = 1, \cdots n;$$

so the matrix $A(x)$ is symmetric for all $x \in \bar{\Omega}$.

[1.1] Lemma 1.1 Assume condition (E). Then

(1) $\Theta \geq a_{ii}(x) \geq \theta > 0 \quad \forall x \in \bar{\Omega},\ i = 1, 2, \cdots n.$

(2) $a_{ij}^2(x) \leq a_{ii}(x)a_{jj}(x) \quad \forall x \in \bar{\Omega},\ i,j = 1, \cdots n$

(3) at each point $x_0 \in \bar{\Omega}$, $\exists$ an $n \times n$ real matrix $D = D_{x_0}$ such that

$$DD^T = D^T D = I$$

and

$$DA(x_0)D^T = \operatorname{diag}(\lambda_1, \cdots, \lambda_n) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

where $\lambda_i \geq \theta \quad i = 1, 2, \cdots n$

Proof (1) Put $\xi = (0, 0, \cdots 1, \cdots 0)$ in condition (E).
$\qquad\qquad\qquad\qquad\qquad\ \ i^{th}$ slot

(2) Let $\lambda > 0$ (to be selected) & put $\xi = (0, \cdots, \underset{j^{th}\text{ slot}}{\lambda}, 0, \cdots \underset{i^{th}\text{ slot}}{\pm 1}, \cdots 0)$ into condition (E):

$$0 \leq a_{kl}\xi_k\xi_l = \lambda^2 a_{jj} + a_{ii} \pm 2\lambda a_{ij}$$

$$\therefore \quad \pm a_{ij} \leq \frac{\lambda}{2}a_{jj} + \frac{a_{ii}}{2\lambda}.$$

Let $\lambda = \left(\dfrac{a_{ii}}{a_{jj}}\right)^{1/2}$ to get $\quad \pm a_{ij} \leq (a_{ii})^{1/2}(a_{jj})^{1/2}$

(3) Standard fact from linear algebra: the matrix $A(x_0)$ is symmetric & positive definite. ||

I. Notation and definitions

(1) ∇u = gradient of $u = (u_{x_1}, u_{x_2}, \cdots u_{x_n})$

(2) Δu = Laplacian of $u = u_{x_i x_i} = u_{x_1x_1} + u_{x_2x_2} + \cdots + u_{x_nx_n}$

(3) L^p and Sobolev norms

$$\|u\|_{L^p(\Omega)} = \left(\int_\Omega |u|^p\,dx\right)^{1/p} \qquad 1 \le p < \infty$$

$$\|u\|_{W^{1,p}(\Omega)} = \left(\int_\Omega |u|^p\,dx + \sum_{i=1}^n \int_\Omega |u_{x_i}|^p\,dx\right)^{1/p}$$

$$\|u\|_{W^{2,p}(\Omega)} = \left(\int_\Omega |u|^p\,dx + \sum_{i=1}^n \int_\Omega |u_{x_i}|^p\,dx + \sum_{i,j=1}^n \int_\Omega |u_{x_ix_j}|^p\,dx\right)^{1/p}$$

The norms $\|\ \|_{L^\infty}$, $\|\ \|_{W^{1,\infty}}$, and $\|\ \|_{W^{2,\infty}}$ are defined similarly.

(4) Hölder norms

$$[u]_\alpha = \sup_{\substack{x,y\in\Omega \\ x\ne y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha} \qquad 0 < \alpha \le 1$$

$$\|u\|_{C^\alpha(\bar\Omega)} = \sup_\Omega |u| + [u]_\alpha$$

$$\|u\|_{C^{1,\alpha}(\bar\Omega)} = \sup_\Omega |u| + \sum_{i=1}^n \left(\sup_\Omega |u_{x_i}| + [u_{x_i}]_\alpha\right)$$

$$\|u\|_{C^{2,\alpha}(\bar\Omega)} = \sup_\Omega |u| + \sum_{i=1}^n \sup_\Omega |u_{x_i}| + \sum_{i,j=1}^n \left(\sup_\Omega |u_{x_ix_j}| + [u_{x_ix_j}]_\alpha\right)$$

(5) Sometimes in the literature $W^{1,2}(\Omega)$ and $W^{2,2}(\Omega)$ are written as "$H^1(\Omega)$" and "$H^2(\Omega)$"

II. Preliminaries

2.1 A. Elementary inequalities

(1) Cauchy's inequality

$$\boxed{ab \le \frac{a^2}{2} + \frac{b^2}{2}} \qquad \forall a,b \ge 0$$

Proof $0 \le (a-b)^2 = a^2 + b^2 - 2ab$ //

(2) Cauchy's inequality with ε

$$\boxed{ab \le \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon}} \qquad \forall a,b \ge 0,\ \forall \varepsilon > 0$$

Proof $ab = (\sqrt{\varepsilon}a)\left(\frac{b}{\sqrt{\varepsilon}}\right) \le \frac{\varepsilon a^2}{2} + \frac{b^2}{2\varepsilon}$ //

(3) Jensen's inequality Let $\phi : \mathbb{R} \to \mathbb{R}$ be convex. Then

$$\boxed{\phi\left(\frac{1}{b-a}\int_a^b f(t)\,dt\right) \le \frac{1}{b-a}\int_a^b \phi(f(t))\,dt}$$

$\forall$ finite interval $[a,b]$ & $\forall$ integrable function $f : [a,b] \to \mathbb{R}$

Proof ✓ Set $\alpha = \frac{1}{b-a}\int_a^b f(t)\,dt$ & let

$y = m(x-\alpha) + \phi(\alpha)$ be a supporting line for ϕ at α.

(ie $\phi(x) \ge m(x-\alpha) + \phi(\alpha)\ \ \forall x$). Put $x = f(t)$ & integrate over $[a,b]$:

$$\int_a^b \phi(f(t))\,dt \ge m \underbrace{\int_a^b (f(t)-\alpha)\,dt}_{=0} + (b-a)\phi(\alpha)$$

//

(7)

(4) Young's inequality

$$\boxed{ab \le \frac{a^p}{p} + \frac{b^q}{q}} \qquad \forall a,b \ge 0,\ 1<p,q<\infty,\ \frac{1}{p}+\frac{1}{q}=1$$

Proof Define, for $a,b>0$,

$$f(t) = \begin{cases} p \log a & 0 \le t \le 1/p \\ q \log b & 1/p < t \le 1 \end{cases}$$

and apply Jensen's inequality over $[0,1]$, with $\phi(x)=e^x$:

$$\underbrace{\phi\left(\int_0^1 f(t)\,dt\right)}_{=ab} \le \underbrace{\int_0^1 \phi(f(t))\,dt}_{=\frac{a^p}{p}+\frac{b^q}{q}} \qquad ||$$

(5) Young's inequality with ε

$$\boxed{ab \le \frac{\varepsilon a^p}{p} + \frac{\varepsilon^{-q/p} b^q}{q}} \qquad \forall a,b\ge 0,\ \forall \varepsilon>0,\ \frac{1}{p}+\frac{1}{q}=1$$

Proof $ab = (\varepsilon^{1/p}a)\left(\frac{b}{\varepsilon^{1/p}}\right) \le \frac{\varepsilon a^p}{p} + \frac{\varepsilon^{-q/p} b^q}{q}$ $||$

(6) Hölder's inequality

$$\boxed{\int_\Omega |uv|\,dx \le \|u\|_{L^p}\|v\|_{L^q}} \qquad 1\le p,q\le\infty,\ \frac{1}{p}+\frac{1}{q}=1$$

Proof By (5)

$$\int_\Omega |uv|\,dx \le \frac{\varepsilon}{p}\int_\Omega |u|^p\,dx + \frac{\varepsilon^{-q/p}}{q}\int_\Omega |v|^q\,dx;$$

choose $\varepsilon = \frac{\|v\|_{L^q}^{q}}{\|u\|_{L^p}^{p-1}}$ $||$ (This value of ε minimizes the R.H.S.)

Rk: So if we have a multiplicative inequality we obtain an additive one by applying Young. Conversely, given an additive we may minimize over ε to obtain a multiplicative.

(8)

(7) General Hölder's inequality

$$\boxed{\int_\Omega |u_1 u_2 \cdots u_k|\,dx \le \|u_1\|_{L^{p_1}}\|u_2\|_{L^{p_2}}\cdots\|u_k\|_{L^{p_k}}} \qquad \frac{1}{p_1}+\frac{1}{p_2}+\dots+\frac{1}{p_k}=1$$

Proof Induction $||$

(8) Inequality of geometric & arithmetic means

$$\boxed{(a_1 a_2\cdots a_k)^{1/k} \le \frac{a_1+a_2+\dots+a_k}{k}} \qquad \forall a_1,a_2,\dots a_k\ge 0$$

Proof Induction, using Young's inequality $||$

(9) Interpolation inequality for L^p spaces

$$\boxed{\|u\|_{L^r} \le \|u\|_{L^s}^{\alpha}\|u\|_{L^t}^{1-\alpha}} \qquad s\le r\le t,\ \frac{1}{r}=\frac{\alpha}{s}+\frac{(1-\alpha)}{t}$$

Proof

$$\int_\Omega |u|^r\,dx = \int_\Omega |u|^{\alpha r+(1-\alpha)r}\,dx$$

$$\le \left(\int_\Omega |u|^{\alpha r\cdot\frac{s}{\alpha r}}dx\right)^{\frac{\alpha r}{s}}\left(\int_\Omega |u|^{(1-\alpha)r\cdot\frac{t}{(1-\alpha)r}}dx\right)^{\frac{(1-\alpha)r}{t}}$$

$\left(\frac{\alpha r}{s}+\frac{(1-\alpha)r}{t}=1\right)$

$$= \|u\|_{L^s}^{\alpha r}\|u\|_{L^t}^{(1-\alpha)r} \qquad ||$$

References:

Inequalities by Hardy, Littlewood, & Polya, Oxford
Inequalities by Bellman and Beckenbach, Springer

B. Integration by parts

1.2

At each point $x \in \partial\Omega$, let $n(x) = (n_1(x), \dots, n_n(x))$ denote the outward unit normal.

Then for each $i = 1, 2, \dots n$ and smooth functions f and g:

$$\boxed{\int_\Omega f_{x_i}(x)\, g(x)\, dx = -\int_\Omega f(x)\, g_{x_i}(x)\, dx + \int_{\partial\Omega} f(x) g(x)\, n_i(x)\, dS_x}$$

Notation
$$\frac{\partial u}{\partial n}(x) = \nabla u(x)\cdot n(x) = u_{x_i}(x) n_i(x) \qquad x \in \partial\Omega$$
= outward normal derivative at $x \in \partial\Omega$

Some applications of integration by parts

Lemma 2.1 (Green's identity)

$$\boxed{\int_\Omega u\,\Delta u\, dx = -\int_\Omega |\nabla u|^2 dx + \int_{\partial\Omega} u \frac{\partial u}{\partial n}\, dS}$$

Proof Fix $1 \le i \le n$ & set $f = u_{x_i}$, $g = u$ into the formula above.
Sum over i. ||

Lemma 2.2 (Interpolation inequality) Suppose that $2 \le p < \infty$ and that u is a smooth function, $u = 0$ on $\partial\Omega$. Then

$$\boxed{\left(\sum_{i=1}^n \int_\Omega |u_{x_i}|^p dx\right)^{2/p} \le C\left(\int_\Omega |u|^s dx\right)^{1/s} \left(\sum_{i,j=1}^n \int_\Omega |u_{x_i x_j}|^r dx\right)^{1/r},}$$

where C is a constant depending only on p. $\frac{2}{p} = \frac{1}{r} + \frac{1}{s}$

(*) $\sum a_i^{\frac{2}{p}} \ge \left(\sum a_i\right)^{\frac{2}{p}}$ since $p \ge 2$. In general if
$0 < p < q$, $0 \le a_i \Rightarrow \left(\sum_{i=1}^n a_i^q\right)^{1/q} \le \left(\sum_{i=1}^n a_i^p\right)^{1/p}$

Proof Fix some $1 \le i \le n$. Then

$$\int_\Omega |u_{x_i}|^p dx = \int_\Omega |u_{x_i}|^{p-1} (\operatorname{sgn} u_{x_i})\, u_{x_i}\, dx$$

$p \ge 2$ used here for sgn x

$$= -(p-1)\int_\Omega |u_{x_i}|^{p-2} u_{x_i x_i} u\, dx \qquad \text{by (1)}$$

$$\le C\left(\int_\Omega |u_{x_i}|^{p-2\cdot\frac{p}{p-2}} dx\right)^{\frac{p-2}{p}} \left(\int_\Omega |u_{x_i x_i}|^r dx\right)^{1/r} \left(\int_\Omega |u|^s dx\right)^{1/s}$$

by the general Hölder inequality $\left(\frac{p-2}{p} + \frac{1}{r} + \frac{1}{s} = 1\right)$. Therefore

$$\left(\int_\Omega |u_{x_i}|^p dx\right)^{2/p} \le C\left(\int_\Omega |u_{x_i x_i}|^r dx\right)^{1/r} \left(\int_\Omega |u|^s dx\right)^{1/s};$$

sum over $i = 1, 2, \dots n$. (*) ||

Corollary 2.3 Suppose $u = 0$ on $\partial\Omega$ and $2 \le p < \infty$. Then for each $\varepsilon > 0$, $\exists$ a constant C_ε depending only on ε and p, such that

$$\boxed{\left(\sum_{i=1}^n \int_\Omega |u_{x_i}|^p dx\right)^{1/p} \le \varepsilon\left(\sum_{i,j=1}^n \int_\Omega |u_{x_i x_j}|^p dx\right)^{1/p} + C_\varepsilon\left(\int_\Omega |u|^p dx\right)^{1/p}}$$

Remark This inequality says that the "intermediate" derivatives u_{x_i} can be estimated by an arbitrarily small number ε times the "higher" derivatives $u_{x_i x_j}$, plus a possibly large constant C_ε times the "lower order" L^p norm of u.

This estimate will be used to justify the heuristic principle that only the highest order derivatives in equation (1) (see p. 3) play a decisive role: the lower order terms can be estimated in terms of these higher derivatives, and hence never cause difficulties.

Proof By the preceding lemma, $(r = s = p)$

$$\left(\sum_{i=1}^n \int |u_{x_i}|^p dx\right)^{1/p} \le C\left(\int_\Omega |u|^p dx\right)^{1/2p} \left(\sum_{i,j=1}^n \int_\Omega |u_{x_i x_j}|^p dx\right)^{1/2p}$$

$$\le \varepsilon \left(\sum_{i,j=1}^{n} \int_\Lambda |u_{x_i x_j}|^p \, dx\right)^{1/p} + \underbrace{\frac{C}{\varepsilon}}_{\equiv C_\varepsilon} \left(\int_\Lambda |u|^p \, dx\right)^{1/p},$$

by Cauchy's inequality with ε. ||

23 C. Sobolev's Inequality

In General: $W^{k,p} \hookrightarrow L^q$

$\frac{1}{p} \ge \frac{1}{q} \ge \frac{1}{p} - \frac{k}{n} > 0$

This true for Ω open + smooth $\partial\Omega$, not necessarily bounded.

Definition Let $1 \le p < n$. The number

$$\boxed{p^* = \frac{pn}{n-p}}$$

is called the Sobolev conjugate of p. Notice

$$p^* > p \quad \text{and} \quad \boxed{\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}}$$

Theorem 23 Suppose $u : \mathbb{R}^n \to \mathbb{R}$ has compact support. Then for $1 \le p < n$,

$$\boxed{\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{1/p^*} \le C \sum_{i=1}^{n} \left(\int_{\mathbb{R}^n} |u_{x_i}|^p dx\right)^{1/p}},$$

for a constant C depending only on p & n.

Proof First assume $p = 1$. We claim

(1) $$\|u\|_{L^{n/n-1}} \le \prod_{i=1}^{n} \|u_{x_i}\|_{L^1}^{1/n}$$

For each fixed $i = 1, 2, \dots n$,

Actually one can do a little bit better.

$$|u(x)| = \left|\int_{-\infty}^{x_i} u_{x_i}(x_1, x_2, \dots, \overset{i\text{th slot}}{\xi_i}, \dots x_n) \, d\xi_i\right|$$

$$\le \int_{-\infty}^{\infty} |u_{x_i}(x_1, x_2, \dots \xi_i \dots x_n)| \, d\xi_i$$

$x = (x_1, x_2, \dots x_n)$

$|u(x)| \le \int_{-\infty}^{x_i} |u_{x_i}(x_1, \dots \xi_i \dots x_n)|$

$|u(x)| \le \int_{x_i}^{\infty} |u_{x_i}(x_1, \dots \xi_i \dots x_n)|$

$|u(x)| \le \frac{1}{2} \int |u_{x_i}|$

Hence

$$|u(x)|^{n/n-1} \le \left(\prod_{i=1}^{n} \int_{-\infty}^{\infty} |u_{x_i}| \, d\xi_i\right)^{1/n-1}$$

Now integrate both sides from $-\infty$ to ∞ with respect to x_1, then x_2, $\dots$ x_n. & Notice that during each integration one of the n terms on the right hand side is independent of the variable being integrated & can be pulled out of the integral. The remaining $n-1$ terms are estimated by the general Hölder inequality (p. 5) with $p_1 = p_2 = \dots = p_{n-1} = n-1$.

This yields

$$\int_{\mathbb{R}^n} |u(x)|^{n/n-1} dx \le \prod_{i=1}^{n} \left(\int_{\mathbb{R}^n} |u_{x_i}| \, dx\right)^{1/n-1} = \left[\prod \left(\int |u_{x_i}| \, dx\right)^{1/n}\right]^{n/n-1}$$

& this gives (1).

From (1) & the inequality of the geometric & arithmetic mean (p. 5), we get

(2) $$\left(\int_{\mathbb{R}^n} |u|^{n/n-1} dx\right)^{n-1/n} \le \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^n} |u_{x_i}| \, dx$$

this proves the theorem for $p = 1$.

In the general case, plug $v = |u|^\gamma$ (γ to be selected) into (2): ($\gamma > 1$)

$$\left(\int_{\mathbb{R}^n} |u|^{\gamma n/n-1} dx\right)^{n-1/n} \le \frac{1}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^n} |(|u|^\gamma)_{x_i}| \, dx$$

(3) $$= \frac{\gamma}{n} \sum_{i=1}^{n} \int_{\mathbb{R}^n} |u|^{\gamma-1} |u_{x_i}| \, dx$$

$$\le \frac{\gamma}{n} \left(\int_{\mathbb{R}^n} |u|^{(\gamma-1)\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \sum_{i=1}^{n} \left(\int_{\mathbb{R}^n} |u_{x_i}|^p dx\right)^{1/p}$$

Now select γ so that (13)

$$\frac{\gamma n}{n-1} = (\gamma-1)\frac{p}{p-1}, \quad \text{i.e. } \gamma = \frac{(n-1)p}{n-p};$$

in which case

$$\frac{\gamma n}{n-1} = \frac{(\gamma-1)p}{p-1} = \frac{pn}{n-p} = p^*.$$

So (3) gives

$$\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{\overbrace{1/p-1/n}^{1/p^*}} \le C \sum_{i=1}^n \left(\int_{\mathbb{R}^n} |u_{x_i}|^p dx\right)^{1/p} \qquad (*)$$

Remark: For $p>n$, we have the estimate (In General: $W^{k,p} \hookrightarrow C^\gamma$ if $0 \le \gamma < k - \frac{n}{p}$)

$$\|u\|_{C^\alpha(\mathbb{R}^n)} \le C\left(\sum_{i=1}^n \left(\int_{\mathbb{R}^n} |u_{x_i}|^p dx\right)^{1/p} + \left(\int_{\mathbb{R}^n} |u|^p dx\right)^{1/p}\right)$$

when $\alpha = 1 - n/p$. See, for example [G-T, p. 148 & 156].
For the case $p = n$, see [G-T, p. 155]. (In General: $W^{k,p} \hookrightarrow L^r \ \forall r < +\infty$ if $kp = n$)

7.4] D. Trace estimates

Question: If you know that u and its derivatives are in some L^p space over all of Ω, what can you say about the restriction of u to $\partial\Omega$??

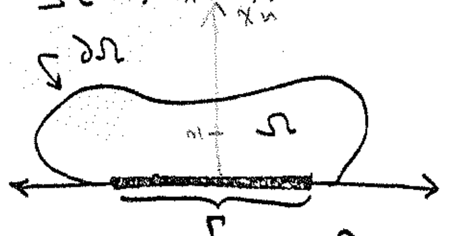
Theorem 25 For every $\varepsilon > 0$, $\exists\, C_\varepsilon$, depending only on ε and Ω, such that

$$\boxed{\left(\int_{\partial\Omega} |u|^2 ds\right)^{1/2} \le \varepsilon\left(\sum_{i=1}^n \int_\Omega |u_{x_i}|^2 dx\right)^{1/2} + C_\varepsilon\left(\int_\Omega |u|^2 dx\right)^{1/2}}$$

Notice this integral is over $\partial\Omega$.

In general: $W^{m,p}(\Omega) \hookrightarrow W^{m-\frac{1}{2},p}(\partial\Omega)$ (Restriction decreases differentiability by $1/2$).

Proof (sketch) Consider first the case that a portion $\Gamma \subset \partial\Omega$ lies in the plane $x_n = 0$, with $\Omega \subset \{x_n > 0\}$. (14)



Let $\eta > 0$ be given. Pick some "cut off" function $\zeta(x) \in C^\infty$, such that

$$\begin{cases} 0 \le \zeta \le 1, \quad \zeta = 1 \text{ on } \Gamma, \quad \zeta = 0 \text{ near } \partial\Omega \setminus \Gamma \cap \{x_n > 0\}, \\ \zeta(x_1, \dots x_n) = 0 \text{ for } x_n > \eta, \quad |\nabla\zeta| \le \frac{2}{\eta} \end{cases}$$

Then for a point $x = (x_1, x_2, \dots x_{n-1}, 0) \in \Gamma$,

$$\begin{aligned} u(x_1, \dots x_{n-1}, 0) &= u(x_1, \dots x_{n-1}, 0)\zeta(x_1, \dots x_{n-1}, 0) \\ &= -\int_0^\eta \big(u(x_1, \dots \xi)\zeta(x_1, \dots \xi)\big)_{x_n} d\xi; \end{aligned}$$

and so

$$|u(x_1, \dots x_{n-1}, 0)| \le \frac{2}{\eta}\int_0^\eta |u(x_1, \dots x_{n-1}, \xi)|\, d\xi + \int_0^\eta |u_{x_n}(x_1, \dots \xi)|\, d\xi. \checkmark$$

Now square both sides, recall $(a+b)^2 \le 2a^2 + 2b^2$, and use Hölder's inequality:

$$|u(x_1, \dots x_{n-1}, 0)|^2 \le \frac{24}{\eta}\int_0^\eta |u(x_1, \dots x_{n-1}, \xi)|^2 d\xi + 2\eta \int_0^\eta |u_{x_n}(x_1, \dots x_{n-1}, \xi)|^2 d\xi$$

Next integrate wrt $x_1, \dots x_{n-1}$ over Γ:

$$(4) \quad \int_\Gamma |u(x_1, \dots x_{n-1}, 0)|^2 dx_1 \cdots dx_{n-1} \le C(\eta)\int_\Omega u^2 dx + 2\eta \int_\Omega |u_{x_n}|^2 dx$$

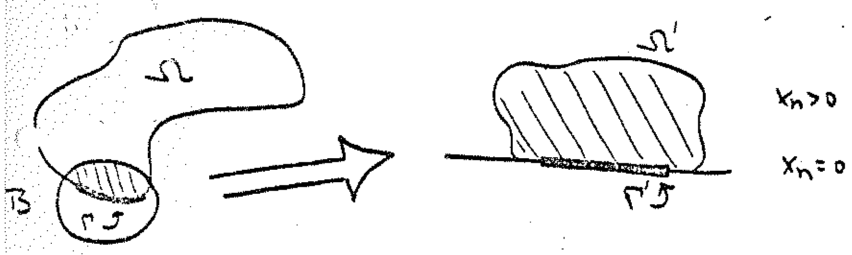
In the general case, we cover $\partial\Omega$ by a finite number of balls B and then map each $B \cap \partial\Omega$ smoothly into $\{x_n = 0\}$, $B \cap \Omega$ into $\{x_n > 0\}$.

(*) Exercise: (Gagliardo–Nirenberg inequality)

Show that $\|f\|_{L^2} \le \|f\|_{L^1}^{\frac{2}{N+2}} \|\nabla f\|_{L^2}^{\frac{N}{N+2}} (C_N)^{N/N+2}$ where $f \in C_c^\infty(\mathbb{R}^N)$, $C_N = \frac{1}{\sqrt{N}}\left(\frac{N-1}{N-2}\right)$

Hint: Use convexity of $\psi(s) = \log \|f\|_{L^s}^s$ and then Theorem 2.4.

(15)



We transform u into a function defined on Ω' & apply inequality (4). Back in the original domain Ω, this gives

$$\|u\|^2_{L^2(\Gamma)} \leq C(\eta)\|u\|^2_{L^2(\Omega)} + \eta C \|\nabla u\|^2_{L^2(\Omega)}.$$

We obtain this inequality for each of the finitely many pieces Γ into which $\partial\Omega$ is subdivided. Adding up the corresponding estimates gives:

$$\|u\|^2_{L^2(\partial\Omega)} \leq C(\eta)\|u\|^2_{L^2(\Omega)} + \eta C \|\nabla u\|^2_{L^2(\Omega)}.$$

Take $\sqrt{\ }$ of both sides, recall that $(a+b)^{1/2} \leq a^{1/2} + b^{1/2}$, and choose $\eta C = \varepsilon$ to finish the proof. ||

Corollary: For every $\varepsilon > 0$, $\exists\, C_\varepsilon$, depending only on ε and Ω, such that (*)

$$\boxed{\left(\int_{\partial\Omega}\left(\frac{\partial u}{\partial n}\right)^2 ds\right)^{1/2} \leq \varepsilon\left(\sum_{i,j=1}^n \int_\Omega u_{x_ix_j}^2\, dx\right)^{1/2} + C_\varepsilon\left(\int_\Omega u^2\, dx\right)^{1/2}}$$

Proof $\frac{\partial u}{\partial n}(x) = u_{x_i}(x)\, n_i(x)$, where $n(x) = (n_1(x), \cdots n_n(x))$ is the outward unit normal at $x \in \partial\Omega$. By the theorem applied to each u_{x_i}, we get

$$\left\|\frac{\partial u}{\partial n}\right\|^2_{L^2(\partial\Omega)} \leq C\sum_{i=1}^n \|u_{x_i}\|^2_{L^2(\partial\Omega)} \leq \varepsilon \sum_{i,j=1}^n \|u_{x_ix_j}\|^2_{L^2(\Omega)} + C_\varepsilon \|\nabla u\|^2_{L^2(\Omega)}. \checkmark$$

(*) Actually a little bit more is true:
if $u(x) \in W^{2,2}(\Omega)$ then $u|_{\partial\Omega}(x) = g(x)$ is in $W^{3/2,2}(\partial\Omega)$ and
$\frac{\partial u}{\partial n}\Big|_{\partial\Omega}(x) = f(x)$ is in $W^{1/2,2}(\partial\Omega)$ which is injected in $L^2(\partial\Omega)$
(i.e. $\frac{\partial u}{\partial n}$ on $\partial\Omega$ has $\frac{1}{2}$ derivative that's $L^2(\partial\Omega)$).

(16)

Now use the interpolation inequality from p. 10 (with $p=2$) ||

III. Maximum principle methods

3.1 The easiest estimates for the equation

$$(*)\begin{cases} -a_{ij}u_{x_ix_j} + b_iu_{x_i} + cu = f & \text{in } \Omega \\ u = \phi \text{ on } \partial\Omega \end{cases}$$

are derived from the

Ω = bounded smooth

(Weak) Maximum Principle

Suppose that $u: \bar{\Omega} \to \mathbb{R}$ is a C^2 function solving the differential inequality

$$(1) \qquad -a_{ij}u_{x_ix_j} + b_iu_{x_i} + cu \leq 0 \quad \text{in } \Omega,$$

where the a_{ij} satisfy condition (E), the b_i & c are bounded, and

$$c(x) \geq 0 \quad \text{in } \Omega.$$

Then

$$(2) \qquad \boxed{\sup_\Omega u \leq \max(0, \sup_{\partial\Omega} u)}.$$

In particular,

$$\boxed{\text{if } u \leq 0 \text{ on } \partial\Omega, \text{ then } u \leq 0 \text{ in } \Omega}$$

For a proof of this standard result, see [G-T, p. 34] or Protter & Weinberger, Maximum Principles in Differential Equations, Prentice Hall.

RK: if $u \leq 0$ on $\partial\Omega \Rightarrow u < 0$ in Ω
(however assuming $c \geq 0$)

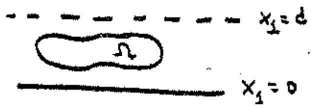
3.2 A. Sup. Norm Estimates from the Maximum Principle

Theorem 3.1 Suppose that u solves $(*)$, where the a_{ij} satisfy the ellipticity condition (E), the b_i and c are bounded, and $c \geq 0$ in Ω.

Then $\exists$ a constant C, depending only on Ω and the coefficients a_{ij}, b_i, and c, such that

$$\boxed{\sup_{\Omega} |u| \leq \sup_{\partial\Omega} |\phi| + C \sup_{\Omega} |f|}$$

Proof Since Ω is bounded, we can assume it lies in the slab $0 < x_1 < d$ for some $d > 0$



Consider the function $e^{\alpha x_1}$, where α is to be selected. We have

$$-a_{ij}(e^{\alpha x_1})_{x_i x_j} + b_i (e^{\alpha x_1})_{x_i} = e^{\alpha x_1}(-\alpha^2 a_{11} + \alpha b_1)$$

$$\leq e^{\alpha x_1}(-\alpha^2 \theta + \alpha \sup_{\Omega} |b_1|)$$

by the lemma on p. (4)

$$\leq -1$$

for α big enough.

Now define the auxillary function

$$v(x) \equiv \sup_{\partial\Omega} |\phi| + (e^{\alpha d} - e^{\alpha x_1}) \sup_{\Omega} |f| \; ;$$

then

$$-a_{ij} v_{x_i x_j} + b_i v_{x_i} + cv \geq \sup_{\Omega} |f| \quad \text{by the above calculation}$$

$$\geq f = -a_{ij} u_{x_i x_j} + b_i u_{x_i} + cu.$$

Exercise. Convexity is the most essential feature of $\varphi(z) = e^{\alpha z}$. Can you obtain result by using $\phi(z) = z^p$, where p is to be chosen? (Might need $d < 1$)

Since $v \geq u$ on $\partial\Omega$, the maximum principle implies

$$v \geq u \text{ in } \Omega;$$

this is,

$$\sup_{\Omega} u \leq \sup_{\Omega} v \leq \sup_{\partial\Omega} |\phi| + e^{\alpha d} \sup_{\Omega} |f|.$$

Then same argument applied to $-u$ gives a similar bound for $\inf_{\Omega} u$ //

3.3 B. Barriers & estimates for the gradient on $\partial\Omega$

Definitions (1) $Lu \equiv -a_{ij} u_{x_i x_j} + b_i u_{x_i} + cu$

(2) Let $x_0 \in \partial\Omega$. A C^2 function $w : \bar{\Omega} \to \mathbb{R}$ is called a barrier at x_0 (with respect to L) if

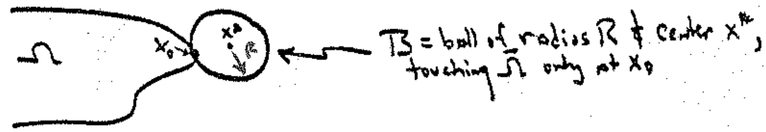
(i) $w(x_0) = 0$

(ii) $w(x) > 0 \quad \forall x \in \partial\Omega \setminus \{x_0\}$

(iii) $Lw \geq 1$ in Ω

(3) Ω satisfies the exterior sphere condition if $\exists R > 0$ such that for each point $x_0 \in \partial\Omega$ there exists a ball $B = B_R(x^*)$ of radius R, with

$$B \cap \bar{\Omega} = \{x_0\}$$



Theorem 3.2 Suppose that Ω satisfies the exterior sphere condition. Assume also that the coefficients of L are bounded, and that the a_{ij} satisfy the ellipticity condition (E). Suppose also $c \geq 0$ in Ω.

Then $\exists$ constants k and p, depending only on R (Radius from exterior sphere condition) and the (and on diameter of Ω)

coefficients of L, such that

x^* = center of B, ball of radius R touching $\bar{\Omega}$ only at $\{x_0\}$

$$(3)\quad \boxed{w(x) \equiv k\left[\frac{1}{R^p} - \frac{1}{|x-x^*|^p}\right]}$$

is a barrier at $x_0 \in \partial\Omega$ (w.r.t L)

<u>Proof</u> Clearly $w(x_0) = 0$, $w(x) > 0$ $\forall x \in \partial\Omega \setminus \{x_0\}$.
We must adjust k & p so that $Lw \geq 1$ in Ω.

WLOG $x^* = 0$, so

$$w(x) = k\left[\frac{1}{R^p} - \frac{1}{|x|^p}\right] = k\left[\frac{1}{R^p} - \left(\sum_{i=1}^n x_i^2\right)^{-p/2}\right]$$

$\therefore$

$$w_{x_i}(x) = kp\frac{x_i}{|x|^{p+2}}$$

$$w_{x_ix_j}(x) = \frac{kp\,\delta_{ij}}{|x|^{p+2}} - kp(p+2)\frac{x_ix_j}{|x|^{p+4}}$$

Therefore

$$Lw = -a_{ij}w_{x_ix_j} + b_iw_{x_i} + cw$$

$$\geq \frac{kp}{|x|^{p+4}}\left((p+2)a_{ij}x_ix_j - a_{ii}|x|^2 + b_ix_i|x|^2\right)$$

$$\geq \frac{kp}{|x|^{p+4}}\left((p+2)\theta|x|^2 - n\Theta|x|^2 - C|x|^3\right)$$

by Condition (E). Now choose $p > 0$ so large that the expression in the ( ) is ≥ 1 for all $x \in \Omega$. This is possible since Ω is bounded & $|x| > R$ for all $x \in \Omega$.

Hence

$$Lw \geq \frac{kp}{|x|^{p+4}} \geq 1,$$ if k is now selected to be large enough. //

<u>Remark</u> We next use the barrier constructed above & the maximum principle to estimate the gradient of a solution of (*) on $\partial\Omega$.

We will assume that $\phi \equiv 0$, ie $u = 0$ on $\partial\Omega$. Note that this really is no restriction. Indeed, if ϕ & $\partial\Omega$ are smooth, if we consider $\bar{u} \equiv u - \Phi$, where Φ is some smooth function defined on $\bar{\Omega}$ that agrees with ϕ on $\partial\Omega$ (cf. [G-T, p. 13?]), then $\bar{u} = 0$ on $\partial\Omega$ & $\bar{u}$ satisfies an equation like (*), with a different function f.

3.4 Theorem 3.3 (<u>Boundary</u> estimate for the gradient)

Assume that u solves

$$(*)\quad \begin{cases} -a_{ij}u_{x_ix_j} + b_iu_{x_i} + cu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the coefficients a_{ij}, b_i, and c are bounded, f is bounded, $c \geq 0$, and the a_{ij} satisfy condition (E). Suppose also that Ω satisfies the exterior sphere condition.

Then $\exists$ a constant C, depending only on a_{ij}, b_i, c, f, and Ω, such that

$$\boxed{\sup_{\partial\Omega}|\nabla u| \leq C}$$

<u>Proof</u> Since $u \equiv 0$ along $\partial\Omega$, $|\nabla u(x)| = \left|\frac{\partial u}{\partial n}(x)\right|$ for $x \in \partial\Omega$.

Pick any $x_0 \in \partial\Omega$ & let w be the function defined by (3) in the previous theorem.

Define

$$v(x) \equiv u(x) - w(x)\left(\sup_\Omega|f|\right) \quad \text{for } x \in \bar{\Omega}.$$

Then

$$v(x) \leq 0 \quad \text{for } x \in \partial\Omega$$

and

$$Lv = Lu - \left(\sup_\Omega|f|\right)Lw$$

$$\leq f - \left(\sup_\Omega|f|\right) \leq 0 \quad \text{in } \Omega.$$

Hence, by the maximum principle, $v \leq 0$ in Ω; that is,

$$u(x) \leq w(x)\left(\sup_\Omega|f|\right) \quad \forall x \in \Omega.$$

But $u(x_0) = 0 = w(x_0)\left(\sup_\Omega|f|\right)$ & so

$$(4)\quad \frac{\partial}{\partial n}u(x_0) \geq \frac{\partial w(x_0)}{\partial n}\left(\sup_\Omega|f|\right) = C$$

since, according to the explicit formula (7), $\frac{\partial w}{\partial n}(x_0)$ can be estimated solely in terms of k, p and R.

Inequality (4) gives a lower bound on $\frac{\partial u}{\partial n}(x_0)$. The same argument with $-w$ in place of w gives an upper bound. ||

3.5 C. Estimates for the gradient in Ω

Lemma 3.4 Suppose that $w: \bar{\Omega} \to \mathbb{R}$ is C^2. If w attains a nonnegative maximum at some interior point $x_0 \in \Omega$, then

$$0 \leq -a_{ij}(x_0) w_{x_ix_j}(x_0) + b_i(x_0) w_{x_i}(x_0) + c(x_0) w(x_0),$$

assuming the a_{ij} satisfy the ellipticity condition (E) & $c \geq 0$.

Proof Since w attains a max at x_0, $w_{x_i}(x_0) = 0$ & so clearly

$$b_i(x_0) w_{x_i}(x_0) + c(x_0) w(x_0) \geq 0.$$

We must show

(4) $$-a_{ij}(x_0) w_{x_ix_j}(x_0) \geq 0.$$

According to Lemma 1.1 (p. 4), $\exists$ an $n \times n$ matrix D such that

$$DD^T = D^T D = I$$

(5) $$D A(x_0) D^T = (\lambda_1, \cdots \lambda_n), \qquad \lambda_i > 0,$$

where $A(x_0) = ((a_{ij}(x_0)))$. If we write $D = ((d_{kl}))$, then (5) reads

(6) $$d_{ki} a_{ij}(x_0) d_{lj} = \delta_{kl} \lambda_k.$$

Now change variables from x to $y = Dx$, ie

(7) $$y_i = d_{ij} x_j \qquad (i = 1, 2, \cdots n)$$

Then, by the chain rule,

$$\begin{aligned} -a_{ij}(x_0) w_{x_ix_j}(x_0) &= -a_{ij}(x_0) w_{y_ky_l}(x_0) \frac{\partial y_k}{\partial x_i} \frac{\partial y_l}{\partial x_j} \\ &= -a_{ij}(x_0) w_{y_ky_l}(x_0) d_{ki} d_{lj} \qquad \text{by (7)} \\ &= -\delta_{kl} \lambda_k w_{y_ky_l}(x_0) \qquad \text{by (6)} \\ &= -\lambda_1 w_{y_1y_1}(x_0) - \lambda_2 w_{y_2y_2}(x_0) \cdots - \lambda_n w_{y_ny_n}(x_0). \end{aligned}$$

Since w attains a max at x_0, the pure second derivatives $w_{y_iy_i}(x_0) \leq 0$. But the $\lambda_i > 0$, and so the last expression above is ≥ 0. This proves (4). ||

3.6 Theorem 3.5 (Estimate for gradient over all of Ω)

Assume that u solves equation (*), where the a_{ij} satisfy condition (E), the a_{ij}, b_i, and c have bounded derivatives, and $c \geq 0$.

Then $\exists$ a constant C, depending only on θ (from condition (E)) and the $W^{1,\infty}$ norms of a_{ij}, b_i, c, such that

$$\boxed{\sup_{\Omega} |\nabla u| \leq \sup_{\partial\Omega} |\nabla u| + C\left(\sup_{\Omega}|u| + \sup_{\Omega}|f| + \sup_{\Omega}|\nabla f|\right)}$$

Remark And so, in particular, since $\sup_{\Omega} |u|$ can be estimated by Theorem 3.1 and $\sup_{\partial\Omega} |\nabla u|$ by Theorem 3.3, this gives a bound on $|\nabla u|$ in Ω solely in terms of known quantities.

Proof We define an auxiliary function

$$\boxed{w(x) = |\nabla u(x)|^2 + \lambda u^2} \quad \text{(where } \lambda \text{ is to be selected).}$$

Then

(8) $$w_{x_i} = 2u_{x_k}u_{x_kx_i} + 2\lambda u u_{x_i}$$

and

(9) $$w_{x_ix_j} = 2u_{x_kx_j}u_{x_kx_i} + 2u_{x_k}u_{x_kx_ix_j} + 2\lambda u_{x_j}u_{x_i} + 2\lambda u u_{x_ix_j}.$$

Notice also that by (*)

(10) $$-a_{ij}u_{x_ix_j} + b_iu_{x_i} + cu = f \quad \text{in } \Omega;$$

and therefore, differentiating with respect to some x_k, we get

(11) $$-a_{ij}u_{x_kx_ix_j} + b_iu_{x_kx_i} + cu_{x_k} = f_{x_k} + a_{ijx_k}u_{x_ix_j} - b_{ix_k}u_{x_i} - c_{x_k}u \qquad (k=1,2,\cdots n)$$

Now, let us consider where in $\bar{\Omega}$ w attains its max. If the max is attained on $\partial\Omega$, then the theorem clearly holds. Otherwise w has a (nonnegative) max at some interior pt. $x_0 \in \Omega$. By Lemma 3.4, therefore, we have

$$0 \le -a_{ij}w_{x_ix_j} + b_iw_{x_i} + cw \quad \underline{\text{at the point } x_0}$$

$$\left.\begin{aligned} &= -2a_{ij}u_{x_kx_j}u_{x_kx_i} - 2\lambda a_{ij}u_{x_j}u_{x_i} \\ &\quad + 2u_{x_k}(-a_{ij}u_{x_kx_ix_j} + b_iu_{x_kx_i} + cu_{x_k}) \\ &\quad + 2\lambda u(-a_{ij}u_{x_ix_j} + b_iu_{x_i} + cu) \\ &\quad - cu_{x_k}u_{x_k} - \lambda cu^2 \end{aligned}\right\} \text{here we used (8), (9), \& the definition of } w$$

$$\left.\begin{aligned} &\le -2\theta u_{x_kx_j}u_{x_kx_j} - 2\lambda\theta u_{x_i}u_{x_i} \\ &\quad + 2u_{x_k}(-a_{ij}u_{x_kx_ix_j} + b_iu_{x_kx_i} + cu_{x_k}) \\ &\quad + 2\lambda u(-a_{ij}u_{x_ix_j} + b_iu_{x_i} + cu) \end{aligned}\right\} \text{by the ellipticity condition (E)}$$

Next use (10) & (11) to substitute for the expressions in the parentheses:

$$\begin{aligned} 0 &\le -2\theta u_{x_kx_j}u_{x_kx_j} - 2\lambda\theta u_{x_i}u_{x_i} \\ &\quad + 2u_{x_k}(f_{x_k} + a_{ijx_k}u_{x_ix_j} - b_{ix_k}u_{x_i} - c_{x_k}u) \\ &\quad + 2\lambda u f \\ &\le -2\theta u_{x_kx_j}u_{x_kx_j} - 2\lambda\theta u_{x_i}u_{x_i} \\ &\quad + \varepsilon u_{x_ix_j}u_{x_ix_j} + C(\varepsilon)u_{x_i}u_{x_i} + C(\lambda^2u^2 + f^2 + f_{x_i}f_{x_i}) \end{aligned}$$

by the Cauchy inequality with ε (p. 6).

We now select $\varepsilon = 2\theta$ & then pick $\lambda > 0$ so large that

$$2\lambda\theta = C(\varepsilon) + 1.$$

This gives

$$|\nabla u|^2 \le C(u^2 + f^2 + |\nabla f|^2), \quad \underline{\text{at the point } x_0}$$

where

$$w = |\nabla u|^2 + \lambda u^2 \quad \text{attained its max.}$$

Clearly, therefore

$$\max_{\Omega} w = w(x_0) \le C(u^2 + f^2 + |\nabla f|^2) \quad \text{at } x_0$$

& so the theorem follows. ||

Remark This trick of applying the maximum principle (or, more precisely, Thm 3.5 ~~Lemma 3.4~~) to

$$w = |\nabla u|^2 + \lambda u^2$$

is due to S. Bernstein. More sophisticated versions of this method & indeed of all the techniques on p. 18 - 24 may be found in [G-T, Chapters 13 & 14].

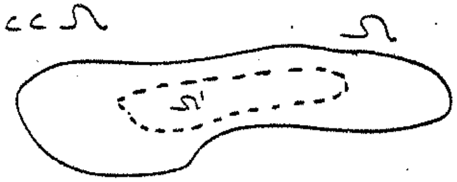
Comment: Perhaps deep reason why this computation goes through is that if $\Delta u = 0 \Rightarrow |\nabla u|^2$ is subharmonic. So for estimating derivatives of $\Delta u = 0$, $u|_{\partial\Omega} = g$ presumably we do not need the λu^2 — that is needed for taking care of rest of operat.

D. Local estimates for the gradient and Higher Order Derivatives (25)

3.7 Sometimes the last theorem is not useful since it may be difficult to estimate $\sup_{\partial\Omega} |\nabla u|$. In this case we consider the auxillary function

$$W'(x) = \zeta^2(x)\,|\nabla u(x)|^2 + \lambda u^2(x)$$

where $\zeta(x)$ is a smooth cutoff function vanishing near $\partial\Omega$. We can choose ζ to be $\equiv 1$ on some strictly interior domain $\Omega' \subset\subset \Omega$



& use the preceeding estimates to bound $\sup_{\Omega'} |\nabla u|$ in terms of $\sup_{\Omega} |u|$, $\sup_{\Omega} |f|$, $\sup_{\Omega} |\nabla f|$.

3.8 Once we have a bound on $|\nabla u|$, bounds on higher derivatives follow the same way. For example we may apply the maximum principle to

$$W'' = u_{x_i x_j} u_{x_i x_j} + \lambda |\nabla u|^2 \qquad (\lambda \text{ to be selected})$$

to obtain a bound for the second derivatives of u. Use

$$W''' = \zeta^2 u_{x_i x_j} u_{x_i x_j} + \lambda |\nabla u|^2$$

to get local second derivative estimates.

(26)

A number of the estimates that follow are important only because of the minimal smoothness assumptions on coefficients. Most results are immediate for $-\Delta$.

IV L^2 estimates

4.1 A. In this chapter we write $(*)$ in the form

$$(*) \begin{cases} -(a_{ij} u_{x_i})_{x_j} + b_i u_{x_i} + cu = f + (f_i)_{x_i} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

Remarks (a) By remark on p. 19, there is no restriction in setting $u = 0$ on $\partial\Omega$ (if ϕ is smooth)

(b) If the a_{ij} are smooth, then

$$-(a_{ij} u_{x_i})_{x_j} + b_i u_{x_i} = -a_{ij} u_{x_i x_j} + \underbrace{(b_i - a_{ij,x_j})}_{\equiv \tilde{b}_i} u_{x_i}$$

& so $(*)$ could be written as before. We'll see that there are advantages to writing it as above.

(c) The right hand side above is "$(f_i)_{x_i}$"; i.e. the sum of the ith derivatives of functions f_i $(i = 1, 2, \dots n)$

(d) We can define u to be a weak solution of $(*)$ if

$$\int_\Omega a_{ij} u_{x_i} \phi_{x_j} + b_i u_{x_i} \phi + cu\phi \, dx = \int_\Omega f\phi - f_i \phi_{x_i} \, dx$$

$\forall \phi \in C_0^\infty(\Omega)$ ⟵ space of smooth functions with compact support in Ω

Definition When the leading term is written as on the last page, we say that $(\star)$ is in divergence form.

4.2 B. $W^{1,2}$ estimates

Definition (1) $c^+(x) = \max(c(x), 0)$

(2) $c^-(x) = -\min(c(x),0) = c^+(x) - c(x)$

Theorem 4.1 Suppose u is a smooth solution of $(\star)$. Assume the a_{ij} satisfy the ellipticity condition (E) & that the a_{ij}, b_i, c are bounded. Suppose $f, f_i \in L^2(\Omega)$.

Then $\exists$ constants C_1 & C_2, depending only on Θ & the coefficients, $\ni$

$$(1)\quad \boxed{\int_\Omega |\nabla u|^2 + c^+ u^2 \, dx \leq C_1 \int_\Omega f^2 + \sum_{i=1}^n f_i^2 \, dx + C_2 \int_\Omega u^2 \, dx}$$

If $\lambda = \min_\Omega c(x)$ is sufficiently large, $\exists\, C_3 \ni$

$$(2)\quad \boxed{\|u\|^2_{W^{1,2}(\Omega)} \leq C_3 \int_\Omega f^2 + \sum_{i=1}^n f_i^2 \, dx}$$

Note This estimate does not require any smoothness of the a_{ij} or of the f_i.

Proof Multiply equation $(\star)$ by u & integrate by parts in the first and last term (as $u=0$ on $\partial\Omega$, we don't get any boundary terms):

$$\int_\Omega a_{ij} u_{x_i} u_{x_j} + b_i u_{x_i} u + c u^2 \, dx = \int_\Omega f u - f_i u_{x_i} \, dx$$

Write $c = c^+ - c^-$ to get

$$\int_\Omega a_{ij} u_{x_i} u_{x_j} + c^+ u^2 \, dx = \int_\Omega c^- u^2 - b_i u_{x_i} u + f u - f_i u_{x_i} \, dx$$

Use condition (E) on the left & Cauchy's inequality with ε on the right:

$$\int_\Omega \Theta |\nabla u|^2 + c^+ u^2 \, dx \leq \varepsilon \int_\Omega |\nabla u|^2 \, dx + \int_\Omega \left(c^- + \frac{\sum_{i=1}^n b_i^2}{4\varepsilon} + \frac{1}{2}\right) u^2 \, dx + \int_\Omega \frac{1}{2} f^2 + \frac{1}{4\varepsilon} \sum_{i=1}^n f_i^2 \, dx.$$

Hence, setting $\varepsilon = \theta/2$, we get

$$\int_\Omega \frac{\theta}{2}|\nabla u|^2 + c^+ u^2\, dx \le C_1 \int_\Omega f^2 + \sum_{i=1}^n f_i^2\, dx + C_2 \int_\Omega u^2\, dx .$$

If $\lambda = \min_\Omega c(x) > C_2$, then $c^+ \ge \lambda > C_2$ & then the term on the right can be subtracted off from the $\int_\Omega c^+ u^2$ term to give estimate (2). ||

<u>Exercise</u> Prove an inequality like (1) assuming only that

$$f \in L^{2n/n+2}(\Omega).$$

[4.3] C. $W^{2,2}$ estimates

Now we assume more about the smoothness of the coefficients & of the f_i, and derive an L^2 estimate for the <u>second</u> derivatives of u.

We will follow [L-U, p. 169-180].

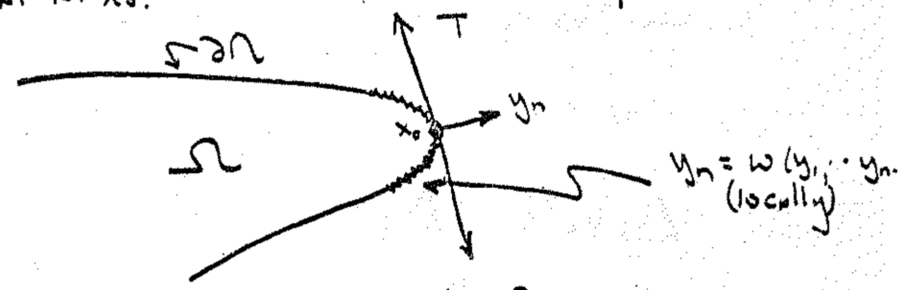
<u>Assumptions on $\partial\Omega$</u>:

(a) for each pt. $x_0 \in \partial\Omega$, $\exists$ a tangent plane T to $\partial\Omega$ at x_0 $\ni$ in some small neighborhood of

x_0, $\partial\Omega$ is represented in local coordinates $(y_1, \cdots y_n)$ of the form

$$y_n = \omega(y_1, \cdots y_{n-1}) \qquad (y_1, \cdots y_{n-1}) \in T$$

We assume that the y_n axis points along the outward normal at x_0:



(b) $\omega: \mathbb{R}^{n-1} \to \mathbb{R}$ is C^2 & $\dfrac{\partial^2 \omega(x_0)}{\partial y_k \partial y_l} = 0 \quad l \ne k$

→ this condition seems wlog. I.e. we can always do this by rotation of the co-ordinate system.

(c) $\exists K$, independent of x_0, $\ni$

-(principal curvatures of $\partial\Omega$ at x_0) $\left\{ \boxed{\dfrac{\partial^2 \omega(x_0)}{\partial y_l^2} \le K} \right. \qquad \forall x_0 \in \partial\Omega,\ l = 1, 2, \cdots n-1$

<u>Remark</u> The bounds we obtain will depend on ω only through K & hence are true, for <u>certain</u> domains (e.g. convex domains) with "corners".

[4.4] As the proofs to follow are somewhat involved, we first present a special case:

(For constant coefficients one can give easy proofs using F.T. However method below is quite general.)

Theorem 4.2 Suppose u is a smooth solution of

$$\begin{cases} \Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\partial\Omega$ satisfies conditions (a)–(c) above. Then $\exists$ C, depending only on Ω, $\ni$

$$(3) \qquad \boxed{\|u\|_{W^{2,2}(\Omega)} \leq C\,\|f\|_{L^2(\Omega)}} \qquad (*)\ (**).$$

Proof $\displaystyle\int_\Omega f^2\,dx = \int_\Omega \Delta u\cdot\Delta u\,dx = \int_\Omega u_{x_jx_j}u_{x_ix_i}\,dx$

$$= -\int_\Omega u_{x_jx_jx_i}u_{x_i}\,dx + \int_{\partial\Omega}\Delta u\frac{\partial u}{\partial n}\,ds$$

$$= \int_\Omega u_{x_ix_j}u_{x_ix_j}\,dx - \int_{\partial\Omega}u_{x_ix_j}u_{x_i}n_j\,ds + \int_{\partial\Omega}\Delta u\frac{\partial u}{\partial n}\,ds.$$

Therefore

$$(4) \qquad \sum_{i,j=1}^n\int_\Omega u^2_{x_ix_j}\,dx = \int_\Omega f^2\,dx - \int_{\partial\Omega}I(x)\,ds,$$

for $\displaystyle I(x) \equiv \Delta u\frac{\partial u}{\partial n} - \frac{\partial^2 u}{\partial n\,\partial x_i}u_{x_i} \qquad (x\in\partial\Omega),$

(*) This estimate argues in favor of Sobolev spaces: one obtains the expected thing (if $f\in L^2 \Rightarrow u\in W^{2,2}$. It is known that if $f\in C \not\Rightarrow u\in C^2$ in general. Hence in this respect C^n are not natural (though C^α ($\alpha\neq$ integer) are natural)

(***) Let $\Delta_{y_i}u(y_1,\dots,y_n)=0$. Let $Ox=y$, $OO^T=I$ $\Rightarrow \Delta_{x_i}u(Ox)=0$, i.e. $\tilde u(x)=u(Ox)$ is harmonic

where $n=(n_1,\dots n_n)$ = outward unit normal.

We now must estimate $I(x)$: more generally $\Delta_x u(Ox) = \Delta_y u(y)$, $y=Ox$ (see back)

Fact $I(x)$ is invariant w.r.t. a linear change of coordinates *** & so, for a fixed $x_0\in\partial\Omega$, let us move to the $(y_1,\dots y_n)$ coordinates:

Then $y_n = n$ = outward unit normal & so

$$I(x_0) = \sum_{i=1}^n\left(\frac{\partial^2 u}{\partial y_i^2}\frac{\partial u}{\partial y_n} - \frac{\partial^2 u}{\partial y_n\partial y_i}\frac{\partial u}{\partial y_i}\right)_{x=x_0}$$

→ obvious!

$$= \sum_{i=1}^{n-1}\left(\frac{\partial^2 u}{\partial y_i^2}\frac{\partial u}{\partial y_n} - \frac{\partial^2 u}{\partial y_n\partial y_i}\frac{\partial u}{\partial y_i}\right)_{x=x_0}$$

Since $u\equiv 0$ on $\partial\Omega$, $u(y_1,\dots y_{n-1},\omega(y_1,\dots y_{n-1}))\equiv 0$; & so

$$(5) \qquad u_{y_k}(y_1,\dots y_{n-1},\omega) + u_{y_n}(y_1,\dots y_{n-1},\omega)\,\omega_{y_k} \equiv 0 \quad (k=1,2,\dots n-1).$$

But $y_1,\dots y_{n-1}\in T$ = tangent plane to $\partial\Omega$ at x_0. Hence

$$(6) \qquad \omega_{y_k}(x_0) = 0 \quad (k=1,2,\dots n-1)$$

& this implies, by (5), that

$$(7) \qquad u_{y_k}(x_0) = 0 \quad (k=1,2,\dots n-1)$$

(**). Let $T: f\to u$. Then $\|Tf\|_{W^{2,2}}\leq C\|f\|_{L^2}$, i.e. $T: L^2\to W^{2,2}$ is bounded. Hence for Ω bounded ($W^{2,2}\subset L^2$ compact) $T: L^2\to L^2$ is compact operator.

Next differentiate (5) again: (33)

$$u_{y_k y_k} + 2u_{y_k y_n}\,\omega_{y_k} + u_{y_n y_n}(\omega_{y_k})^2 + u_{y_n}\,\omega_{y_k y_k} \equiv 0 \qquad (k=1,2,\dots n-1)$$

At x_0, (6) implies

2^{nd} derivatives expressed in terms of first! Reason: $y_n = \omega(y_1,\dots,y_{n-1})$ "$\Rightarrow$" derivatives are dependent *

$$(8)\quad u_{y_k y_k}(x_0) = -u_{y_n}(x_0)\,\omega_{y_k y_k}(x_0) \qquad (k=1,\dots n-1)$$

Now we use (7) & (8) to simplify the expression for $I(x_0)$:

$$I(x_0) = -\left(u_{y_n}(x_0)\right)^2 \sum_{i=1}^{n-1} \omega_{y_i y_i}(x_0)$$

$$\geq -\left(\frac{\partial u}{\partial n}(x_0)\right)^2 (n-1)K \qquad \text{by assumption (c) on } \partial\Omega.$$

Plug this estimate into (4):

$$\int_\Omega u_{x_i x_j} u_{x_i x_j}\,dx \leq \int_\Omega f^2\,dx + (n-1)K\int_{\partial\Omega}\left(\frac{\partial u}{\partial n}\right)^2 ds$$

$$\leq \varepsilon \int_\Omega u_{x_i x_j} u_{x_i x_j}\,dx + \int_\Omega f^2\,dx + C(\varepsilon)\int_\Omega u^2 \qquad \text{by Corollary 2.6 (p. 15).}$$

Hence

$$(9)\quad \int_\Omega u_{x_i x_j} u_{x_i x_j}\,dx \leq C\int_\Omega f^2\,dx + C\int_\Omega u^2\,dx \qquad (9)$$

* Try this with $f(x,y(x),y'(x)) \equiv 0 \Rightarrow y' = (\dots)$ first derivatives.

Now we get rid of the last term in (9). (34)

Since $u=0$ on $\partial\Omega$, we may apply Sobolev's inequality to calculate

$$\int_\Omega u^2\,dx \leq C\left(\int_\Omega |u|^{2^*}dx\right)^{2/2^*} \leq C\int_\Omega |\nabla u|^2\,dx$$

← not regular Poincaré's inequal.

$$= -C\int_\Omega \Delta u \cdot u\,dx \qquad \text{by Green's identity (p. 9)}$$

$$\leq \varepsilon\int_\Omega u^2\,dx + C(\varepsilon)\int_\Omega \underbrace{(\Delta u)^2}_{=f^2}dx$$

This proves

$$(10)\quad \int u^2\,dx \leq C\int f^2\,dx;$$

plug this into (9) to estimate $\int_\Omega u_{x_i x_j} u_{x_i x_j}\,dx \leq C\int_\Omega f^2\,dx$

Then $\int_\Omega |\nabla u|^2\,dx \leq C\int f^2\,dx$ by (10) & Corollary 2.3 (or by the calculation at the top of this page). ||

$\boxed{4.5}$ We'll use the same method of proof to derive $W^{2,2}$ estimates for a solution of (*): see Theorem 4.4.

Lemma 4.3 Let $A = ((a_{ij}))$ & $B = ((b_{ij}))$ be a real symmetric $n\times n$ matrices. Suppose that A is positive definite, with smallest eigenvalue $\geq \theta > 0$.

Then
$$\boxed{a_{ij}\,b_{ik}\,a_{kl}\,b_{jl} \geq \theta^2\,b_{ij}\,b_{ij}}$$

Exercise: For $L = \sum a_{ij}u_{x_i x_j} + \sum b_i u_{x_i} + cu$ the following estimate holds for u / $u|_{\partial\Omega} = 0$

$\|u\|_{W^{2,2}} \leq C\left(\|Lu\|_2 + \|u\|_{L^2}\right)$. Show that for $n > 2$ one can not hope that estimate holds by looking at the following: (continued)

Proof Suppose first that $A = \operatorname{diag}(\lambda_1, \cdots \lambda_n)$. (35)
Then

(11) $$a_{ij} b_{ik} a_{k\ell} b_{j\ell} = \lambda_i \lambda_k b_{ik} b_{ik} \geq \theta^2 b_{ik} b_{ik}.$$

In general, we note that

$$a_{ij} b_{ik} a_{k\ell} b_{j\ell} = \operatorname{trace}(ABAB)$$

Choose an orthonormal matrix $D \ni DAD^T = A'$ $= \operatorname{diag}(\lambda_1, \cdots \lambda_n)$. Since trace is an invariant, we have

$$\operatorname{trace}(ABAB) = \operatorname{trace}(D(ABAB)D^T)$$
$$= \operatorname{trace}\left[(DAD^T)(DBD^T)(DAD^T)(DBD^T)\right]$$
$$= \operatorname{trace}\left[A'B'A'B'\right] \qquad B' \equiv DBD^T$$
$$\geq \theta^2 b'_{ik} b'_{ik} \quad \text{by (11)}$$
$$= \theta^2 b_{ik} b_{ik} \qquad //$$

Theorem 4.4 Suppose u is a smooth solution of (*) (p. 26). Assume that $\partial\Omega$ satisfies the hypotheses (a)-(c) on p. 29 - 30. Suppose that the a_{ij} satisfy the ellipticity condition (E) and that the a_{ij} have a bounded gradient on Ω. Assume finally that the b_i and c are bounded & that $f + (f_i)_{x_i} \equiv \bar{f} \in L^2(\Omega)$.

Then $\exists$ C_1 and C_2, depending only on the coefficients and on Ω, $\ni$

(36)

(12) $$\boxed{\sum_{i,j=1}^{n} \int_\Omega u_{x_i x_j}^2 \, dx \leq C_1 \int_\Omega \bar{f}^2 \, dx + C_2 \int_\Omega u^2 \, dx}$$

If $\lambda \equiv \min_\Omega c(x)$ is sufficiently large, $\exists\, C_3 \ni$

(13) $$\boxed{\|u\|_{W^{2,2}(\Omega)} \leq C_3 \|\bar{f}\|_{L^2(\Omega)}} \qquad \bar{f} \equiv f + (f_i)_{x_i}$$

Proof Since $|a_{ij,x_k}| \leq C$, we may write (*) as

$$-a_{ij} u_{x_i x_j} + \bar{b}_i u_{x_i} + cu = \bar{f}$$

for $\bar{b}_i = (b_i - a_{ij,x_j})$, $\bar{b}_i$ bounded. Then

$$\int_\Omega \bar{f}^2 \, dx = \int_\Omega (-a_{ij} u_{x_i x_j} + \bar{b}_i u_{x_i} + cu)(-a_{k\ell} u_{x_k x_\ell} + \bar{b}_k u_{x_k} + cu) \, dx$$

$$\geq \int_\Omega a_{ij} u_{x_i x_j} a_{k\ell} u_{x_k x_\ell} \, dx - \varepsilon \|u\|^2_{W^{2,2}} - C(\varepsilon) \|u\|^2_{W^{1,2}}$$

$$= \int_\Omega a_{ij} u_{x_i x_k} a_{k\ell} u_{x_\ell x_j} \, dx + \underbrace{\int_{\partial\Omega} a_{ij} u_{x_i} a_{k\ell} u_{x_k x_\ell} n_j - a_{ij} u_{x_i} a_{k\ell} u_{x_\ell x_j} n_k \, d\sigma}_{\text{Call the integrand } I(x)}$$

$$- \varepsilon' \|u\|^2_{W^{2,2}} - C(\varepsilon') \|u\|^2_{W^{1,2}}$$

The last step above follows from two integrations by parts. We also used Cauchy's inequality with ε to estimate

(continuation of Ex) (Important: In the estimate c should be such that for $\Omega = B(0;R)$, the same c holds $\forall R \leq R_0$.)

$$Lu = a_{ij} u_{x_i x_j} = 0, \quad a_{ij} = \delta_{ij} + b \frac{x_i x_j}{r^2} \qquad r = |x| \qquad b = -1 + \frac{n-1}{1-\lambda}$$

a) Show that this has a solution $r^\lambda - R^\lambda = u$

b) Show that for $\lambda < 1$ L is elliptic (i.e. satisfies (E))

c) Show that $u \in W^{2,2}$ for $n \geq 3$ and

$$\|u\|_{W^{2,2}} \sim R^{\lambda - 2 + \frac{n}{2}}, \quad \|u\|_{L^2} \sim R^{\lambda + \frac{n}{2}}$$

hence derive result: $a_{ij} \in L^\infty$ not enough.

the lower order terms.

Now we apply Lemma 4.3, with $A = ((a_{ij}(x)))$ & B = Hessian matrix of u = $((u_{x_ix_j}(x)))$, in the first term on the right. This gives, after rearranging,

$$\theta^2 \int_\Omega u_{x_ix_j} u_{x_ix_j}\, dx \le \int_\Omega \bar{f}^2\, dx + \varepsilon' \|u\|^2_{W^{2,2}}$$

$$(14) \qquad + C(\varepsilon') \|u\|^2_{W^{1,2}} - \int_{\partial\Omega} I(x)\, dx.$$

We must now estimate $I(x)$

$$(15) \qquad I(x) \equiv a_{ij} a_{kl} (u_{x_i} u_{x_kx_l} n_j - u_{x_i} u_{x_lx_j} n_k) \qquad (x \in \partial\Omega)$$

Estimate of $I(x)$ Pick any $x_0 \in \partial\Omega$: WLOG $x_0 = 0$. Choose an orthogonal matrix $C = ((c_{kl}))$ to convert from the x-variables to the y-variables (defined at x_0 by hypothesis (a) on $\partial\Omega$):

Ex: Exercise above indicates that $n=2$ may be different. Actually this is true. Estimate holds under hypothesis of $L^\infty(\Omega)$-boundedness on coefficients. Verify the following steps: Consider the $Lu = \sum_{i,j=1}^{2} a_{ij} u_{x_ix_j} + \sum^{2} a_i u_{x_i} + au = f(x)$, L satisfies (E), $a_i \in L_4(\Omega)$, $a \in L_2(\Omega)$, $\|a_i\|_{L_4}, \|a\|_{L_2} \le \Theta$

a) Rewrite equation as

$$\frac{a_{11}}{a_{22}} u^2_{x_1x_1} + \frac{2a_{12}}{a_{22}} u_{x_1x_2} u_{x_1x_2} + u^2_{x_1x_2} = F \frac{u_{x_1x_1}}{a_{22}} + u^2_{x_1x_2} - u_{x_1x_1} u_{x_2x_2}, \qquad F(x) = f(x) - a_i u_{x_i} - au$$

b) Derive from this $\frac{\theta}{2\Theta}(u^2_{x_1x_1} + u^2_{x_1x_2}) \le \frac{1}{2}\frac{\Theta}{\theta^3} F^2 + u^2_{x_1x_2} - u_{x_1x_1} u_{x_2x_2}$

(continued)

$$y_k = c_{kl} x_l \qquad (k = 1, \cdots n);$$

Since C is orthonormal,

$$(16) \qquad x_l = c_{kl} y_k \qquad (l = 1, 2, \cdots n)$$

From this we have that at x_0,

$$(17) \qquad n_l(x_0) = l^{th} \text{ component of outward unit normal at } x_0 = c_{nl} \qquad (l = 1, \cdots n)$$

(This follows since the y_n-axis points along the outward normal at x_0)

We plug (17) into the definition of $I(x_0)$ to get

$$I(x_0) = a_{ij} a_{kl} (u_{x_i} u_{x_kx_l} c_{nj} - u_{x_i} u_{x_lx_j} c_{nk})$$

$$= a_{ij} a_{kl} (c_{mi} u_{y_m} c_{pk} c_{ql} u_{y_py_q} c_{nj} - c_{mi} u_{y_m} c_{pj} c_{ql} u_{y_py_q} c_{nk})$$

by the chain rule & (16) (N.B. Implicit summation over 7 indices!)

Hence

$$(17) \quad I(x_0) = b_{mn} b_{pq} u_{y_m} u_{y_py_q} - b_{mp} b_{nq} u_{y_m} u_{y_py_q},$$

where

$$(18) \qquad b_{pq} \equiv a_{kl} c_{pk} c_{ql} \qquad (p, q = 1, 2, \cdots n)$$

Now we simplify. In view of (7) (in proof of Theorem 4.2), we have

$$I(x_0) = [b_{nn}b_{pq} - b_{np}b_{nq}]\, u_{y_n} u_{y_p y_q}.$$

For $p=n$ & any q or for $q=n$ and any p, the terms in the bracket cancel out. Hence

$$I(x_0) = \sum_{p,q=1}^{n-1} [b_{nn}b_{pq} - b_{np}b_{nq}]\, u_{y_n} u_{y_p y_q}.$$

Next we see that the proof of (8) on p 33 is easily modified to show

$$u_{y_k y_\ell}(x_0) = -u_{y_n}(x_0)\,\omega_{y_k y_\ell}(x_0) \qquad (k,\ell = 1,2,\dots n-1).$$

Plug this into the equality above to get

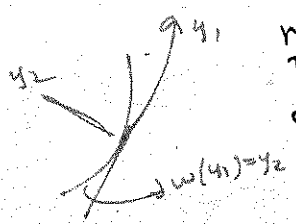
$$I(x_0) = -\sum_{p,q=1}^{n-1} [b_{nn}b_{pq} - b_{np}b_{nq}]\,(u_{y_n})^2\,\omega_{y_p y_q}$$

$$(19) \qquad = -\sum_{p=1}^{n-1} [b_{nn}b_{pp} - b_{np}^2]\,(u_{y_n})^2\,\omega_{y_p y_p}$$

by hypothesis (b) on $\partial\Omega$.

Now the matrix $B = ((b_{pq}))$ is given by $B = CAC^T$, according to (18). Thus B satisfies the ellipticity condition (E) & so, by Lemma 1.1,

$$0 \le b_{nn}b_{pp} - b_{np}^2 \le \Theta^2.$$



(continued)

c) Arguing as in Proof of Th. show that

$$J_1 = \int_\Omega (u_{x_1x_1} u_{x_2x_2} - u_{x_1x_2}^2)\,dx = -\frac{1}{2}\int_S \frac{d^2 w(y_1)}{dy_1^2}\Big|_{y_1=0} \left(\frac{\partial u}{\partial n}\right)^2 dS.$$

Using that $\dfrac{d^2 w}{dy_1^2}\Big|_{y_1=0} \le K$

Derive result.

Hence (19) & hypothesis (c) on $\partial\Omega$ give

$$(20) \qquad I(x_0) \ge -K(n-1)\,\Theta^2 \left(\frac{\partial u}{\partial n}\right)^2 \qquad (x_0 \in \partial\Omega)$$

Now use (20) in estimate (14) to obtain

$$\Theta^2 \int_\Omega u_{x_i x_j} u_{x_i x_j}\,dx \le \int_\Omega \bar{f}^2\,dx + \varepsilon'' \|u\|^2_{W^{2,2}} + C(\varepsilon'')\|u\|^2_{W^{1,2}} + C\int_{\partial\Omega} \left(\frac{\partial u}{\partial n}\right)^2 ds$$

$$\le \int_\Omega \bar{f}^2\,dx + \varepsilon \|u\|^2_{W^{2,2}} + C(\varepsilon)\|u\|^2_{W^{1,2}}$$

by Corollary 2.6

The rest is standard: choose $\varepsilon = \frac{\Theta^2}{2}$ & employ Corollary 2.3 //

Remarks (a) Estimate (12) is true if we write (*) in nondivergence form & assume only a_{ij} are continuous: see [L-U, p. 190-193]. For merely bounded a_{ij} estimate (12) is in general false ([L-U, p 19-20]), but does hold in $n=2$ dimensions [L-U, p. 223-227].

(b) By introducing a "cutoff" function ζ^2, we can obtain interior $W^{2,2}$ estimates, even when $u|_{\partial\Omega}$ is badly behaved.

(c) $W^{2,2}$ estimates for Δu with general (possibly nonlinear) boundary conditions may be found on p. 63 of Barbu, Nonlinear Semigroups & differential equations in Banach spaces, Noordhoff International Publishing.

V Bounds & Hölder Continuity for equations in divergence form

[5.1] A. As in Chapter IV, we'll write (*) in the divergence form

$$\begin{cases} -(a_{ij}u_{x_i})_{x_j} + b_i u_{x_i} + cu = f + (f_i)_{x_i} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

& will make NO assumptions as to the smoothness of the a_{ij} (only that the ellipticity condition (E) holds). The remarkable conclusion will be that nevertheless the C^γ norm of u can be estimated (for some $\gamma > 0$) in terms of known quantities. Examples presented later will show how important this fact is for applications to nonlinear problems.

For simplicity of the exposition we'll study a simpler form of the above problem, namely

$$(*) \begin{cases} -(a_{ij}u_{x_i})_{x_j} = (f_i)_{x_i} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Remarks about writing the equation in the simpler form (*):

There's really not much loss of generality in studying equation (*):

(a) If f is a given function & we define

$$V(x) \equiv \frac{-1}{n(n-2)\omega_n} \int_\Omega f(y) \frac{1}{|x-y|^{n-2}}\,dy$$

(ω_n = measure of unit ball in $\mathbb{R}^n$), then V (as we'll see in Chapter VI) solves

$$\Delta V = f \text{ in } \Omega, \text{ with } \|\nabla V\|_{L^{p^*}(\Omega)} \le C\|f\|_{L^p(\Omega)}$$

Thus

$$f = (f_i)_{x_i} \quad \text{for } f_i \equiv V_{x_i} \;\&$$

so the right hand side of (*) includes as a special case a simple function f.

(b) The simplified form (*) also really includes the case of first order terms by means of this trick:

Let $\tilde{\Omega} \equiv \{(x_1, \cdots x_n, y) \mid x = (x_1, \cdots x_n) \in \Omega,\ 0 < y < d\}$ for $d > 0$ to be selected. Then a problem of the form

$$-(a_{ij}u_{x_i})_{x_j} + b_i u_{x_i} = 0 \qquad \text{in } \Omega$$

can be written as

$$-\sum_{i,j=1}^n (a_{ij}u_{x_i})_{x_j} + \sum_{i=1}^n (b_i y u_{x_i})_y + \sum_{i=1}^n (b_i y u_y)_{x_i} - u_{yy} = 0 \quad \text{in } \tilde{\Omega}$$

since $u_y = u_{yy} \equiv 0$;

That is,

$$-(\tilde{a}_{ij} u_{x_i})_{x_j} = 0 \text{ in } \tilde{\Omega} \qquad (y = x_{n+1}),$$

for

$$A = ((\tilde{a}_{ij})) = \begin{pmatrix} a_{11} & \cdots & a_{1n} & -b_1 y \\ \vdots & & \vdots & -b_2 y \\ & & & \vdots \\ a_{n1} & \cdots & a_{nn} & -b_n y \\ -b_1 y & \cdots & -b_n y & 1 \end{pmatrix}$$

Claim: If d is small enough, the $\tilde{a}_{ij}$ satisfy condition (LE) in $\tilde{\Omega}$.

Pf $\xi = (\xi_1, \dots \xi_{n+1}) \in \mathbb{R}^{n+1}$

$$\sum_{i,j=1}^{n+1} \tilde{a}_{ij}\xi_i\xi_j = \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j - 2\sum_{i=1}^{n} b_i \xi_i \xi_n y + \xi_{n+1}^2$$

$$\geq \theta \sum_{i=1}^{n} \xi_i^2 - Cy|\xi|^2 + \xi_{n+1}^2$$

$$\geq \min(\theta, 1)|\xi|^2 - Cd|\xi|^2$$

$$\geq \frac{1}{2}\min(\theta,1)|\xi|^2 = \theta'|\xi|^2 \text{ for } d \text{ small enough} \quad //$$

Note, however, that we don't have $u = 0$ on $\partial\tilde{\Omega}$.

In this chapter we will not assume $a_{ij} = a_{ji}$; this is important for certain applications (see p. 81)

5.2 B. Global L^∞ estimates

Definition $W_0^{1,2}(\Omega)$ = closure in $W^{1,2}(\Omega)$ of $C_0^\infty(\Omega)$

Lemma 5.1 (Truncation lemma): Let $G: \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitz (i.e. $\exists K \geq 0 \ni$
$$|G(s) - G(t)| \leq K|s-t| \quad \forall s,t \in \mathbb{R})$$
Spse $G(0) = 0$. Assume $u \in W_0^{1,2}(\Omega)$.

Then

(1) $G(u) \in W_0^{1,2}(\Omega)$

and

(2) if G' has only finitely many pts. of discontinuity, then
$$[G(u)]_{x_i} = G'(u) u_{x_i} \quad \text{a.e.} \quad (i = 1, .. n)$$

We now follow Stampacchia (reference [S] on p. 1) to prove that if u solves (*), with the f_i in a high enough L^p space, then $u \in L^\infty$. The idea will be to prove that
$$\phi(k) \equiv \text{mes}\{x \in \Omega \mid u(x) > k\}$$
is identically equal to zero for k large enough.

45

Lemma 5.2 (Technical lemma). Let $\phi : [0,\infty) \to [0,\infty)$ be non increasing. Suppose $\exists$ constants $C \geq 0$, $\alpha > 0$ and $\boxed{\beta > 1}$ such that

(1) $$\boxed{\phi(h) \leq \frac{C}{(h-k)^\alpha}[\phi(k)]^\beta} \quad \forall h > k \geq 0.$$

Then
$$\phi(d) = 0$$
for
$$d \equiv \left(C\phi(0)^{\beta-1} 2^{\alpha\beta/(\beta-1)} \right)^{1/\alpha}.$$

Proof Define d as above & set
$$k_s = d\left(1 - 1/2^s\right) \qquad s = 0,1,2,\dots$$

By (1) $$\phi(k_{s+1}) \leq \frac{C}{(k_{s+1}-k_s)^\alpha}\phi(k_s)^\beta$$

(2) $$= \frac{C2^{\alpha(s+1)}}{d^\alpha}\phi(k_s)^\beta.$$

Claim

(3) $$\phi(k_s) \leq \frac{\phi(0)}{2^{-s\mu}} \quad \text{for } \mu = \frac{\alpha}{1-\beta};$$

(3) Is clearly true for $s=0$. Assume it's true for some s & we'll prove it for $s+1$

46

By (2)
$$\phi(k_{s+1}) \leq \frac{C2^{\alpha(s+1)}}{d^\alpha}\phi(k_s)^\beta$$
$$\leq \frac{C2^{\alpha(s+1)}}{d^\alpha}\left(\frac{\phi(0)}{2^{-s(\frac{\alpha}{1-\beta})}}\right)^\beta \quad \text{by induction hypothesis}$$
$$= \frac{C\,2^{\alpha(s+1)}}{C\,\phi(0)^{\beta-1}2^{\alpha\beta/\beta-1}}\;\frac{\phi(0)^\beta}{2^{-s(\alpha\beta/1-\beta)}} \quad \text{definition of } d$$
$$= \phi(0)\left(2^{\alpha(s+1)+(s+1)\frac{\alpha\beta}{1-\beta}}\right) = \phi(0)\,2^{\frac{\alpha(s+1)}{1-\beta}}$$
$$= \frac{\phi(0)}{2^{-(s+1)\mu}} \quad \left(\mu = \frac{\alpha}{1-\beta}\right)$$

This proves (3) for all $s = 1,2,\dots$. Since ϕ is $\searrow$ & $k_s \leq d$ $\forall s$, we have
$$0 \leq \phi(d) \leq \lim_{s\to\infty}\phi(k_s) \leq \lim_{s\to\infty}\frac{\phi(0)}{2^{-s\mu}} = 0$$
(since $\beta > 1$ $\Leftrightarrow \mu < 0$) //

Definitions For $k \geq 0$,
$$A(k) \equiv \{x \in \Omega \mid u(x) > k\}$$
$$\phi(k) \equiv \text{mes } A(k)$$
$$(u-k)^+ = \begin{cases} u-k & \text{if } u \geq k \\ 0 & \text{if } u < k \end{cases}$$

5.3 Theorem 5.3 (Global L^∞ estimate).(**)

Let u be a smooth solution of

$$(*)\begin{cases} -(a_{ij}u_{x_i})_{x_j} = (f_i)_{x_i} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume the a_{ij} satisfy the ellipticity condition (E) & that $f_i \in L^p(\Omega)$ $(i = 1, \cdots n)$ for some $\boxed{p > n}$.

Then $\exists C$, depending only on p, n, and θ, $\ni$ *

$$(4) \qquad \boxed{\|u\|_{L^\infty(\Omega)} \leq C \sum_{i=1}^n \|f_i\|_{L^p(\Omega)} \operatorname{mes}(\Omega)^{1/n - 1/p}}$$

Remark The dependence of the bound on $\operatorname{mes}(\Omega)$ will be important in later developments (See p. 68)

Proof Multiply (*) by $(u-k)^+$ (for some $k \geq 0$) & integrate by parts

$$\int_\Omega a_{ij}u_{x_i}(u-k)^+_{x_j}dx = -\int_\Omega f_i (u-k)^+_{x_i} dx$$

Hence, by Lemma 5.1,

$$\int_{A(k)} a_{ij}u_{x_i}u_{x_j}dx = -\int_{A(k)} f_i u_{x_i} dx$$

(**) Consider
$$-\sum \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{m-1}\frac{\partial u}{\partial x_i}\right) = \sum_{i=1}^n \frac{\partial}{\partial x_i}(f_i(x))$$
$u\big|_{\partial\Omega} = 0$. Let $n < m+1$ $\left(\begin{array}{l} n > m+1 \text{ trivial} \\ \text{because of } W^{1,m+1} \subset L^\infty \end{array}\right)$ Then if $f_i \in L^p$ $p > \frac{n}{m}$
$$\Rightarrow \|u\|_{\infty} \leq \left(\sum \|f_i\|_{L^p}\right)^{\frac{1}{m}} (m(\Omega))^{\frac{1}{m} - \frac{1}{mp}}$$

* The difficulty seems to be in fact we assume nothing for a_{ij}. For example consider the $-\Delta u = \sum \frac{\partial f_i}{\partial x_i}$, $u\big|_{\partial\Omega} = 0$, $f_i \in L^p$ instead.

Now, $W^{-1,q} = \{T \,/\, T = \sum_{|\alpha| \leq 1} D^\alpha f_\alpha,\ f_\alpha \in L^q,\ \frac{1}{p} + \frac{1}{q} = 1\}$.

$-\Delta : W^{1,p} \to W^{-1,q}$ isomorphism. Hence solution exists in $W_0^{1,p}$.

Since $p > n$ by Sobolev imbedding $u \in L^\infty(\Omega)$.

Use the ellipticity condition (E) to get

$$\theta \int_{A(k)} |\nabla u|^2 dx \leq \int_{A(k)} |f_i||u_{x_i}| dx$$
$$\leq C \sum_{i=1}^n \left(\int_{A(k)} |f_i|^2 dx\right)^{1/2} \left(\int_{A(k)} |\nabla u|^2 dx\right)^{1/2}.$$

Cancel to get

$$\theta^2 \int_{A(k)} |\nabla u|^2 dx \leq C \sum_{i=1}^n \int_{A(k)} |f_i|^2 dx$$
$$(5) \qquad \leq C \sum_{i=1}^n \left(\int_{A(k)} |f_i|^p dx\right)^{2/p} \underbrace{(\operatorname{mes} A(k))^{1-2/p}}_{= \phi(k)^{1-2/p}}.$$

By Sobolev's inequality,

$$\left(\int_{A(k)} (u-k)^{2^*} dx\right)^{2/2^*} = \left(\int_\Omega (u-k)^{+\,2^*} dx\right)^{2/2^*}$$
$$(6) \qquad \leq C \int_\Omega |\nabla (u-k)^+|^2 dx$$
$$= C \int_{A(k)} |\nabla u|^2 dx \qquad \left(2^* = \frac{2n}{n-2}\right)$$
$$\leq C \sum_{i=1}^n \|f_i\|^2_{L^p(\Omega)} \phi(k)^{1-2/p} \quad \text{by (5)}.$$

$\Delta : W_0^{1,p} \to W^{-1,p}$.

$(W_0^{1,p})^* = W^{-1,q}$.

Irrelevant but Important

In the $L^1 \Rightarrow L^\infty$ estimate we go through for parabolic operator satisfying (E) but do not need smoothness.

Now let $h > k$; then

$$(h-k)^2 (\text{mes } A(h))^{2/2^*} \le \left(\int_{A(h)} (u-k)^{2^*} dx\right)^{2/2^*}$$

$$\le \left(\int_{A(k)} (u-k)^{2^*} dx\right)^{2/2^*}$$

$$\le C \sum_{i=1}^{n} \|f_i\|^2_{L^p(\Omega)} \phi(k)^{1-2/p} \quad \text{by (6)}$$

Thus

$$(7) \quad \phi(h) \le C \sum_{i=1}^{n} \|f_i\|^{2^*}_{L^p(\Omega)} \frac{\phi(k)^\beta}{(h-k)^\alpha}$$

for $\alpha = 2^*$ & $\beta = (1-2/p)\frac{2^*}{2} = \frac{1-2/p}{1-2/n}$.

Since $p > n$, $\beta > 1$ & so Lemma 5.2 applies:

$$\phi(d) = 0 \text{ for } d \le C \sum_{i=1}^{n} \|f_i\|_{L^p} \phi(0)^{\frac{\beta-1}{\alpha}}.$$

But $\frac{\beta-1}{\alpha} = \frac{1}{n} - \frac{1}{p}$ & so $\phi(0)^{\beta-1/\alpha} \le \text{mes}(\Omega)^{1/n - 1/p}$.

This proves the stated estimate as an upper bound on u in Ω; the same method applies to $-u$ to give a lower bound //

Remark(*) A different proof, based on the isoperimetric inequality, is given in H.F. Weinberger, Symmetrization in uniformly elliptic problems, in Studies in Math Analysis & Related Topics, Stanford U. Press

The $p=n$ case can be shown with different argument (~~…~~ not ~~…~~ Stampacchia's) and obtain the boundedness of Orlicz norm. See notes.

5.4 C. Local L^∞ estimates for subsolutions

The key idea for proving that a solution u of (*) is Hölder continuous depends upon various local L^∞ estimates for certain subsolutions of (*).

Here we follow:

J. Moser, A new proof of de Giorgi's theorem concerning the regularity problem for elliptic differential equations, Comm. Pure Appl. Math. 13 (1960), p. 457-468.

$\boxed{\text{Hyp: } u \in W^{1,2}_{loc}(\Omega)}$

We first consider the case that $f_i = 0$ (so u solves

$$(*) \qquad -(a_{ij} u_{x_i})_{x_j} = 0 \quad \text{in } \Omega \qquad);$$

but we make no assumptions about $u|_{\partial\Omega}$.

Definition v is called a subsolution of (*) if

$$-(a_{ij} v_{x_i})_{x_j} \le 0 \quad \text{in } \Omega.$$

Lemma 5.4 If u solves (*) & ϕ is convex, smooth, then

$$v = \phi(u)$$

is a subsolution of (*).

Comment: cutting and Fatou's is involved in a number of places.

Proof

$$-(a_{ij} v_{x_i})_{x_j} = -(\phi'(u) a_{ij} u_{x_i})_{x_j}$$

$$= -\phi'(u) \underbrace{(a_{ij} u_{x_i})_{x_j}}_{=0} - \phi''(u) a_{ij} u_{x_i} u_{x_j} \le 0 \text{ by (E)} \quad //$$

Assume a priori that $v, \nabla v$ are $L^2(\Omega)$ (i.e. $v \in W^{1,2}(\Omega)$)

v subsolution $\Longrightarrow \forall \phi \in C_c^\infty(\Omega)$, $\phi \ge 0$; we have $\int a_{ij} v_{x_i} \phi_{x_j} dx \le 0$, (a_{ij}) elliptic (E). By density this is o.k. $\forall \phi \in W_0^{1,2}$

Definition $B(x_0,R)$ = ball of radius R, centered at x_0

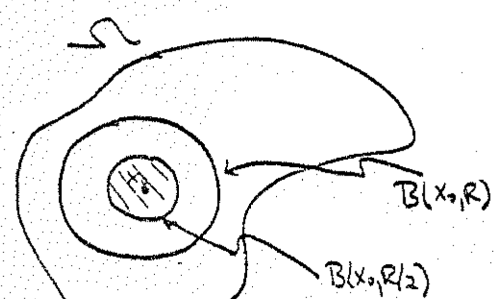
Theorem 5.5 Let v be a nonnegative subsolution of (*), the coefficients a_{ij} of which satisfy the ellipticity condition (E).

Choose any $x_0 \in \Omega$ & $R>0 \ni B(x_0,R) \subset \Omega$.

Then $\exists\, C$, depending only on n, θ, and $\Theta \ni$

$$\boxed{\max_{B(x_0,R/2)} v \leq C\left[\frac{1}{R^n}\int_{B(x_0,R)} v^2\,dx\right]^{1/2}}$$

Ω



The L^∞ norm of v in the smaller ball is estimated by the L^2 norm on the larger ball.

Proof (Moser iteration method)

Idea We will consider an infinite sequence of balls $B(x_0,R_k)$ between $B(x_0,R)$ & $B(x_0,R/2)$. We estimate the L^p norm in each ball by a slightly "weaker" L^q norm in the next larger ball & then pass to limits.

Let $p \geq 2$ & $\zeta \in C_0^\infty(\Omega)$, $0 \leq \zeta \leq 1$. We have

$$-(a_{ij}v_{x_i})_{x_j} \leq 0 \text{ in } \Omega;$$

multiply by $\zeta^2 v^{p-1}$ & integrate by parts:

$$\int_\Omega a_{ij}v_{x_i}(\zeta^2 v^{p-1})_{x_j}\,dx \leq 0.$$

(Here let $g(s) = \begin{cases} s^{p-1}, & |s| \leq \\ M & \text{otherwise} \end{cases}$ $\Rightarrow g(v) \in W_0^{1,2}$. Then $\epsilon \to 0$.)

Therefore

$$(p-1)\int_\Omega a_{ij}v_{x_i}v_{x_j}v^{p-2}\zeta^2\,dx \leq -2\int_\Omega \underbrace{a_{ij}v_{x_i}v^{p-1}\zeta\zeta_{x_j}}_{*}\,dx$$

By (E) & Cauchy's inequality with ε,

$$(p-1)\,\theta\int_\Omega |\nabla v|^2 v^{p-2}\zeta^2\,dx \leq \varepsilon\int_\Omega |\nabla v|^2 v^{p-2}\zeta^2\,dx + \frac{C}{\varepsilon}\int_\Omega |\nabla\zeta|^2 v^p\,dx$$

Set $\varepsilon = \frac{(p-1)\theta}{2}$ to get

$$(7)\quad \frac{(p-1)\theta}{2}\int_\Omega |\nabla v|^2 v^{p-2}\zeta^2\,dx \leq \frac{2C}{(p-1)\theta}\int_\Omega |\nabla\zeta|^2 v^p\,dx.$$

But $|\nabla v|^2 v^{p-2} = \frac{4}{p^2}|\nabla(v^{p/2})|^2$ & $\Rightarrow \frac{(p-1)^2\theta^2}{4}\int.$

so $\frac{(p-1)^2\theta^2}{4}|\nabla v|^2 v^{p-2} \geq |\nabla(v^{p/2})|^2.$ $\leq C\int$

Plug this into (7) to obtain

$\left(\frac{(p-1)^2\theta^2}{4} \geq \frac{p^2}{4}\right)$

$K = \frac{1}{\theta}\left(\frac{p}{p-1}\right)^2$ essent. indep of p.

$$* \quad a_{ij}v_{x_i}v^{p-1}\zeta\zeta_{x_j} = a_{ij}v_{x_i}v^{\frac{p-2}{2}}\zeta\, v^{\frac{p}{2}}\zeta_{x_j}$$

$$\leq \varepsilon |\nabla v|^2 v^{p-2}\zeta^2 + \frac{1}{\varepsilon}v^p|\nabla\zeta|^2$$

$$\int_\Omega \zeta^2 |\nabla(v^{p/2})|^2 dx \le C\int_\Omega |\nabla \zeta|^2 v^p dx$$

& so $\nabla(z v^{p/2}) = (\nabla z) v^{p/2} + z\nabla(v^{p/2}) \Rightarrow |\nabla(z v^{p/2})|^2 \le 2|\nabla z|^2 v^p + 2z^2|\nabla(v^{p/2})|^2$

$$\int_\Omega |\nabla(\zeta v^{p/2})|^2 dx \le C\int_\Omega |\nabla \zeta|^2 v^p dx.$$

Now use Sobolev's inequality:

$$\left(\int_\Omega (\zeta v^{p/2})^{2^*} dx\right)^{2/2^*} \le C\int_\Omega |\nabla(\zeta v^{p/2})|^2 dx$$

$$(8) \qquad \le C\int_\Omega |\nabla \zeta|^2 v^p dx.$$

This inequality holds $\forall p \ge 2$, $\forall \zeta \in C_0^\infty$ & the constant C does not depend on ζ or p.

Now we iterate inequality (8) for various choices of p & ζ.

Define

$$\boxed{R_k \equiv \frac{R}{2}\left(1 + \frac{1}{2^k}\right)} \qquad (k = 0, 1, 2, \dots)$$

& choose $\zeta = \zeta_k \ni$

$$\zeta = \begin{cases} 1 & \text{on } B(x_0, R_{k+1}) \\ 0 & \text{on } \Omega \setminus B(x_0, R_k), \end{cases} \qquad 0 \le \zeta \le 1,$$

and

$$(9) \qquad |\nabla \zeta| \le \frac{2}{R_k - R_{k+1}} = \frac{2^{k+3}}{R}.$$

By (8) & (9), we get $\left(2^* = \frac{2n}{n-2}\right)$

$$(10) \qquad \left(\int_{B(x_0, R_{k+1})} v^{\frac{pn}{n-2}} dx\right)^{\frac{n-2}{n}} \le \frac{C 4^k}{R^2} \int_{B(x_0, R_k)} v^p dx.$$

Now define $\boxed{p_k = 2\left(\frac{n}{n-2}\right)^k}$ $\quad k = 0, 1, 2, \dots$

Take p^{th} roots on both sides of (10) & set $p = p_k$:

$$(11) \qquad \left(\int_{B(x_0, R_{k+1})} v^{p_{k+1}} dx\right)^{1/p_{k+1}} \le \frac{C^{1/p_k} 4^{k/p_k}}{R^{2/p_k}} \left(\int_{B(x_0, R_k)} v^{p_k} dx\right)^{1/p_k}$$

Set

$$a_k \equiv \|v\|_{L^{p_k}(B(x_0, R_k))}$$

and

$$\gamma_k \equiv \frac{C^{1/p_k} 4^{k/p_k}}{R^{2/p_k}};$$

then (11) says $\boxed{a_{k+1} \le \gamma_k a_k}$ $\quad k = 0, 1, 2, \dots$

Iterate this estimate for $k = 0, 1, \dots n$:

$$a_{n+1} \le \gamma_0 \gamma_1 \cdots \gamma_n a_0,$$

(12) $$a_{n+1} \le \left[\frac{C^{\sum_{k=1}^{n} 1/p_k}\, 4^{\sum_{k=0}^{n} k/p_k}}{R^{\sum_{k=0}^{n} 2/p_k}} \right] a_0 .$$

Now let $n \to \infty$ in (12). Since $\sum_{k=0}^{\infty} 1/p_k < \infty$ & $\sum_{k=0}^{\infty} k/p_k < \infty$ (ratio test), the constants on the right have a finite limit. In fact

$$\sum_{k=0}^{\infty} 1/p_k = \frac{1}{2}\sum_{k=0}^{\infty} \frac{1}{\left(\frac{n}{n-2}\right)^k} = \frac{1}{2}\cdot\frac{n}{2} = \frac{n}{4}.$$

Now $$a_0 = \left(\int_{B(x_0,R)} v^2\, dx\right)^{1/2}$$

and $$\lim_{n\to\infty} a_n = \|v\|_{L^\infty(B(x_0,R/2))}.$$

Therefore passing to limits in (12) gives

$$\|v\|_{L^\infty(B(x_0,R/2))} \le \frac{C}{R^{n/2}}\left(\int_{B(x_0,R)} v^2\, dx\right)^{1/2} \qquad //$$

$(\bar{a}_{ij}(|Du|^2)\, u_{x_i})_{x_j}$

$\bar{a}_{ij} = \delta_{ij}\left(\frac{\partial u}{\partial x_i}\right)^2$

$a_{ij}(|Du|^2)$

Corollary 5.6 Suppose u solves $(*)$. Then

$$\max_{B(x_0,R/2)} |u| \le C\left[\frac{1}{R^n}\int_{B(x_0,R)} u^2\, dx\right]^{1/2}$$

Proof Apply Theorem 5.5 to $v = \phi(u)$ for $\phi(x) = |x|$; ϕ is convex & so v is a subsolution (Lemma 5.4) //

~~but need a~~ ~~are subsolution~~. smoothening involved. See comment

5.5 D. Technical lemmas

Definition Let $x_0 \in \Omega$. Choose $R_0 > 0$ ∋ $B(x_0,R_0) \subset \Omega$. Set

$$\boxed{\omega(R) = \max_{B(x_0,R)} u - \min_{B(x_0,R)} u} \qquad 0 < R \le R_0$$

ω is the oscillation of u (with respect to x_0)

Definition $\Omega' \subset \Omega$ is said to be compactly contained in Ω if $\bar{\Omega}' \subset \Omega$. We write $\Omega' \subset\subset \Omega$.

Definition $u : \Omega \to \mathbb{R}$ is called locally Hölder continuous with exponent γ if $\forall\, \Omega' \subset\subset \Omega$, $\exists$ a constant C, ∋

$$\boxed{\omega(R) \le C R^\gamma} \qquad \forall\, 0 < R, \text{ sufficiently small}$$

Here the oscillation is computed w.r.t. any $x_0 \in \Omega'$.

Lemma 5.7 Suppose $\exists$ constants $C_1 \ge 0$, $0 < \alpha \le 1$, and $\boxed{0 < \eta < 1}$ such that

(13) $$\boxed{\omega(R/4) \le \eta\, \omega(R) + C_1 R^\alpha} \qquad \forall\, 0 < R \le R_0 < 1.$$

Then $\exists$ $0 < \gamma < 1$ & $C_2 \ge 0$, depending only on

Exercise: Consider the ($m > 1$)

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{m-1} \frac{\partial u}{\partial x_i}\right) = 0 \qquad (*)$$

(a) Prove a Lemma 5.4 for this

(b) Prove that if v is subsol. ≥ 0 then $$\max_{B(x_0,R/2)} v \le c\left(\frac{1}{R}\right)^{\frac{n}{m+1}}\left(\int_{B(x_0,R)} v^{m+1}\, dx\right)^{\frac{1}{m+1}}$$

Hints: (i) Multiply by $z^{m+1} v^{p-m}$

(ii) Use identity $$\left|\frac{\partial v}{\partial x_i}\right|^{m+1} v^{p-m+1} = \left(\frac{m+1}{p}\right)^{m+1} \cdot \left|\frac{\partial}{\partial x_i}\left(v^{p/(m+1)}\right)\right|^{m+1}$$

C_1, α, η, $\sup \omega(R)$, such that

$$\boxed{\omega(R) \le C_2 \left(\frac{R}{R_0}\right)^\gamma} \quad \forall\, 0 < R \le R_0$$

If, in particular, the inequalities (13) hold for $\omega = \operatorname{osc} u$, u is locally Hölder continuous with exponent γ.

Proof Pick any $\eta < a < 1$ & then choose $0 < \beta < 1 \ni$ $4^\beta \eta = a < 1$. Set $\gamma = \min(\alpha, \beta)$.

Define $M \equiv \sup_{\frac{R_0}{4} \le R \le R_0} \frac{\omega(R)}{R^\gamma}$; then

(14) $\omega(R) \le M R^\gamma$ for $\frac{R_0}{4} \le R \le R_0$.

Now let $\frac{R_0}{4^2} \le R \le \frac{R_0}{4}$; (13) & (14) imply

$$\begin{aligned} \omega(R) &\le \eta\, \omega(4R) + C_1 (4R)^\alpha \\ &\le \eta M (4R)^\gamma + C_1 4^\alpha R^\alpha \\ &\le [M 4^\gamma \eta + C] R^\gamma . \end{aligned}$$

this C taken care of j=0 term. (i.e. $C \ge C_1 4^\alpha$.)

In general, if $\frac{R_0}{4^{i+1}} \le R \le \frac{R_0}{4^i}$, we have (inductively)

$$\begin{aligned} \omega(R) &\le \left[M (4^\gamma \eta)^i + C \sum_{j=0}^{i-1} (4^\gamma \eta)^j\right] R^\gamma \\ &\le \left[M (\underbrace{4^\beta \eta}_{=a})^i + C \sum_{j=0}^{\infty} (\underbrace{4^\beta \eta}_{=a})^j\right] R^\gamma \\ &\le \left[M + \frac{C}{1-a}\right] R^\gamma \\ &\le C_2 \left(\frac{R}{R_0}\right)^\gamma \quad \| \end{aligned}$$

(*) In general: $m \ge 1$
$$\int_{B(x_0;R)} u^{m+1}\, dx \le C \left(\frac{R^n}{m(N)}\right)^{m+1} R^{m+1} \int_{B(x_0;R)} |\nabla u|^{m+1}\, dx$$

5.6 **Lemma 5.8** Suppose $N \subset B(x_0,R)$ is a subset of positive measure on which $u \equiv 0$. Then $\exists\, C \ni$

R^{n+1}

$$(*) \quad \boxed{\int_{B(x_0,R)} u^2\, dx \le C \left(\frac{\cancel{R^n}}{\operatorname{mes}(N)}\right)^2 R^2 \int_{B(x_0,R)} |\nabla u|^2\, dx}$$

C depends only on n & not on u or R.

Proof WLOG $x_0 = 0$. Pick any $x \in B(R) = B(0,R)$ & $y \in N$. We convert to polar coordinates centered at x. Then $y = x + r\xi$ for some $|\xi| = 1$ & $r = |x-y|$.

Therefore

$$\begin{aligned} u(x) = u(x) - u(y) &= -\int_0^r \frac{d}{dt} u(x+t\xi)\, dt \\ &= -\int_0^r u_{x_i}(x+t\xi)\, \xi_i\, dt ; \end{aligned}$$

and so

$$|u(x)| \le \int_0^r |\nabla u(x+t\xi)|\, dt .$$

We integrate this inequality wrt y over N:

$$|u(x)|\,\mathrm{mes}(N) \le \int_N \left(\int_0^r |\nabla u|\,dt \right) dy$$

$$\le \int_{B(R)} \left(\int_0^r |\nabla u|\,dt \right) dy.$$

Now write $dy = r^{n-1}\,dr\,d\sigma$ ($d\sigma$ = surface element on unit sphere)

Then

$$|u(x)|\,\mathrm{mes}(N) \le \int_0^R \int_{|\xi|=1} \left(\int_0^r |\nabla u(x+t\xi)|\,dt \right) d\sigma\, r^{n-1}\,dr.$$

like polar coordinates, ξ = angle

Write $z = x + t\xi$, $t = |x-z|$; this gives $dz = t^{n-1}\,dt\,d\theta$.

$$|u(x)|\,\mathrm{mes}(N) \le \int_0^R \left(\int_0^r \int_{|\xi|=1} \frac{|\nabla u(z)|}{|x-z|^{n-1}} \underbrace{t^{n-1}\,d\sigma\,dt}_{dz} \right) r^{n-1}\,dr$$

$$\le \frac{R^n}{n} \left(\int_{|z|\le R} \frac{|\nabla u(z)|}{|x-z|^{n-1}}\,dz \right).$$

Divide by $\mathrm{mes}(N)$ & then integrate wrt x over B_R:

$$\int_{B(R)} |u(x)|\,dx \le \frac{CR^n}{\mathrm{mes}(N)} \int_{|x|\le R} \left(\int_{|z|\le R} \frac{|\nabla u(z)|}{|x-z|^{n-1}}\,dz \right) dx$$

(15)

$$\le \frac{CR^n}{\mathrm{mes}(N)} \sup_{|z|\le R} \left(\int_{|x|\le R} \frac{dx}{|x-z|^{n-1}} \right) \int_{B(R)} |\nabla u(z)|\,dz$$

$$\frac{CR^n}{m(N)} \int_{|z|\le R} \left(\int_{|x|\le R} \frac{dx}{|x-z|^{n-1}} \right) |\nabla u(z)|\,dz \le$$

$$\le \left(\sup_{|z|\le R} \int_{|x|\le R} \frac{dx}{|x-z|^{n-1}} \right) \left(\int_{|z|\le R} |\nabla u(z)|\,d\ldots \right)$$

But $\displaystyle\int_{|x|\le R} \frac{dx}{|x-z|^{n-1}} \le \int_{|x|\le 2R} \frac{dx}{|x|^{n-1}} \le CR$

& so (15) gives

$$\int_{B(R)} |u(x)|\,dx \le \frac{CR^{n+1}}{\mathrm{mes}(N)} \int_{B(R)} |\nabla u(x)|\,dx.$$

Slight mistake. but $g(s) = \begin{cases} |\nabla u| \\ 0 \text{ if } s \in B(0,R) \end{cases}$

Now replace u by u^2 in this inequality:

$$\int_{B(R)} u^2\,dx \le \frac{CR^{n+1}}{\mathrm{mes}(N)} \int_{B(R)} |u|\,|\nabla u|\,dx$$

$$\le \frac{1}{2} \int_{B(R)} u^2\,dx + \frac{CR^{2n+2}}{\mathrm{mes}(N)^2} \int_{B(R)} |\nabla u|^2\,dx$$

//

$\boxed{5.7}$ E. Interior Hölder Continuity

We now consider a solution u of

$$(*) \qquad -(a_{ij}u_{x_i})_{x_j} = 0 \quad \text{in } \Omega$$

& derive the key inequality —

Proposition 5.9 Assume that $0 \le u \le 1$ is a solution of $(*)$ in $B(x_0, 2R) \subset \Omega$.

Suppose that

$u \in W^{1,2}_{loc}$ suffices for 5.9

$$\boxed{\operatorname{mes}\{x \in B(x_0,R) \mid u(x) \geq 1/2\} \geq \frac{1}{2}\operatorname{mes} B(x_0,R)} \quad (*)$$

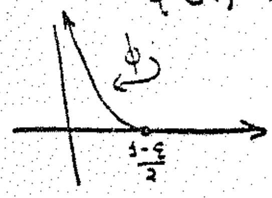
Then $\exists\, C > 0$, depending only on n, θ & Θ, $\ni$

$$\boxed{\min_{B(x_0,R/2)} u \geq C > 0}$$

Remark So if u is $\geq 1/2$ on more than half of $B(x_0,R)$, it's bounded away from 0 on the smaller ball $B(x_0,R/2)$.

Proof Let $\varepsilon > 0$. Define

$$\phi(x) \equiv \max(-\log(2x+\varepsilon), 0)$$



Then for $x > \frac{1-\varepsilon}{2}$, $\phi \equiv 0$ & for $x < \frac{1-\varepsilon}{2}$,

$$\phi(x) = -\log(2x+\varepsilon)$$
$$\phi'(x) = \frac{-2}{2x+\varepsilon}$$
$$\phi''(x) = \frac{4}{(2x+\varepsilon)^2} = (\phi'(x))^2$$

Hence ϕ is convex, $\phi \geq 0$, and

(16) $$\phi'' = (\phi')^2.$$

Choose ζ^2 to be a cutoff function with support in $B(2R)$ (WLOG $x_0 = 0$).

(*) This can be proved for solutions of $\sum \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{m-1}\frac{\partial u}{\partial x_i}\right) = 0$

Multiply (*) by $\phi'(u)\zeta^2$ & integrate by parts

(17) $$0 = \int_\Omega a_{ij} u_{x_i} (\phi'(u)\zeta^2)_{x_j} = \int_\Omega a_{ij} u_{x_i} u_{x_j} \phi''(u)\zeta^2 + 2a_{ij} u_{x_i}\phi'(u)\zeta_{x_j}\zeta\, dx$$

Define $v = \phi(u)$;

then

$$\int_\Omega a_{ij} v_{x_i} v_{x_j} \zeta^2 dx = \int_\Omega a_{ij} u_{x_i} u_{x_j} (\phi')^2 \zeta^2$$
$$= \int_\Omega a_{ij} u_{x_i} u_{x_j} \phi'' \zeta^2 \quad \text{by (16)}$$
$$= -2\int_\Omega a_{ij} v_{x_i} \zeta_{x_j} \zeta \quad \text{by (17)}.$$

From (E) & Cauchy's inequality we get

$$\theta \int_\Omega |Dv|^2 \zeta^2 dx \leq \frac{\theta}{2}\int_\Omega |Dv|^2 \zeta^2 dx + C\int_\Omega |D\zeta|^2 dx.$$

Now choose $\zeta \equiv 1$ on $B(R)$, $|D\zeta| \leq 2/R$. Then this implies

(18) $$\int_{B(R)} |Dv|^2 dx \leq C R^{n-2}.$$

Now we collect various inequalities proved earlier:

Since ϕ is convex, $v = \phi(u)$ is a subsolution (Lemma 5.4). Hence Theorem 5.5 implies

$$(19) \quad \max_{B(R/2)} v^2 \leq \frac{C}{R^n} \int_{B(R)} v^2 \, dx .$$

By hypothesis, $v = \phi(u) = 0$ for

$$x \in N \equiv \{x \in B(R) \mid u(x) \geq 1/2\} \ \&$$

$$\text{mes}(N) \geq \frac{1}{2} \text{mes}\, B(R)$$

We apply Lemma 5.8 to discover

$$(20) \quad \int_{B(R)} v^2 \, dx \leq C R^2 \int_{B(R)} |\nabla v|^2 \, dx .$$

Now combine inequalities (18)–(20):

$$\max_{B(R/2)} v^2 \leq \frac{C}{R^n} \int_{B(R)} v^2 \, dx \leq \frac{C}{R^{n-2}} \int_{B(R)} |\nabla v|^2 \, dx$$

$$\leq \frac{C}{R^{n-2}} \cdot R^{n-2} = C;$$

the constant does not depend on R or ε.

By the definition of v and u, we have

$$-\log(2u(x) + \varepsilon) \leq C \quad \forall x \in B(R/2)$$

$$\therefore \quad u(x) + \frac{\varepsilon}{2} \geq \frac{e^{-C}}{2} > 0 \quad \forall x \in B(R/2).$$

Let $\varepsilon \to 0$ & the Proposition is proved. ||

$\boxed{5.8}$ Theorem 5.10 (de Giorgi–Moser) Let u be solution of

$$(*) \quad -(a_{ij} u_{x_i})_{x_j} = 0 \quad \text{in } \Omega,$$

the coefficients of which satisfy the ellipticity condition (E).

Let $\Omega' \subset\subset \Omega$. Then $\exists\, C \geq 0$ and $0 < \gamma < 1$ such that

$$\boxed{|u(x) - u(y)| \leq C|x-y|^\gamma}^{(*)} \quad \forall x, y \in \Omega'.$$

C depends only on θ, Θ, n, $\|u\|_2$, and $\text{dist}(\Omega', \partial\Omega)$; γ depends on the same quantities, except for $\text{dist}(\Omega', \partial\Omega)$.

Hence a solution of $(*)$ is uniformly Hölder continuous on compact subsets of Ω, irrespective of the behavior of $u|_{\partial\Omega}$.

(*) True for solution of $\qquad m > 1$

$$-\sum \frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{m-1} \frac{\partial u}{\partial x_i} \right) = 0$$

<u>Proof</u> Pick any $x_0 \in \Omega'$ & let

$$\omega(R) = \max_{B(x_0,R)} u - \min_{B(x_0,R)} u$$

be the oscillation of u wrt x_0. Set $R_0 = \frac{1}{2}\operatorname{dist}(x_0, \partial\Omega)$ & consider only $0 < R \le R_0$.

By Corollary 5.6, u is bounded in Ω' & so we may WLOG assume

$$(21) \qquad \max_{B(x_0,R)} u = 1, \quad \min_{B(x_0,R)} u = 0$$

(If not, then notice that if u solves $(*)$, so does $\tilde{u} \equiv a(u+b)$ $\forall a,b \in \mathbb{R}$. Adding b does not change the oscillation. Multiplication by a <u>does</u> change the oscillation, but in the main estimates (22) & (23) the effects of this multiplication cancel out.)

By (21) we know that <u>either</u> <u>u</u> <u>or</u> <u>$1-u$</u> (also a solution of $(*)$) satisfies the hypothesis of Proposition 5.9, depending upon whether $u \ge 1/2$ or $u \le 1/2$ more often in $B(x_0, R/2)$.

$\boxed{\text{Case 1}}$ $\operatorname{mes}\{x \in B(x_0, R/2) \mid u(x) \ge 1/2\} \ge \frac{1}{2} \operatorname{mes} B(x_0, R/2)$.

Then, by Proposition 5.9, $\exists\, C > 0 \ni$

$$u(x) \ge C \qquad \forall x \in B(x_0, R/4); \text{ ie}$$

$$\min_{B(x_0,R/4)} u \ge C$$

Hence

$$\omega(R/4) = \max_{B(x_0,R/4)} u - \min_{B(x_0,R/4)} u$$

$$(22) \qquad \le \max_{B(x_0,R)} u - C \le 1 - C$$

$$= \eta\, \omega(R) \quad \text{for } \eta \equiv 1 - C < 1$$

(independent of R).

$\boxed{\text{Case 2}}$ $\operatorname{mes}\{x \in B(x_0, R/2) \mid 1-u \ge 1/2\} \ge 1/2 \operatorname{mes} B(x_0, R/2)$.

Since $1-u$ solves $(*)$, Proposition 5.9 implies

$$1 - u(x) \ge C > 0 \quad \forall x \in B(x_0, R/4), \text{ ie}$$

$$\max_{B(x_0,R/4)} u \le 1 - C.$$

Hence

$$\omega(R/4) = \max_{B(x_0,R/4)} u - \min_{B(x_0,R/4)} u$$

$$(23) \qquad \le 1 - C - \min_{B(x_0,R)} u$$

$$= 1 - C = \eta\, \omega(R) \qquad \eta = 1 - C.$$

Now, inequalities (22) & (23) both imply

$$\omega(R/4) \le \eta\, \omega(R) \quad \forall\, 0<R\le R_0,\ \eta<1.$$

$C_1 = 0$ here.

By Lemma 5.7, therefore, $\exists\, C$ & $0<\gamma<1$ $\ni$

$$(24) \quad \omega(R) \le C\left(\frac{R}{R_0}\right)^{\gamma} \quad \forall\, 0<R\le R_0 = \tfrac{1}{2}\operatorname{dist}(x_0,\partial\Omega)$$

Hence if $x,y\in\Omega'$, $|x-y| = R \le R_0^* = \frac{1}{2}\operatorname{dist}(\Omega',\partial\Omega)$, we have the estimate

$$|u(x)-u(y)| \le \frac{C}{R_0^{*\gamma}}|x-y|^{\gamma} \le C|x-y|^{\gamma}$$

If $|x-y| \ge R_0^*$, then

$$|u(x)-u(y)| \le 2\max|u| \le C|x-y|^{\gamma} \text{ as well.}$$

$\|$ $*$

$\boxed{5.9}$ **Theorem 5.11** (*) (Stampacchia) Suppose u solves

$$(*) \quad -(a_{ij}u_{x_i})_{x_j} = (f_i)_{x_i} \text{ in } \Omega,$$

where the a_{ij} satisfy the ellipticity condition (E) & the $f_i \in L^p(\Omega)$ for some $\boxed{p>n}$. Let $\Omega' \subset\subset \Omega$.

Then $\exists\, C\ge 0$ & $0<\gamma<1$ such that

$$\boxed{|u(x)-u(y)| \le C|x-y|^{\gamma}} \quad \forall x,y\in\Omega'.$$

C depends on θ, Θ, n, $\|u\|_{L^2}$, $\|f_i\|_{L^p}$, and $\operatorname{dist}(\Omega',\partial\Omega)$; γ depends on all these quantities except for $\operatorname{dist}(\Omega',\partial\Omega)$.

Proof Write $u = v+w$, where v solves

$$(**) \quad \begin{cases} -(a_{ij}v_{x_i})_{x_j} = (f_i)_{x_i} & \text{in } B(x_0,2R) \\ v = 0 & \text{on } \partial B(x_0,2R) \end{cases}$$

(‡) and w solves

$$-(a_{ij}w_{x_i})_{x_j} = 0 \quad \text{in } B(x_0,2R).$$

By Theorem 5.3,

$$(25) \quad \|v\|_{L^\infty(B(x_0,2R))} \le C\left[\operatorname{mes} B(x_0,2R)\right]^{1/n-1/p} = C R^{1-n/p}.$$

Furthermore the proof of Theorem 5.10 gives

$$(26) \quad \bar\omega_w(R/4) \le \eta\, \bar\omega_w(R) \quad \forall\, 0<R\le R_0,$$

where $\bar\omega_w$ = oscillation of w.

(*) Result trivial for Δ since $(f_i)_{x_i} \in W^{-1,p}$

$$\Delta: W^{1,p} \to W^{-1,p} \text{ isom.}$$

so $\Delta^{-1}(f_i)_{x_i} \in W^{1,p} \hookrightarrow C^{\gamma}$ since $p>n$

$(0\le\gamma\le k-n/p \Rightarrow W^{k,p} \hookrightarrow C^{\gamma}(\Omega))$

~~(‡) Restrict u on $\partial B(x_0,2R)$. Then~~

(**) Here we assume solvability and we are concerned with regularity. Also u is assume weak solution of (*) in some Sobolev space. Given u let w be defined as follows: $\left.\begin{array}{l} -(a_{ij}u_{x_i})_{x_j} = 0 \\ w|_{\partial B} = u|_{\partial B} \end{array}\right\}$ ← solvability of this is assumed

Then define $v = u - w$. Clearly v satisfies (**) above

Hence

$$\omega(R/4) \leq \omega_v(R/4) + \omega_w(R/4)$$
$$\leq CR^{1-n/p} + \omega_w(R/4) \quad \text{by } (25)$$
$$\leq CR^{1-n/p} + \eta\, \omega_w(R) \quad \text{by } (26) \quad \left(\text{pass again from } w \text{ to } u, v\right) \quad (28)$$
$$\leq CR^{1-n/p} + \eta\, \omega(R) \quad \text{by } (25) \text{ again.}$$

Now $\alpha = 1 - n/p > 0$ since $p > n$. Therefore the hypotheses of Lemma 5.7 hold, and so

(27) $$\omega(R) \leq C\left(\frac{R}{R_0}\right)^{\gamma} \quad \forall\, 0 < R \leq R_0 = \tfrac{1}{2}\operatorname{dist}(x_0, \partial\Omega)$$

for some C & γ.

The rest of the proof is similar to that given before. ||

5.10 F. Hölder continuity near $\partial\Omega$

Next we prove that a solution u of

$$(*)\begin{cases} -(a_{ij}u_{x_i})_{x_j} = (f_i)_{x_i} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

is Hölder continuous on all of $\bar{\Omega}$, provided $f_i \in L^p$ ($p > n$) and $\partial\Omega$ satisfies a weak regularity

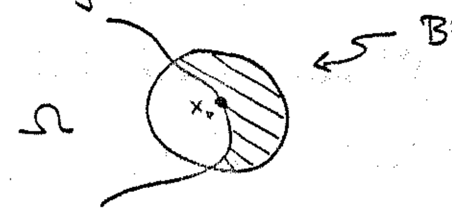
condition:

Definition $\partial\Omega$ is admissible if $\exists\, A > 0$ & $R_0 > 0$

$$\boxed{\frac{\operatorname{mes}\left[B(x_0,R) \cap \Omega^c\right]}{\operatorname{mes} B(x_0,R)} \geq A}$$

$\forall x_0 \in \partial\Omega$, $\forall\, 0 < R \leq R_0$ (Ω^c = complement of Ω).

This says Ω^c is not "thin" at $x_0 \in \partial\Omega$.



Theorem 5.12 Let $x_0 \in \partial\Omega$. Suppose that $v \geq 0$ and

$$-(a_{ij}v_{x_i})_{x_j} \leq 0 \quad \text{in } \Omega$$

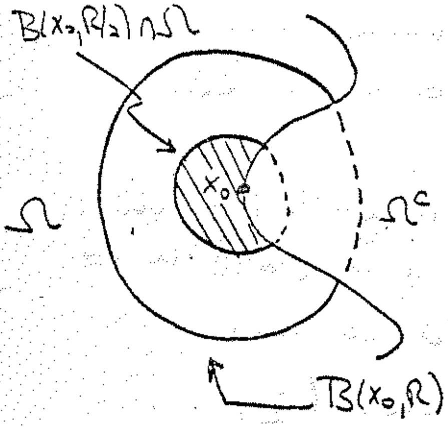
(i.e. v is a subsolution of $(*)$). Assume also that

$$v = 0 \quad \text{on } \partial\Omega \cap B(x_0, R_0)$$

for some $R_0 > 0$.

Then $\exists\, C$, depending only on n, θ, and Θ, $\ni$

$$\boxed{\max_{B(x_0,R/2)\cap\Omega} v \leq C\left[\frac{1}{R^n}\int_{B(x_0,R)\cap\Omega} v^2\,dx\right]^{1/2}} \quad \forall\, 0<R\leq R_0$$



The L^∞ norm of v in the smaller ball is estimated by the L^2 norm in the larger ball

Proof This follows by the Moser iteration method almost exactly as in the proof of Theorem 5.5 (p. 51). We choose the cut off functions ζ in the same way – Then the ζ don't vanish along $B(x_0,R)\cap\partial\Omega$; but since $v=0$ there, there are no boundary terms when we integrate by parts. Similarly the use of the Sobolev inequality (p. 53) is OK, since the product $\zeta v^{p/2}$ vanishes on $\partial\Omega$. ||

Definition Let $x_0\in\partial\Omega$.

(a) $\omega^+(R) = \max_{B(x_0,R)\cap\Omega} u$

(b) $\omega^-(R) = \min_{B(x_0,R)\cap\Omega} u$

Since we'll assume $u=0$ on $\partial\Omega$, we have

$$\omega^+\geq 0,\quad \omega^-\leq 0$$

5.11 **Proposition 5.13** Suppose that $\partial\Omega$ is admissible and $x_0\in\partial\Omega$. Assume that u solves

$$(*)\quad \begin{cases} -(a_{ij}u_{x_i})_{x_j} = 0 & \text{in } B(x_0,R_0)\cap\Omega \\ u=0 & \text{on } B(x_0,R_0)\cap\partial\Omega \end{cases}$$

for some $R_0>0$.
Choose $0<R<R_0/2$. If

$$\max_{B(x_0,2R)\cap\Omega} u = 1,$$

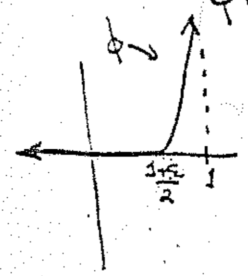
then $\exists\, C>0$, depending only on n,θ,Θ, and A $\ni$

$$\boxed{\max_{B(x_0,R/2)\cap\Omega} u \leq 1-C}$$

Proof This is similar to the proof of Proposition 5.9 (p. 60).

Define

$$\phi(x) \equiv \max\left(-\log(2(1-x)+\varepsilon), 0\right)$$



Then $\phi \equiv 0$ for $x < \frac{1+\varepsilon}{2}$ & for $x > \frac{1+\varepsilon}{2}$, $\phi(x) = -\log(2(1-x)+\varepsilon)$,

$$\phi'(x) = \frac{2}{2(1-x)+\varepsilon},$$

$$\phi''(x) = \frac{4}{(2(1-x)+\varepsilon)^2} = (\phi'(x))^2.$$

Hence ϕ is convex, $\phi \geq 0$, and

$$\phi'' = (\phi')^2.$$

Choose ζ to be a cutoff function with support in $B(2R)$ (WLOG $x_0 = 0$). Multiply $(*)$ by $\phi'(u)\zeta^2$ & integrate by parts over $B(2R) \cap \Omega$. (Since $u = 0$ & $\therefore$ $\phi'(u) = 0$ on $\partial\Omega \cap B(2R)$, there are no boundary terms.)

Then calculations exactly like those on p. 62 show that

new element

$$v = \phi(u)$$

satisfies the estimate

$$(28) \quad \int_{B(R)\cap\Omega} |\nabla v|^2 \, dx \leq C R^{n-2}.$$

Now, by Theorem 5.12, we have

$$(29) \quad \max_{B(R/2)\cap\Omega} v^2 \leq \frac{C}{R^n} \int_{B(R)\cap\Omega} v^2 \, dx.$$

Next, extend u to be zero on Ω^c. Then $v \equiv 0$ on $\Omega^c \cap B(R)$, a set of measure $\geq A \operatorname{mes} B(R)$ according to (28). Hence Lemma 5.8 gives us

$$(30) \quad \int_{B(R)\cap\Omega} v^2 \, dx \leq C R^2 \int_{B(R)\cap\Omega} |\nabla v|^2 \, dx.$$

We now combine (28) – (30) to find

$$\max_{B(R/2)\cap\Omega} v^2 \leq C, \text{ independently of } R \,\&\, \varepsilon.$$

75

Then

$$-\log(2(1-u(x))+\varepsilon) \le C \qquad \forall x\in B(R/2)\cap\Omega$$

& so

$$u(x) \le 1-\frac{e^{-C}}{2}+\frac{\varepsilon}{2} \qquad \forall x\in B(R/2)\cap\Omega.$$

Let $\varepsilon \to 0$. ||

$\boxed{5.12}$ Theorem 5.14 (Global Hölder continuity) Let u solve

$$(*)\begin{cases} -(a_{ij}u_{x_i})_{x_j} = (f_i)_{x_i} & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where the a_{ij} satisfy the ellipticity condition (E), $f_i \in L^p(\Omega)$ $(i=1,\dots n)$ for some $\boxed{p>n}$, and $\partial\Omega$ is admissible.

Then $\exists\, C\ge 0$ & $0<\gamma<1$, depending only on n, θ, Θ, A, $\mathrm{mes}(\Omega)$, and $\|f_i\|_{L^p}$, $\ni$

$$\boxed{|u(x)-u(y)| \le C|x-y|^{\gamma}} \qquad \forall x,y\in\bar\Omega.$$

Proof First consider $x_0\in\partial\Omega$. Let $R_0>0$ be the number mentioned in the definition of admissibility of $\partial\Omega$ & from now on consider

76

only $0<R\le \frac{R_0}{2}$.

First we claim

$$(31)\qquad |u(x)| \le C|x-x_0|^{\gamma} \quad \text{if } x\in\Omega,\ |x-x_0|\le R_0/2$$

for some constants C & γ.

To see this we write $u=v+w$, where v solves

$$\begin{cases} -(a_{ij}v_{x_i})_{x_j} = (f_i)_{x_i} & \text{in } B(x_0,2R)\cap\Omega \\ v=0 & \text{on } \partial(B(x_0,2R)\cap\Omega) \end{cases}$$

for some $R>0$ & w solves

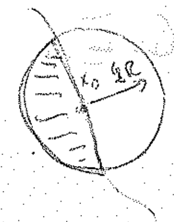
$$\begin{cases} -(a_{ij}w_{x_i})_{x_j} = 0 & \text{in } B(x_0,2R)\cap\Omega \\ w=0 & \text{on } B(x_0,2R)\cap\partial\Omega. \end{cases}$$

$w=u$ on $\partial B\cap\Omega$

By Theorem 5.3,

$$\|v\|_{L^{\infty}(B(x_0,2R)\cap\Omega)} \le C\,\mathrm{mes}(B(x_0,2R)\cap\Omega)^{1/n-1/p}$$

$$(32)\qquad \le C R^{1-n/p}.$$



Now let $\omega_w^+(R) \equiv \max_{B(x_0,R)\cap\Omega} w$. (77)

By Proposition 5.13 we have

(33) $\omega_w^+(R/4) \le \eta\, \omega_w^+(R)$ for (homogeneity eq. + b.c.)

some $\eta < 1$. Indeed, if $\omega_w^+(R) = 0$ this is clear & if not, we may WLOG multiply w by a constant to ensure $\max_{B(x_0,R)\cap\Omega} w = 1$.

From (32), (33), Lemma 5.7, & calculations like those in the proof of Theorem 5.11, we have

$$\omega^+(R) \le CR^\gamma \qquad 0<R\le R_0/2$$

for certain constants C & $0<\gamma<1$. Similarly

$$\omega^-(R) \ge -CR^\gamma.$$

These inequalities together yield (31).

Now by Theorem 5.3 again we know u is bounded & so from (31) we have

(34) $|u(x)| \le C|x-x_0|^\gamma$ $\forall x\in\Omega, x_0\in\partial\Omega$.

(In particular, u attains its boundary values continuously) (78)

Now for $x,y\in\Omega$, define

$$d_x \equiv \text{dist}(x,\partial\Omega)$$
$$d_y \equiv \text{dist}(y,\partial\Omega)$$
$$d_{xy} = \max(d_x,d_y)$$

In the proof of Theorem 5.11 we showed ((27))

(35) $\omega(R) \le \dfrac{CR^\gamma}{(d_x)^\gamma}$

where ω is the oscillation of u about the point x.

Choose any $x,y\in\Omega$. If

$$|x-y| \le (d_{xy})^2,$$

then assume WLOG $|x-y| \le (d_x)^2$. Hence by (35)

$$|u(x)-u(y)| \le \omega(R) \qquad R=|x-y|$$
$$\le \frac{C|x-y|^\gamma}{(d_x)^\gamma}$$

(36) $$\le \frac{C|x-y|^{\gamma/2}}{(d_x)^\gamma}|x-y|^{\gamma/2}$$

(36) $\leq C|x-y|^{\gamma'} \qquad \gamma' = \gamma/2$

If, on the other hand,

$$|x-y| > (d_{xy})^2,$$

then

$$|u(x)-u(y)| \leq |u(x)| + |u(y)|$$
$$\leq C[(d_x)^\gamma + (d_y)^\gamma] \text{ by (34)}$$

(37) $\leq C\,(d_{xy})^\gamma$

$\leq C\,|x-y|^{\gamma'} \qquad \gamma' = \gamma/2.$

Both cases (36) & (37) yield the inequality

$$|u(x) - u(y)| \leq C|x-y|^{\gamma'} \quad \forall x,y \in \Omega.$$

//

Remark Theorems 5.11 & 5.14 have various intermediate versions, for the cases that only part of $\partial\Omega$ is admissible or that $u|_\Gamma = 0$ for only some piece $\Gamma \subset \partial\Omega$. See [G-T].

5.13 G. Typical applications

Example 1 Suppose $f: \mathbb{R}^n \to \mathbb{R}$ satisfies

(38) $\theta|\xi|^2 \leq f_{x_i x_j}(x)\xi_i\xi_j \leq \Theta|\xi|^2$

for some $0 < \theta \leq \Theta$ & $\forall x \in \mathbb{R}^n$, $\xi = (\xi_1, \dots \xi_n)$. (ie f is uniformly strictly convex).

Consider the problem

$$(*) \begin{cases} (f_{x_i}(\nabla u))_{x_i} = 0 & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega. \end{cases}$$

(This is the Euler equation for the problem of minimizing

$$\int_\Omega f(\nabla u)\,dx$$

subject to the condition $u = \phi$ on $\partial\Omega$).

Assume we know $\|\nabla u\|_{L^2(\Omega)} \leq$ Const; then we claim

$\exists\ 0 < \gamma < 1 \ni$

(39) $\|u\|_{C^{1,\gamma}(\Omega')} \leq C(\Omega') \quad \forall \Omega' \subset\subset \Omega$

(that is, u has a Hölder continuous gradient).

For this, set $w = u_{x_k}$ (for some fixed k) & differentiate $(*)$ wrt x_k:

$$\left(f_{x_i x_j}(\nabla u) w_{x_j} \right)_{x_i} = 0 \quad \text{in } \Omega.$$

Apply Theorem 5.10 to w. (The Schauder estimates now give a $C^{2,\gamma}$ estimate: see Chapter VII). //

<u>Example 2</u> Suppose u solves the <u>2-dimensional</u> problem

$$(*) \begin{cases} a u_{xx} + 2b u_{xy} + c u_{yy} = f & \text{in } \Omega \subset \mathbb{R}^2 \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

We assume that f is bounded, that

$$\theta|\xi|^2 \le a\xi_1\xi_1 + 2b\xi_1\xi_2 + c\xi_2\xi_2 \le \Theta|\xi|^2$$

$\forall\, \xi = (\xi_1, \xi_2)$, but make no assumptions about the smoothness of a, b & c.
(For example a, b, c may depend on $u, \nabla u$, etc).

<u>Claim</u> $\exists\ 0 < \gamma < 1 \ni$

$$\|u\|_{C^{1,\gamma}(\Omega')} \le C(\Omega') \qquad \forall\, \Omega' \subset\subset \Omega.$$

To see this, rewrite $(*)$ as

$$\frac{a}{c} u_{xx} + \frac{2b}{c} u_{xy} + u_{yy} = \frac{f}{c}$$

Now differentiate wrt x to find that $w = u_x$ solves

$$\left(\frac{a}{c} w_x + \frac{2b}{c} w_y \right)_x + w_{yy} = \left(\frac{f}{c} \right)_x .$$

The matrix

$$A = \begin{pmatrix} \frac{a}{c} & \frac{2b}{c} \\ 0 & 1 \end{pmatrix}$$

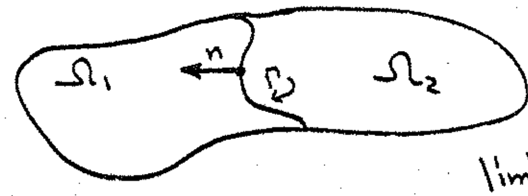
satisfies Condition (E) & so, by Theorem 5.11, $w = u_x$ is Hölder Continuous on $\Omega' \subset\subset \Omega$.
A similar argument proves the same for u_y //

<u>Example 3</u> Let $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$.

a_{ij}, b_{ij} constants

$$(*) \begin{cases} a_{ij} u_{x_i x_j} = 0 & \text{in } \Omega_1 \\ b_{ij} u_{x_i x_j} = 0 & \text{in } \Omega_2 \\ a_{ij} u^+_{x_i} n_j = b_{ij} u^-_{x_i} n_j & \\ u^+ = u^- & \end{cases} \text{ on } \Gamma = \text{boundary between } \Omega_1 \text{ \& } \Omega_2$$

(n = normal to Γ pointing into Ω_1)



Here the superscript "+" means the limit from within Ω_1 & "−" means the limit from within Ω_2.

The conditions stated on Γ are called the *transmission conditions* & $(*)$ is called a *diffraction problem*. (83)

We have
$$\|u\|_{C^{\gamma}(\Omega')} \le C(\Omega') \quad \forall \Omega' \subset\subset \Omega,$$
since $(*)$ is equivalent to
$$-(\gamma_{ij} u_{x_i})_{x_j} = 0 \quad \text{in } \Omega,$$
where
$$\gamma_{ij} = \begin{cases} a_{ij} & x \in \Omega_1 \\ b_{ij} & x \in \Omega_2 \end{cases}$$

(84)

VI L^p estimates

$\boxed{6.1}$ In this Chapter we write $(*)$ as
$$(*)\begin{cases} -a_{ij} u_{x_i x_j} + b_i u_{x_i} + cu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$
and will assume $f \in L^p(\Omega)$ for some $1 < p < \infty$. Under the assumption that *the* a_{ij} *are continuous* on $\bar{\Omega}$ (& of course, satisfy the ellipticity condition (E)), we will prove that the $W^{2,p}$ norm of u can be estimated by the L^p norm of f.

We'll prove the estimate first for Δu on all of $\mathbb{R}^n$ (sections A–C) and then, via a perturbation argument, for a solution u of $(*)$ on Ω (sections D & E).

$\boxed{6.2}$ A. Interpolation Theorem | Decomposition Lemma

We will show the required estimate for the case $p = 2$ & in a weak form for $p = 1$;

then the interpolation theorem below will imply the estimate for all $1<p<\infty$. (85)

Definitions (a) A linear mapping $T: L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is called (strong) type (p,p) if $\exists\, C \ni$

(1) $$\boxed{\|T(f)\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}} \quad \forall f\in L^p$$

(b) T is called weak type (p,p) if $\exists\, C \ni$

(2) $$\boxed{\text{mes}\{x \mid |Tf(x)|>\alpha\} \le \left(\frac{C\|f\|_{L^p(\mathbb{R}^n)}}{\alpha}\right)^p}$$

$\forall\, \alpha>0$ & $\forall f\in L^p$, $1\le p<\infty$.

Lemma 6.1 If T is strong type (p,p), it is weak type (p,p) $\quad 1\le p<\infty$.

Remark The converse is false.

(86)

Proof

$$\text{mes}\{x \mid |Tf|>\alpha\}\,\alpha^p \le \int_{\{|Tf|>\alpha\}} |Tf|^p\,dx \le \int_{\mathbb{R}^n} |Tf|^p\,dx \le C\|f\|_{L^p}^p$$

since T is strong type (p,p) //

Definitions (a) $L^p(\mathbb{R}^n)+L^q(\mathbb{R}^n)$

$$= \{f \mid f=f_1+f_2, \text{ for some } f_1\in L^p, f_2\in L^q\}$$

(b) $$\phi(\alpha) = \text{mes}\{x \mid |g(x)|>\alpha\} \qquad \alpha>0$$

is the distribution function of g

Lemma 6.2

$$\boxed{\int_{\mathbb{R}^n} |g(x)|^p\,dx = p\int_0^\infty \alpha^{p-1}\phi(\alpha)\,d\alpha}$$

$1\le p<\infty$

Note: $\displaystyle\int_{\mathbb{R}^n}|g(x)|^p dx = -\int_0^\infty \alpha^p\, d\phi(\alpha) =$

$\displaystyle = -\alpha^p\phi(\alpha)\Big|_0^\infty + p\int_0^\infty \phi(\alpha)\alpha^{p-1}\,d\alpha$. $\displaystyle\lim_{\alpha\to+\infty}\phi(\alpha)\,\alpha^p = 0$ since

$\displaystyle\int_\alpha^\infty |g(x)|^p dx \ge \alpha^p\, \text{meas}\{|g(x)|>\alpha\} = \alpha^p\phi(\alpha)$.

Proof Let $p=1$ first. Define
$$G = \{(x,y)\in\mathbb{R}^{n+1} \mid 0 \le y \le |g(x)|\};$$
then
$$\int_{\mathbb{R}^n} |g(x)|\,dx = \iint_G dy\,dx = \iint_G dx\,dy$$
$$= \int_0^\infty \phi(y)\,dy \quad \text{by Fubini's theorem.}$$
If $p>1$, apply this equality to $h=|g|^p$ & change variables. ||

(G3) Marcinkiewicz Interpolation Theorem

Let $1 \le p < q < \infty$. Let T be a linear mapping on $L^p(\mathbb{R}^n)+L^q(\mathbb{R}^n)$, which is both weak type (p,p) & weak type (q,q).

Then T is strong type (r,r) for each $p<r<q$; that is, $\exists\ C(r) \ni$
$$\boxed{\|T(f)\|_{L^r(\mathbb{R}^n)} \le C(r)\,\|f\|_{L^r(\mathbb{R}^n)}}$$
$p<r<q$, $\forall f\in L^r(\mathbb{R}^n)$

Proof Let $f\in L^r$ & write
$$f = f_1 + f_2 \in L^p + L^q,$$
where
$$f_1(x) = \begin{cases} f(x) & \text{if } |f(x)|>\alpha \\ 0 & \text{if } |f(x)|\le\alpha, \end{cases}$$
$$f_2(x) = \begin{cases} 0 & \text{if } |f(x)|>\alpha \\ f(x) & \text{if } |f(x)|\le\alpha. \end{cases}$$
Then $|Tf| \le |Tf_1| + |Tf_2|$ & so
$$\phi(\alpha) = \text{mes}\{x \mid |Tf|>\alpha\} \le \text{mes}\{x \mid |Tf_1|>\alpha/2\} + \text{mes}\{x \mid |Tf_2|>\alpha/2\}.$$
Since T is weak type (p,p) & (q,q), we have
$$\phi(\alpha) \le \frac{C}{(\alpha/2)^p}\int_{\mathbb{R}^n}|f_1|^p\,dx + \frac{C}{(\alpha/2)^q}\int_{\mathbb{R}^n}|f_2|^q\,dx$$
$$\le \frac{C}{\alpha^p}\int_{|f|>\alpha}|f|^p\,dx + \frac{C}{\alpha^q}\int_{|f|\le\alpha}|f|^q\,dx$$
by the definitions of f_1 & f_2

(89)

According to Lemma 6.2,

$$\int_{\mathbb{R}^n} |Tf|^r dx = \int_0^\infty r\alpha^{r-1}\phi(\alpha)\, d\alpha$$

$$\leq C\int_0^\infty r\,\alpha^{r-1-p}\Big(\int_{|f|>\alpha} |f|^p dx\Big)\, d\alpha$$

(3)

$$+\, C\int_0^\infty r\,\alpha^{r-1-q}\Big(\int_{|f|\leq\alpha} |f|^q dx\Big)\, d\alpha$$

But $\int_0^\infty \alpha^{r-1-p}\Big(\int_{|f|>\alpha} |f|^p dx\Big) d\alpha = \int |f|^p\Big(\int_0^{|f|}\alpha^{r-1-p} d\alpha\Big) dx$

$$= \frac{1}{r-p}\int_{\mathbb{R}^n} |f|^p |f|^{r-p} dx = \frac{1}{r-p}\int_{\mathbb{R}^n} |f|^r dx$$

&, similarly,

$$\int_0^\infty \alpha^{r-1-q}\Big(\int_{|f|\leq\alpha} |f|^q dx\Big) d\alpha = \int_{\mathbb{R}^n} |f|^q\Big(\int_{|f|}^\infty \alpha^{r-1-q} d\alpha\Big) dx$$

$$= \frac{1}{q-r}\int_{\mathbb{R}^n} |f|^q |f|^{r-q} dx$$

$$= \frac{1}{q-r}\int_{\mathbb{R}^n} |f|^r dx.$$

(90)

We combine these equalities with (3) to get

$$\int_{\mathbb{R}^n} |Tf|^r dx \leq C\int_{\mathbb{R}^n} |f|^r dx \;;$$

C depends on p, q & r, but not on f. //

6.4 **Lemma 6.3** (Decomposition Lemma)

Let $f \geq 0$, $f \in L^1(\mathbb{R}^n)$. Suppose $\alpha > 0$ is given.

Then $\exists$ 2 sets F & $\Omega \ni$

(i) $\mathbb{R}^n = F \cup \Omega$, $F \cap \Omega = \phi$

(ii) $\boxed{f(x) \leq \alpha \quad \text{a.e. on } F}$

(iii) $\Omega = \bigcup_{k=1}^\infty Q_k$, where the Q_k are cubes with disjoint interiors $\ni$

$$(4) \qquad \alpha < \frac{1}{\operatorname{mes}(Q_k)}\int_{Q_k} f\, dx \leq 2^n\alpha$$

In particular,

(91)

$$(5)\quad \boxed{\begin{aligned} &\operatorname{mes}(\Omega) \le \frac{1}{\alpha}\|f\|_{L^1(\mathbb{R}^n)} \\ &\frac{1}{\operatorname{mes}(Q_k)}\int_{Q_k} f\,dx \le C\alpha \end{aligned}}$$

$C = 2^n$

$\frac{1}{|Q_k|}\int f\,dx \sim \alpha$
Q_k on Ω
$f \le \alpha$ on F

<u>Proof</u> Decompose $\mathbb{R}^n$ into a mesh of equal cubes with common diameter so large that

$$\frac{1}{\operatorname{mes}(Q')}\int_{Q'} f\,dx \le \alpha \quad \forall \text{ such cube } Q'.$$

Pick any cube Q' & divide it into 2^n new cubes Q'' by bisecting each side of Q'.

<u>Case 1</u> $$\frac{1}{\operatorname{mes}(Q'')}\int_{Q''} f\,dx \le \alpha$$

<u>Case 2</u> $$\frac{1}{\operatorname{mes}(Q'')}\int_{Q''} f\,dx > \alpha.$$

If Case 2 holds, choose Q'' to be one of the cubes Q_k mentioned in the statement of the lemma; (4) clearly holds for it since

(92)

$$\frac{1}{\operatorname{mes}(Q'')}\int_{Q''} f\,dx \le \frac{1}{2^{-n}\operatorname{mes}(Q')}\int_{Q'} f\,dx \le 2^n\alpha.$$

If Case 1 holds, continue the subdivision process until you reach the second case (if this happens)

Define Ω = union of cubes Q_k for which Case 2 held at some time in the above procedure.

Let $F \equiv \mathbb{R}^n \setminus \Omega$. <u>Claim</u> $f(x) \le \alpha$ a.e. in F.

Indeed, if $x \in F$, then $\exists$ cubes $Q_\ell \ni$

$$x \in Q_\ell,\ \operatorname{mes}(Q_\ell) \to 0,$$

& Case 1 holds for each ℓ.

By Lebesgue's differentiation thm,

$$f(x) = \lim_\ell \frac{1}{\operatorname{mes}(Q_\ell)}\int_{Q_\ell} f\,dx \quad \text{a.e. } x$$

$$\le \alpha \quad \text{by Case 1 for each } \ell. \quad \|$$

<u>Remark</u> The material presented above is from Chapter 1 in Stein (reference [ST]) - The next section is based on Chapter 2 in [ST].

B. Singular Integrals

6.5 **Definition** Let $f \in L^2(\mathbb{R}^n)$. Then $\hat{f}$, the Fourier transform of f, is defined by

$$\boxed{\hat{f}(y) = \lim_{n\to\infty} \int_{|x|\le n} e^{2\pi i x\cdot y} f(y)\, dy},$$

the limit taking place in $L^2(\mathbb{R}^n)$.

Basic properties of Fourier transforms

(i) $\|f\|_{L^2(\mathbb{R}^n)} = \|\hat{f}\|_{L^2(\mathbb{R}^n)}$

(ii) $\widehat{f*g} = \hat{f}\,\hat{g}$ ($*$ = convolution)

(iii) $\widehat{f\circ\tau_\varepsilon} = \varepsilon^{-n}\,\hat{f}\circ\tau_{\frac{1}{\varepsilon}}$, ✓

where τ_ε represents the dilation of length ε (ie $f\circ\tau_\varepsilon(x) = f(\varepsilon x)$)

6.6 We'll now study an operator T, defined to be the convolution with a kernel K. Under hypotheses listed below, we will prove T is of strong type $(2,2)$ & is of weak type $(1,1)$. From these facts the Marcinkiewicz

Interpolation Thm (plus a duality argument) will show that T is strong type (p,p) $\forall\ 1<p<\infty$. In Section C we'll apply these estimates to Δ on $\mathbb{R}^n$.

Hypotheses on the kernel K

Assume $K:\mathbb{R}^n\setminus\{0\}\to\mathbb{R}$ & $\exists$ a constant $C \ni$

(a) $\boxed{|K(x)| \le \dfrac{C}{|x|^n}}$ $\quad\forall x\ne 0$

(b) $\boxed{\displaystyle\int_{|x|\ge 2|y|} |K(x-y)-K(x)|\,dx \le C}$ $\quad\forall y\ne 0$

(c) $\boxed{\displaystyle\int_{R_1\le|x|\le R_2} K(x)\,dx = 0}$ $\quad\forall\ 0<R_1<R_2<\infty$ (Cancellation property)

Definition Let $\varepsilon>0$

$$K_\varepsilon(x) = \begin{cases} K(x) & \text{if } |x|\ge\varepsilon \\ 0 & \text{if } |x|<\varepsilon \end{cases}$$

Note that $K_\varepsilon \in L^2(\mathbb{R}^n)$ ✓

6.7 **Lemma 6.4** $\exists$ constant C, independent of ε, $\ni$ 95

$$\boxed{\sup_{y\in\mathbb{R}^n} |\hat{K}_\varepsilon(y)| \le C}$$

Proof Consider first the case $\varepsilon = 1$.

Claim: the function $K_1(x)$ satisfies the same hypotheses as K (with a new constant C' depending only on C & n).

That (a) & (c) hold for K_1 is clear. For (b), we have

$$\int_{|x|\ge 2|y|} |K_1(x-y) - K_1(x)|\, dx$$

$$= \int_{\substack{|x|\ge 2|y|\\ |x|\ge 1\\ |x-y|\ge 1}} |K(x-y)-K(x)|\, dx + \int_{\substack{|x|\ge 2|y|\\ |x|\ge 1\\ |x-y|<1}} |K(x)|\, dx$$

96

$$+ \int_{\substack{|x|\ge 2|y|\\ |x|<1\\ |x-y|\ge 1}} |K(x-y)|\, dx .$$

$\Rightarrow |y| < \frac{1}{2} \Rightarrow |x-y| < \frac{3}{2}$

$\int_{1\le|z|\le 3/2} |K(z)|\, dz \le C$

The first integral is bounded, by hypothesis (b) on K. Also

$$\int_{\substack{|x|\ge 2|y|\\ |x|\ge 1\\ |x-y|<1}} |K(x)|\, dx \le \int_{1\le|x|\le 2} |K(x)|\, dx$$

$$\le \int_{1\le|x|\le 2} \frac{C}{|x|^n}\, dx \quad \text{by (a)}$$

$$\le C;$$

& the third integral above is estimated similarly. This proves the claim.

Now for $y \ne 0$,

$$(6) \quad \hat{K}_1(y) = \lim_{n\to\infty} \int_{|x|\le n} e^{2\pi i x\cdot y} K_1(x)\, dx$$

$|x| - |y| < 1 \Rightarrow$

$|x| < |y| + 1 \Rightarrow$ ① but $|y| + 1 > 2|y| \Rightarrow |y| > 1$

$$= \int_{|x| \le 1/|y|} e^{2\pi i x\cdot y} K_1(x)\,dx$$ (97)

(6)

$$+ \lim_{n\to\infty} \int_{1/|y| \le |x| \le n} e^{2\pi i x\cdot y} K_1(x)\,dx \equiv I_1 + I_2$$

Estimate of I_1

$$I_1 = \int_{|x| \le 1/|y|} \left[e^{2\pi i x\cdot y} - 1\right] K_1(x)\,dx$$ by the cancellation property (c);

hence

$$|I_1| \le C|y| \int_{|x| \le 1/|y|} |x|\,|K_1(x)|\,dx$$

$$\le C|y| \int_{|x| \le 1/|y|} |x|^{-n+1}\,dx \quad \text{by (a)}$$

(7)

$$= C|y| \int_0^{1/|y|} \int_{|\xi|=1} \frac{1}{r^{n-1}}\, r^{n-1}\, d\sigma\, dr$$

$= C$, the constant independent of y.

Estimate of I_2 Set (98)

$$z = \frac{1}{2}\frac{y}{|y|^2}, \qquad |z| = \frac{1}{2|y|}$$

so that $e^{2\pi i y\cdot z} = -1$.

Then

$$\int_{\mathbb{R}^n} K_1(x)\, e^{2\pi i y\cdot x}\,dx = \int_{\mathbb{R}^n} K_1(x-z)\, e^{2\pi i y\cdot(x-z)}\,dx$$

$$= -\int_{\mathbb{R}^n} K_1(x-z)\, e^{2\pi i y\cdot x}\,dx;$$

therefore

$$\int_{\mathbb{R}^n} K_1(x)\, e^{2\pi i y\cdot x} = \frac{1}{2}\int_{\mathbb{R}^n} (K_1(x) - K_1(x-z))\, e^{2\pi i x\cdot y}\,dx.$$

Thus we have

$$I_2 = \lim_{n\to\infty} \int_{1/|y| \le |x| \le n} K_1(x)\, e^{2\pi i y\cdot x}\,dx$$

$$= \frac{1}{2}\lim_{n\to\infty} \int_{1/|y| \le |x| \le n} (K_1(x) - K_1(x-z))\, e^{2\pi i x\cdot y}\,dx$$

(8)

$$- \frac{1}{2}\int_{|x| \le 1/|y|} (K_1(x) - K_1(x-z))\, e^{2\pi i x\cdot y}\,dx$$

$$\equiv J_1 + J_2 .$$ (99)

We have

$$|J_1| \leqslant \frac{1}{2}\int_{1/|y|\leqslant|x|} |K_1(x)-K_1(x-z)|\,dx$$

(9) $$= \frac{1}{2}\int_{2|z|\leqslant|x|} |K_1(x)-K_1(x-z)|\,dx$$

$\leqslant C$, by the condition analagous to hypothesis (b) which holds for K_1.

$$J_2 = -\frac{1}{2}\int_{|x|\leqslant 1/|y|} K_1(x)\,e^{2\pi i x\cdot y}\,dx$$

$$+\frac{1}{2}\int_{|x+z|\leqslant 1/|y|} K_1(x)\,e^{2\pi i y\cdot x}\,\underbrace{e^{2\pi i y\cdot z}}_{=-1}\,dx$$

$$= -\frac{1}{2}\int_{\substack{|x|\leqslant 1/|y|\\ |x+z|\geqslant 1/|y|}} K_1(x)\,e^{2\pi i x\cdot y}\,dx$$

$$+\frac{1}{2}\int_{\substack{|x+z|\leqslant 1/|y|\\ |x|\geqslant 1/|y|}} K_1(x)\,e^{2\pi i x\cdot y}\,dx$$

The region of the first integration (100) is contained in the set $\frac{1}{2|y|}\leqslant|x|\leqslant\frac{1}{|y|}$ & the region of the second integration is contained in $1/|y|\leqslant|x|\leqslant\frac{3}{2|y|}$.

Hence

$$|J_2| \leqslant \frac{1}{2}\int_{\frac{1}{2|y|}\leqslant|x|\leqslant\frac{3}{2|y|}} |K_1(x)|\,dx$$

$$\leqslant C\int_{\frac{1}{2|y|}\leqslant|x|\leqslant\frac{3}{2|y|}} |x|^{-n}\,dx \quad \text{by hypothesis (a)}$$

$$= C\left(\log\left(\tfrac{3}{2|y|}\right)-\log\left(\tfrac{1}{2|y|}\right)\right)$$

$$= C\log 3 = C, \text{ independently of } y.$$

We use this estimate & (9) in equation (8) to find

$$|I_2| \leqslant C.$$

This & inequality (7) complete the proof that

(10) $$|\hat{K}_1(y)| \leqslant C \quad \forall y\in\mathbb{R}^n.$$

We must prove (10) for any K_ε in place of K_1. (101)

Set $K'(x) = \varepsilon^n K(\varepsilon x) \qquad x \neq 0;$

then K' satisfies the same hypotheses (a)-(c) as K, and <u>with the same constants C</u>.

Thus inequality (10) gives

$$(10)' \quad |\hat{K}_1'(y)| \leq C \qquad \forall y \in \mathbb{R}^n.$$

The Fourier transform of

$K_\varepsilon(x) = \varepsilon^{-n} \cdot K_1'(\frac{x}{\varepsilon})$ is $\hat{K}_1'(\varepsilon y)$ & this is bounded, uniformly in y, by (10)'.

Hence

$$|\hat{K}_\varepsilon(y)| \leq C \qquad \forall \varepsilon > 0, \forall y \in \mathbb{R}^n$$

//

$\boxed{6.8}$ <u>Definitions</u> For $f \in L^p(\mathbb{R}^n)$, we define (102)

(a) $$T_\varepsilon f(x) = \int_{|y| \geq \varepsilon} K(y) f(x-y)\, dy = K_\varepsilon * f$$

(b) $$Tf(x) = \lim_{\varepsilon \to 0} \int_{|y| \geq \varepsilon} K(y) f(x-y)\, dy,$$

whenever these expressions make sense.

<u>Theorem 6.5</u> (Calderon-Zygmund) Assume that K satisfies the hypotheses (a)-(c) listed on p. (94).

Then for each $f \in L^p(\mathbb{R}^n)$ $(1 < p < \infty)$, $T_\varepsilon f$ is defined & $\exists$ a constant $C \ni$

$$(11) \quad \boxed{\|T_\varepsilon f\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}}$$

The constant C depends only on p & not on f or $\varepsilon > 0$.

Corollary 6.6 Assume K satisfies hypotheses (a) – (c).

Then for each $f \in L^p(\mathbb{R}^n)$ $(1<p<\infty)$, the limit

$$\lim_{\varepsilon \to 0} T_\varepsilon f \equiv Tf$$

exists in $L^p(\mathbb{R}^n)$ & T satisfies the estimate

(12) $$\boxed{\|Tf\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}}$$

Remark Inequalities (11) & (12) are, in general, false for $p=1$ or ∞.

Proof of Theorem 6.5 Let $\varepsilon>0$ be fixed

Step 1: T_ε is strong type (2,2)

We have

$$\|T_\varepsilon f\|_{L^2} = \|K_\varepsilon * f\|_{L^2} = \|\widehat{K_\varepsilon * f}\|_{L^2}$$

$$= \|\hat{K}_\varepsilon \hat{f}\|_{L^2} \leq$$

$$\leq \|\hat{K}_\varepsilon\|_{L^\infty} \|\hat{f}\|_{L^2} \leq C \|\hat{f}\|_{L^2}$$

by Lemma 6.4

$$= C\|f\|_{L^2}.$$

This proves that T_ε is strong type (2,2); accordingly it is of weak type (2,2) as well (Lemma 6.1) & so

(13) $$\operatorname{mes}\{x \mid |T_\varepsilon f(x)| > \alpha\} \leq \frac{C}{\alpha^2} \int_{\mathbb{R}^n} f^2\, dx,$$

$\forall\, \alpha>0$. & $\forall\, f \in L^2(\mathbb{R}^n)$.

Step 2: T_ε is weak type (1,1)

Fix $\alpha>0$; we must show $\exists\, C \ni$

(14) $$\operatorname{mes}\{x \mid |T_\varepsilon f(x)| > \alpha\} \leq \frac{C}{\alpha} \int_{\mathbb{R}^n} |f|\, dx$$

$\forall\, f \in L^1(\mathbb{R}^n)$.

For this we use Lemma 6.3 to decompose f into the sum of 2 functions g & b defined

As follows:

By Lemma 6.3 we may write $\mathbb{R}^n = F \cup \Omega$, where

$$(15)\begin{cases} |f(x)| \le \alpha & \text{a.e. } x \in F, \\ \Omega = \bigcup_{k=1}^{\infty} Q_j, \ \operatorname{mes}(\Omega) \le \frac{1}{\alpha}\|f\|_{L^1} \\ \frac{1}{\operatorname{mes}(Q_k)} \int_{Q_k} |f|\,dx \le C\alpha. \end{cases}$$

Define

$$g(x) \equiv \begin{cases} f(x) & x \in F \\ \frac{1}{\operatorname{mes}(Q_k)} \int_{Q_k} f\,dx & x \in Q_k \end{cases}$$

& $b(x) \equiv f(x) - g(x)$.

Then

$$(16)\begin{cases} |g(x)| \le C\alpha & \text{a.e. } x \in \mathbb{R}^n \\ b(x) = 0 & x \in F \\ \int_{Q_k} b(x)\,dx = 0 & \forall \text{ cube } Q_k \end{cases}$$

Then $T_\varepsilon f = T_\varepsilon g + T_\varepsilon b$ & so

$$\operatorname{mes}\{x \mid |T_\varepsilon f| > \alpha\} \le \operatorname{mes}\{x \mid |T_\varepsilon g| > \tfrac{\alpha}{2}\}$$

$$(17) \qquad + \operatorname{mes}\{x \mid |T_\varepsilon b| > \alpha/2\}.$$

<u>Estimate of $T g$</u>: We have $g \in L^2(\mathbb{R}^n)$: indeed,

$$\|g\|_{L^2}^2 = \int_{\mathbb{R}^n} |g|^2\,dx = \int_F |g|^2\,dx + \int_\Omega |g|^2\,dx$$

$$\le \alpha \int_F |f|\,dx + \alpha^2 C \operatorname{mes}(\Omega)$$

$$\le C\alpha \|f\|_{L^1}, \text{ by (15)}.$$

Now apply inequality (13) to g:

$$\operatorname{mes}\{x \mid |T_\varepsilon g| > \alpha/2\} \le \frac{C}{\alpha^2}\|g\|_{L^2}^2$$

$$(18) \qquad \le \frac{C}{\alpha}\|f\|_{L^1}.$$

Estimate of Tb

Write

$$b_j = b\chi_{Q_j} = \begin{cases} b & \text{on } Q_j \\ 0 & \text{otherwise} \end{cases};$$

then $b = \sum_{j=1}^{\infty} b_j$ & $T_\varepsilon b = \sum_{j=1}^{\infty} T_\varepsilon b_j$

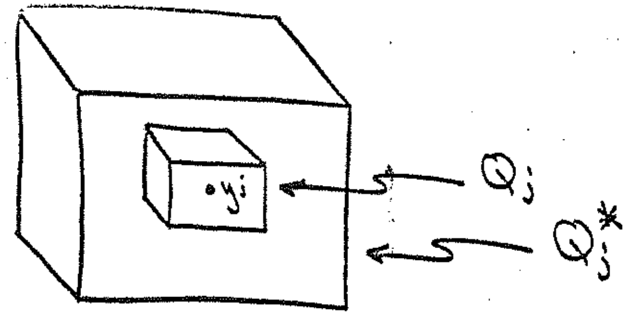
Let y^j denote the center of the cube Q_j. Then

$$T_\varepsilon b_j(x) = \int_{Q_j} K_\varepsilon(x-y)\, b_j(y)\, dy$$

(19)
$$= \int_{Q_j} [K_\varepsilon(x-y) - K_\varepsilon(x-y^j)]\, b_j(y)\, dy$$

(since $\int_{Q_j} b_j\, dy = 0$, by (16)).

Now define Q_j^* to be the cube obtained by expanding Q_j by a factor of $2\sqrt{n}$:



Define $\Omega^* \equiv \bigcup_{j=1}^{\infty} Q_j^*$, $F^* = (\Omega^*)^c$.

Then

(20) $\operatorname{mes}(\Omega^*) \leq (2\sqrt{n})^n \operatorname{mes}(\Omega)$

and if $x \notin Q_j^*$, then

(21) $|x - y^j| \geq 2|y - y^j| \quad \forall\, y \in Q_j$.

Hence by (19) we have

$$\int_{(Q_j^*)^c} |T_\varepsilon b_j(x)|\, dx \leq \sup_{y \in Q_j} \left(\int_{(Q_j^*)^c} |K_\varepsilon(x-y) - K_\varepsilon(x-y^j)|\, dx \right) \cdot \int_{Q_j} |b_j(y)|\, dy$$

$$\leq \sup_{y \in Q_j} \int_{|x'| \geq 2|y'|} |K_\varepsilon(x'-y') - K_\varepsilon(x')|\,dx' \cdot \int_{Q_j} |b(y)|\,dy$$

by (21) $(x' = x - y^j,\ y' = y - y^j)$

$$\leq C \int_{Q_j} |b(y)|\,dy \quad \text{by hypothesis (b).}$$

Therefore

$$\int_{F^*} |T_\varepsilon b(x)|\,dx \leq \sum_{j=1}^{\infty} \int_{(Q_j^*)^c} |T_\varepsilon b_j(x)|\,dx \longleftarrow (Q_j^*)^c \supseteq F^*$$

$$\leq C \sum_{j=1}^{\infty} \int_{Q_j} |b(y)|\,dy$$

$$\leq C \|b\|_{L^1} = C \|f\|_{L^1};$$

hence

(22) $$\text{mes}\{x \in F^* \mid |T_\varepsilon b| > \alpha/2\} \leq \frac{C}{\alpha} \|f\|_{L^1}.$$

But (15) & (20) together imply

$$\text{mes}(\Omega^*) \leq \frac{C}{\alpha} \|f\|_{L^1}.$$

This & (22) give

$$\text{mes}\{x \mid |T_\varepsilon b| > \alpha/2\} \leq \frac{C}{\alpha} \|f\|_{L^1}.$$

Now we combine this estimate with (18) & (17); this gives us inequality (14).

Step 3 L^p estimates for $1 < p \leq 2$

In steps 1 & 2 we proved that T_ε is strong type (2,2) & weak type (1,1): the Marcinkiewicz Interpolation Thm (p. 87) implies that T_ε is strong type (p,p) $\forall\ 1 < p \leq 2$; ie $\exists\ C = C(p) \ni$

(23) $$\|T_\varepsilon f\|_{L^p(\mathbb{R}^n)} \leq C(p) \|f\|_{L^p(\mathbb{R}^n)}$$

$$\forall f \in L^p(\mathbb{R}^n),\ 1 < p \leq 2.$$

Step 4 L^p estimates for $2<p<\infty$

We choose $f \in L^p(\mathbb{R}^n)$ for $2<p<\infty$ & take any $\phi \in L^q(\mathbb{R}^n)$ $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, with

$$\|\phi\|_{L^q} \le 1.$$

Then

$$\int_{\mathbb{R}^n} (T_\varepsilon f)\,\phi\,dx = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} K_\varepsilon(x-y)\, f(y)\,\phi(x)\,dx\,dy$$

$$= \int_{\mathbb{R}^n} f(y)\,(\overline{T}_\varepsilon \phi)\,dy,$$

where $\overline{T}_\varepsilon \phi \equiv K_\varepsilon(-x) * \phi$. Hence by inequality (23) we have

$$\left|\int_{\mathbb{R}^n} (T_\varepsilon f)\,\phi\right| dx \le \|f\|_{L^p}\,\|\overline{T}_\varepsilon \phi\|_{L^q}$$

$$\le C(q)\,\|f\|_{L^p}\,\|\phi\|_{L^q}$$

$$\le C(q)\,\|f\|_{L^p}.$$

This holds $\forall\, \phi \in L^q$ with $\|\phi\|_{L^q} \le 1$.

Hence

$$T_\varepsilon f \in L^p(\mathbb{R}^n) \ (\equiv \text{dual space of } L^q(\mathbb{R}^n)),$$

with

$$\|T_\varepsilon f\|_{L^p(\mathbb{R}^n)} \le C(p)\,\|f\|_{L^p} \quad (2<p<\infty)$$

This holds $\forall\, f \in L^p(\mathbb{R}^n)$. ||

Proof of Corollary 6.6 By estimate (11)

$$\|T_\varepsilon f\|_{L^p} \le C(p)\,\|f\|_{L^p} \quad \forall\, f \in L^p(\mathbb{R}^n),\ 1<p<\infty$$

and the constant C does not depend on $\varepsilon>0$.

Suppose now that $f \in C_0^\infty(\mathbb{R}^n)$. Then

$$T_\varepsilon f(x) = \int_{|y|\ge\varepsilon} K(y)\, f(x-y)\,dy$$

$$= \int_{|y|\ge 1} K(y)\, f(x-y)\,dy + \int_{1\ge|y|\ge\varepsilon} K(y)\left[f(x-y) - f(x)\right] dy$$

by the cancellation property (c).
The first integral represents a fixed function in L^p. The integrand of the second integral can be estimated by

$$\frac{|f(x-y)-f(x)|}{|x|^n} \leq \frac{C}{|x|^{n-1}} \quad \text{\& so}$$

converges uniformly in x as $\varepsilon \to 0$.
Hence $T_\varepsilon(f) \longrightarrow T(f)$ in $L^p(\mathbb{R}^n)$ as $\varepsilon \to 0$, $\forall f \in C_0^\infty(\mathbb{R}^n)$.
The T_ε thus converge to T on a dense subset of $L^p(\mathbb{R}^n)$ & are uniformly bounded: this implies

$$T_\varepsilon f \to Tf \text{ in } \underline{L^p} \text{ as } \varepsilon \to 0,$$

$\forall f \in L^p(\mathbb{R}^n)$ ||

C. Application to Δ on $\mathbb{R}^n$

6.9 Next we apply the Calderon-Zygmund inequality to obtain L^p estimates for the second derivatives of a solution u of $\Delta u = f$ $(f \in L^p)$ on all of $\mathbb{R}^n$ $(n \geq 3)$

Definitions

(a) $$\boxed{\Gamma(x) = \frac{-1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}}} \qquad x \neq 0$$

(ω_n = volume of unit ball in $\mathbb{R}^n$) is called the fundamental solution of Laplace's equation.

(b) Let f be bounded & integrable. Then

$$\boxed{u(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, dy} = \Gamma * f$$

is the Newtonian potential of f.

Facts about Γ

(i) $$\boxed{\Gamma_{x_i}(x) = \frac{1}{n\omega_n} \frac{x_i}{|x|^n}} \qquad x \neq 0$$

(ii) $\boxed{\Gamma_{x_ix_j}(x) = \frac{1}{n\omega_n}\left(\frac{\delta_{ij}}{|x|^n} - \frac{n x_i x_j}{|x|^{n+2}}\right)}$ $x \neq 0$

(iii) $\Delta\Gamma(x) = 0$ $x \neq 0$

Remark We'll prove in chapter VII that

Any function $u \in C^\infty(\overline{\mathbb{R}^n})$ can be written as

$$\boxed{u(x) = \int_{\mathbb{R}^n} \Gamma(y)\,\Delta u(x-y)\,dy} \quad x \in \mathbb{R}^n;$$

so a smooth function can be written as the Newtonian potential of its Laplacian.

Definitions (a) Fix some $1 \le i,j \le n$.

$$\boxed{K(x) = \Gamma_{x_ix_j}(x)} \quad x \neq 0$$

(b) Let $f \in C_0^\infty(\mathbb{R}^n)$. We define

$$T(f) = \lim_{\varepsilon \to 0} \int_{|y| \ge \varepsilon} K(y) f(x-y)\,dy$$

$$= u_{x_ix_j} - \frac{f(x)\delta_{ij}}{n}$$

where u is the Newtonian potential of f & so

$$\Delta u = f \quad \text{in } \mathbb{R}^n$$

We'll show that T is of strong type (p,p) $(1<p<\infty)$ & so obtain the estimate

$$\|u_{x_ix_j}\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \quad 1<p<\infty.$$

Lemma 6.7 The kernel K satisfies the hypotheses listed on p. 94

Proof (a) Since $K = \Gamma_{x_ix_j}$ (for some fixed $1 \le i,j \le n$)

$$= C\left(\frac{\delta_{ij}}{|x|^n} - \frac{n x_i x_j}{|x|^{n+2}}\right),$$

clearly $|K(x)| \le \dfrac{C}{|x|^n}$ $x \neq 0$.

(b) For $x \neq 0$, we may differentiate K to discover

(24) $$|\nabla K(x)| \le \frac{C}{|x|^{n+1}} \quad x \neq 0.$$

Therefore, by the Mean Value Theorem,

$$(25)\quad \int_{|x|\geq 2|y|} |K(x-y)-K(x)|\,dx \leq \int_{|x|\geq 2|y|} |y|\,|\nabla K(z)|\,dx$$

for some point z between $x-y$ & x, say

$$z = x - \lambda y \quad (0 \leq \lambda \leq 1)$$

(z depends on x). Since $|y| \leq \frac{|x|}{2}$, we have

$$|z| \geq |x| - \lambda|y| \geq |x| - \lambda\frac{|x|}{2} \geq \frac{|x|}{2}.$$

Hence (24) & (25) imply

$$\int_{|x|\geq 2|y|} |K(x-y)-K(x)|\,dx \leq C|y| \int_{|x|\geq 2|y|} \frac{dx}{|x|^{n+1}}$$

$$\leq C|y| \cdot \frac{1}{|y|} = C,$$

the constant independent of y //

(c) To prove

$$\int_{R_1 \leq |x| \leq R_2} K(x)\,dx = 0 \qquad 0 < R_1 < R_2,$$

it suffices to convert to polar coordinates & prove

$$\int_{\partial B(0,1)} \Omega(x)\,ds = 0$$

for $\Omega(x) \equiv \delta_{ij} - n\frac{x_i x_j}{|x|^2}$; that is,

$$\int_{\partial B(0,1)} x_i x_j\,ds = 0 \quad \text{if } i \neq j,$$

$$\int_{\partial B(0,1)} x_i^2\,ds = \frac{\text{mes}(\partial B(0,1))}{n} \qquad i = 1, 2, \cdots n.$$

These facts can be checked by an explicit evaluation of the surface integrals. //

By this lemma & Corollary 6.6 we now know that

$$T(f) = \lim_{\varepsilon\to 0} \int_{|y|\geq\varepsilon} \Gamma_{x_i x_j}(y)\,f(x-y)\,dy$$

is defined $\forall f \in L^p(\mathbb{R}^n)$ (if $1 < p < \infty$).
We'll see in the next chapter that (for a

smooth function f) the right hand side of this expression represents $u_{x_i x_j}$, where u is the Newtonian potential of f & so solves $\Delta u = f$ in $\mathbb{R}^n$.

Hence the L^p estimate provided by Corollary 6.6 & the remark on p. 115 give us

Theorem 6.8 Let $u \in C_0^\infty(\mathbb{R}^n)$. Then $\forall\ 1<p<\infty$, $\exists$ a constant $C(p) \ni$

$$\boxed{\|u_{x_i x_j}\|_{L^p(\mathbb{R}^n)} \leq C(p)\|\Delta u\|_{L^p(\mathbb{R}^n)}}$$

$\forall\ 1 \leq i,j \leq n$. The constant depends only on n & p.

Otherwise stated: if $u \in C_0^\infty(\mathbb{R}^n)$ solves

$$(*)\ \left\{ \Delta u = f \quad \text{in } \mathbb{R}^n, \right.$$

then

$$\boxed{\sum_{i,j=1}^n \|u_{x_i x_j}\|_{L^p(\mathbb{R}^n)} \leq C(p)\|f\|_{L^p(\mathbb{R}^n)}}$$

Remark This estimate in general fails for $p=1$ or $p=\infty$. By approximation, the estimate holds for $u \in W^{2,p}(\mathbb{R}^n)$ ||

D. Global $W^{2,p}$ estimates

$\boxed{6.10}$ In this section we use the results from §C to study a solution u of

$$(*)\ \left\{ \begin{aligned} -a_{ij}u_{x_i x_j} + b_i u_{x_i} + cu &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \right.$$

Assumptions on the Coefficients

(a) a_{ij}, b_i, c are bounded on Ω

(b) the a_{ij} satisfy the ellipticity condition (E)

(c) the functions a_{ij} are continuous on $\overline{\Omega}$

First are two lemmas covering the cases that Ω is a ball or half ball of small radius R.

Lemma 6.8 Assume that the coefficients a_{ij}, b_i & c satisfy hypotheses (a)-(c) (with Ω replaced by some ball centered at the origin). Choose $1<p<\infty$.

Then $\exists$ constants $R_0>0$, $C_1, C_2>0$, depending only on the bounds of the coefficients, the modulus of continuity of the a_{ij}, n, θ, Θ, and $p \ni$

if

$$0 < R \leq R_0$$

& u solves $(*)$ in $\Omega = B(R)$, with $u \equiv 0$ near $\partial B(R)$, then

$$\boxed{\sum_{i,j=1}^{n} \|u_{x_ix_j}\|_{L^p(B(R))} \leq C_1 \|f\|_{L^p(B(R))} + C_2 \|u\|_{W^{1,p}(B(R))}}$$

Proof We may rewrite $(*)$ as

$$(26) \quad a_{ij}(0)\, u_{x_ix_j} = (a_{ij}(0) - a_{ij}(x))\, u_{x_ix_j} + b_i u_{x_i} + cu - f \quad \text{in } B(R)$$

By Lemma 1.1 (cf. also the proof of Lemma 3.4), $\exists$ an orthogonal matrix $B \ni$ if we change to new variables $y = Bx$, (26) becomes

$$a'_{ij}(0)\, u_{y_iy_j} = (a'_{ij}(0) - a'_{ij}(y))\, u_{y_iy_j} + b'_i u_{y_i} + cu - f \quad \text{in } B(R),$$

where $A'(0) = BA(0)B^T = \operatorname{diag}(\lambda_1, \cdots \lambda_n)$ (that is,

$a'_{ij}(0) = \lambda_i \delta_{ij}$.)

We now follow this by a "stretching"

$$z_k = \frac{1}{\sqrt{\lambda_k}}\, y_k$$

& this change of variable converts (26) into the form

$$(27) \quad \left. \begin{aligned} \Delta u &= (a''_{ij}(0) - a''_{ij}(z))\, u_{z_iz_j} \\ &\quad + b''_i u_{z_i} + cu - f \end{aligned} \right\} \equiv f'(z)$$

(& $B(R)$ is mapped onto an ellipsoid B' centered at 0)

Thus

$$\begin{cases} \Delta u = f' & \text{in } B' \\ u \equiv 0 & \text{near } \partial B'. \end{cases}$$

We may extend u to be $\equiv 0$ on all of $\mathbb{R}^n \setminus B'$. To this function we now apply the estimate provided by Theorem 6.8:

$$\sum_{i,j=1}^{n} \|u_{z_iz_j}\|_{L^p(B')} \leq C \|f'\|_{L^p(B')}$$

Now use the definition of f' (see (27)) to obtain

$$\sum_{i,j=1}^{n} \|u_{z_i z_j}\|_{L^p(B')}$$

(28) $$\leq C \sup_{\substack{i,j\\ z\in B'}} |a''_{ij}(0) - a''_{ij}(z)| \sum_{i,j=1}^{n} \|u_{z_i z_j}\|_{L^p(B')}$$

$$+ C\|f\|_{L^p(B')} + C\|u\|_{W^{1,p}(B')}.$$

Recall now that the a_{ij} (& hence the a''_{ij}) are continuous. We choose $R_0>0$ so small that if $0<R\leq R_0$, then

(29) $$C \sup_{\substack{i,j\\ z\in B'}} |a''_{ij}(0) - a''_{ij}(z)| \leq \frac{1}{2}$$

(The choice of R_0 depends only on the quantities listed in the statement of the lemma).

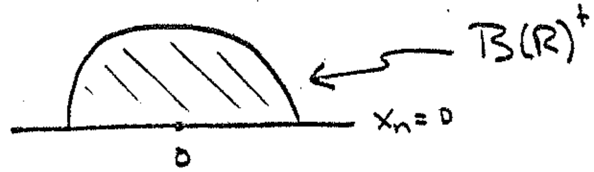
We plug (29) into (28) to get

$$\sum_{i,j=1}^{n} \|u_{z_i z_j}\|_{L^p(B')} \leq C\|f\|_{L^p(B')} + C\|u\|_{W^{1,p}(B')}$$

We now convert back to the original coordinates x & note that the various norms change at most by a bounded quantity, to finish the proof. ||

6.11 Definition

$$B(R)^+ = \{x\in\mathbb{R}^n \mid |x|\leq R,\ x_n\geq 0\}$$



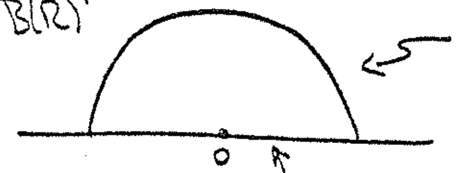
Lemma 6.9 Under the same assumptions as for Lemma 6. , $\exists$ constants $R_0>0$, $C_1, C_2 \geq 0$ $\ni$ if

$$0<R\leq R_0$$

and u solves $(*)$ in $\Omega = B(R)^+$, with $u\equiv 0$ near $\partial B(R)^+ \cap \{x_n>0\}$ & $u=0$ on $\{x_n=0\}$, then

$$\boxed{\sum_{i,j=1}^{n} \|u_{x_i x_j}\|_{L^p(B(R)^+)} \leq C_1\|f\|_{L^p(B(R)^+)} + C_2\|u\|_{W^{1,p}(B(R)^+)}}$$

$B(R)^+$



$u \equiv 0$ near the curved part of the boundary

$u = 0$ along $\{x_n = 0\}$, but its derivatives need not vanish here

Proof By means of the changes of variable indicated in the proof of Lemma 6.8, we may reduce to the case that u solves

$$(30) \quad \begin{cases} \Delta u = f' & \text{in } (B')^+ \\ u = 0 & \text{near } \partial(B')^+ \cap \{z_n > 0\} \\ u = 0 & \text{on } \{z_n = 0\}, \end{cases}$$

where $(B')^+ = B' \cap \{z_n \geq 0\}$ &

f' is defined as in (27).

Now extend u by reflection to all of B'; ie define

$$(31) \quad \bar{u}(z) = \begin{cases} u(z) & \text{if } z_n \geq 0 \\ -u(z_1, \cdots z_{n-1}, -z_n) & \text{if } z_n < 0 \end{cases} \quad z \in B'.$$

Then $\bar{u}$ is not as smooth as u was, but $\bar{u}$ & its first derivatives are continuous across $\{z_n = 0\}$ & so

$$\bar{u} \in W^{2,p}(B'), \ \bar{u} \equiv 0 \text{ near } \partial B'.$$

Also (30) & (31) imply that

$$(32) \quad \Delta \bar{u} = \bar{f}' \quad \text{in } B',$$

where

$$\bar{f}'(z) = \begin{cases} f'(z) & \text{if } z_n \geq 0 \\ -f'(z_1, \cdots z_{n-1}, -z_n) & \text{if } z_n < 0 \end{cases} \quad z \in B'.$$

Since $\bar{f}' \in L^p(B')$ we now use the techniques of the proof before to obtain the estimate

$$\sum_{i,j=1}^{n} \|\bar{u}_{z_i z_j}\|_{L^p(B')} \leq C\|\bar{f}'\|_{L^p(B')} + C\|\bar{u}\|_{W^{1,p}(B')}$$

& so

$$\sum_{i,j=1}^{n} \|u_{z_i z_j}\|_{L^p((B')^+)} \leq C\|f'\|_{L^p((B')^+)} + C\|u\|_{W^{1,p}((B')^+)}$$

if $0 < R \leq R_0$ & R_0 is small enough. We

Convert back to the original coordinates to finish the proof. ||

6.12 We now prove the $W^{2,p}$ estimates for a solution u of $(\ast)$ on an arbitrary smooth domain Ω. The idea, due to Korn, is to use a partition of unity & local changes of coordinates to reduce to one of the cases covered by Lemmas 6.8 & 6.9.

Theorem 6.10 Assume that u is a smooth solution of $(\ast)$ on a bounded domain Ω with smooth boundary $\partial\Omega$. Suppose that the coefficients satisfy the hypotheses listed on p. 120.

Let $\boxed{1<p<\infty}$.

Then $\exists$ constants C_1 & C_2, depending only on the bounds of the coefficients, the modulus of continuity of the a_{ij}, Ω, n, θ, Θ, & p $\ni$

$$\boxed{\|u\|_{W^{2,p}(\Omega)} \le C_1\|f\|_{L^p(\Omega)} + C_2\|u\|_{L^p(\Omega)}}$$

If $\min_{\Omega} c(x) \ge \lambda$ & λ is sufficiently large, we may take $C_2 = 0$.

__Proof__ We take a finite number N of C_0^∞ functions ζ_k $(k=1,\dots N)$ $\ni$

$$(33)\qquad \sum_{k=1}^{N} \zeta_k(x) = 1 \qquad \forall x \in \bar{\Omega},$$

$$(34)\qquad \operatorname{diam}(\operatorname{supp}\zeta_k) \le R \le \frac{R_0}{2} \qquad k=1,2,\dots N,$$

where $R_0>0$ is the constant from Lemmas 6.8 & 6.9.

Define

$$(35)\qquad u_k(x) = \zeta_k(x)\,u(x) \qquad x\in\bar{\Omega}$$

then u_k is supported in some ball $B_k = B(x_k, R)$, intersected with $\bar{\Omega}$.

__Case 1__ __$B_k \subset \Omega$__

Then by (35) & $(\ast)$ we have

$$(36)\qquad L(u_k) = \underbrace{f\zeta_k - a_{ij}(2u_{x_i}\zeta_{kx_j} + u\zeta_{kx_ix_j}) + b_i u\zeta_{kx_i}}_{\equiv f_k}$$

in B_k, where L denotes the elliptic operator

$$Lu = -a_{ij}u_{x_ix_j} + b_iu_{x_i} + cu$$

Hence u_k solves

$$(*)_k \begin{cases} Lu_k = f_k & \text{in } B_k \\ u_k \equiv 0 & \text{near } \partial B_k. \end{cases}$$

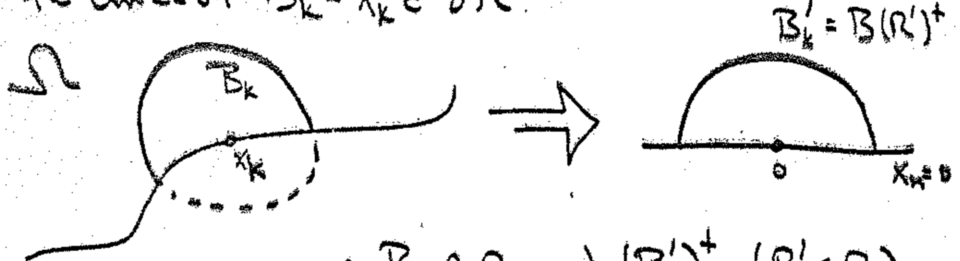
We apply Lemma 6.8 to find

$$\|u_k\|_{W^{2,p}(\Omega)} \leq C\|f_k\|_{L^p(\Omega)} + C\|u_k\|_{W^{1,p}(\Omega)}$$

$$(37) \qquad \leq C\|f\|_{L^p(\Omega)} + C\|u\|_{W^{1,p}(\Omega)},$$

according to the definition of f_k (see (36)).

Case 2 $B_k \cap \partial\Omega \neq \phi$ We can arrange things $\ni$ the balls B_k which intersect $\partial\Omega$ are such that the center of $B_k = x_k \in \partial\Omega$.



We may now map $B_k \cap \Omega$ onto $(R')^+$ $(R' \leq R_0)$ by a local, smooth change of coordinates. Straight-

forward & standard calculations now show that (the transform of) u_k satisfies in $B(R')^+$ an equation of the form

$$(*)'' \begin{cases} L'u_k' = f_k' & \text{in } B(R')^+ \\ u_k = 0 & \text{near } \partial B(R')^+ \cap \{x_n > 0\} \\ u_k = 0 & \text{along } \{x_n = 0\}, \end{cases}$$

where L' is an elliptic operator with the same properties as L.

We may apply Lemma 6.9 to find

$$\|u_k'\|_{W^{2,p}(B(R')^+)} \leq C\|f_k'\|_{L^p(B(R')^+)} + C\|u_k'\|_{W^{1,p}(B(R')^+)}.$$

We transform back into the original variables to find

$$(38) \qquad \|u_k\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)} + C\|u\|_{W^{1,p}(\Omega)}.$$

Now add inequalities (37) & (38) for $k=1,\dots,N$ & recall, by (33) & (35), that $u = \sum_{k=1}^{N} u_k$:

$$(39)\quad \|u\|_{W^{2,p}(\Omega)} \leq C\|f\|_{L^p(\Omega)} + C\|u\|_{W^{1,p}(\Omega)}.$$

But by Lemma 6.11 from below we have

$$\|u\|_{W^{1,p}(\Omega)} \leq \varepsilon\|u\|_{W^{2,p}(\Omega)} + C(\varepsilon)\|u\|_{L^p(\Omega)}$$

$\forall\, \varepsilon>0$; choose ε small & plug this into (39) to finish the proof. ||

6.13 <u>Lemma 6.11</u> Let $1<p<\infty$. For each $\varepsilon>0$, $\exists\ C(\varepsilon)$, depending only on ε, p, and Ω $\ni$

$$\boxed{\|u\|_{W^{1,p}(\Omega)} \leq \varepsilon\|u\|_{W^{2,p}(\Omega)} + C(\varepsilon)\|u\|_{L^p(\Omega)}} \quad \forall u \in W$$

<u>Proof</u> We give an indirect proof, based on the compactness assertion:

(39) if Ω is bounded, then bounded subsets of $W^{1,p}(\Omega)$ are precompact in $L^p(\Omega)$.

For proof of this standard result, see [G-T, p. 160].

We may apply assertion (39) to u & its first derivatives to prove that

(40) bounded subsets of $W^{2,p}(\Omega)$ are precompact in $W^{1,p}(\Omega)$.

Now suppose the lemma were false; then $\forall$ integer n $\exists\ u^n \ni$

$$(41)\quad \|u^n\|_{W^{1,p}} > \varepsilon\|u^n\|_{W^{2,p}} + n\|u^n\|_{L^p}$$

We may normalize if necessary to obtain

$$(42)\qquad \|u^n\|_{W^{1,p}} = 1. \qquad \forall n$$

Then (41) implies

$$\|u^n\|_{W^{2,p}} \leq \frac{1}{\varepsilon} \qquad \forall n$$

& so, by (40), $\exists$ a subsequence (which we denote also as "u^n") $\ni$

$$u^n \longrightarrow u \quad \text{strongly in } W^{1,p}$$

By (42) we have

(43) $$\|u\|_{W^{1,p}} = 1.$$

But (41) and (42) also imply

$$\|u^n\|_{L^p} \leq \frac{1}{n}$$

& so $u = \lim u^n = 0$, a contradiction to (43). Hence (41) cannot hold $\forall$ integer n. ||

E. Local $W^{2,p}$ estimates

$\boxed{6.14}$ We conclude this chapter by proving that the $W^{2,p}(\Omega')$ norm of a solution u of

(*) $$\left\{ -a_{ij}u_{x_ix_j} + b_iu_{x_i} + cu = f \text{ in } \Omega \right.$$

can be estimated for each $\Omega' \subset\subset \Omega$, even if we do not assume u to be well behaved on $\partial\Omega$ (or that $\partial\Omega$ is smooth).

These estimates are somewhat more delicate than the global estimates presented in section D.

For simplicity of the exposition we consider only $\boxed{2 \leq p < \infty}$

First we need a refinement of Lemma 6.11:

<u>Lemma 6.12</u> There exists a constant C, depending only on n & p ∋

(44) $$\boxed{\sum_{i=1}^n \|u_{x_i}\|_{L^p(B(R))} \leq \varepsilon \sum_{i,j=1}^n \|u_{x_ix_j}\|_{L^p(B(R))} + \frac{C}{\varepsilon}\|u\|_{L^p(B(R))}}$$

for all $\varepsilon > 0$ and $u \in W^{2,p}(B(R))$.

Note, in particular, that C does not depend on ε <u>or</u> <u>R</u>.

<u>Proof</u> We may assume WLOG that u is smooth. Consider first the case

that $R = 1$. Then by standard theory (see, for example, p. 10 in A. Friedman, <u>Partial Differential Equations</u>, Holt, Rinehart & Winston, Inc.) $\exists$ a smooth function $\bar{u} : \mathbb{R}^n \to \mathbb{R} \ni$

$$(45)\quad \begin{cases} \bar{u} = u \quad \text{on } B(1) \\ \bar{u} \equiv 0 \quad \text{on } \mathbb{R}^n \setminus B(2) \\ \|\bar{u}\|_{L^p(\mathbb{R}^n)} \leq C \|u\|_{L^p(B(1))} \\ \sum_{i,j=1}^{n} \|\bar{u}_{x_i x_j}\|_{L^p(\mathbb{R}^n)} \leq C \|u\|_{W^{2,p}(B(1))}; \end{cases}$$

$\bar{u}$ is called an <u>extension</u> of u to $\mathbb{R}^n$.

Then Lemma 2.2 & Young's inequality imply

$$\sum_{i=1}^{n} \|\bar{u}_{x_i}\|_{L^p(B(2))} \leq \varepsilon' \sum_{i,j=1}^{n} \|\bar{u}_{x_i x_j}\|_{L^p(B(2))} + \frac{C}{\varepsilon'} \|\bar{u}\|_{L^p(B(2))}$$

$\forall\, \varepsilon' > 0$; and so (45) gives

$$\sum_{i=1}^{n} \|u_{x_i}\|_{L^p(B(1))} \leq \sum_{i=1}^{n} \|\bar{u}_{x_i}\|_{L^p(B(2))}$$

$$\leq \varepsilon' C \|u\|_{W^{2,p}(B(1))} + \frac{C}{\varepsilon'} \|u\|_{L^p(B(1))}.$$

We set $\varepsilon = \varepsilon' C$ to prove (44) for the case $R = 1$.

For the general case of u defined on $B(R)$, we let

$$v(x) \equiv u(Rx) \qquad x \in B(1)$$

so that v is defined on $B(1)$ &

$$v_{x_i}(x) = R\, u_{x_i}(Rx) \qquad x \in B(1)$$

$$v_{x_i x_j}(x) = R^2\, u_{x_i x_j}(Rx) \qquad x \in B(1).$$

Hence

$$(46)\quad \|v\|_{L^p(B(1))} = R^{-n/p} \|u\|_{L^p(B(R))},$$

$$(47)\quad \|v_{x_i}\|_{L^p(B(1))} = R^{1-n/p}\|u_{x_i}\|_{L^p(B(R))},$$

and

$$(48)\quad \|v_{x_ix_j}\|_{L^p(B(1))} = R^{2-n/p}\|u_{x_ix_j}\|_{L^p(B(R))}.$$

Now use (46)-(48) & inequality (44) for v on $B(1)$ to get

$$R^{1-n/p}\sum_{i=1}^{n}\|u_{x_i}\|_{L^p(B(R))} \le \varepsilon' R^{2-n/p}\sum_{i,j=1}^{n}\|u_{x_ix_j}\|_{L^p(B(R))} + \frac{C}{\varepsilon'}R^{-n/p}\|u\|_{L^p(B(R))}$$

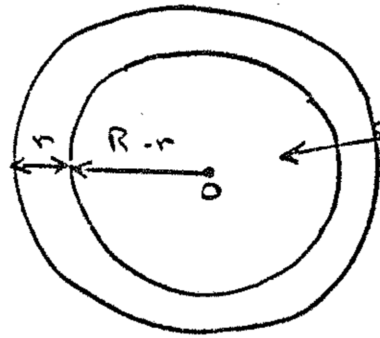
Let $\varepsilon' = \frac{\varepsilon}{R}$ to obtain (44) in the general case. //

<u>Proposition 6.13</u> Let u be a smooth solution of

$$(*)\left\{\ Lu = -a_{ij}u_{x_ix_j} + b_iu_{x_i} + cu = f\right.$$

in $B(R)$, where $R \le R_0$ & R_0 is the constant mentioned in Lemma 6.8. Then $\exists\, C$, depending only on n, p, R_0, & the coefficients of L $\ni$

$$(49)\quad \boxed{\begin{aligned}&\sum_{i,j=1}^{n}\|u_{x_ix_j}\|_{L^p(B(R-r))}\\ &\quad\le \frac{C}{r^2}\left[\|f\|_{L^p(B(R))} + \|u\|_{L^p(B(R))}\right]\end{aligned}}$$



The $W^{2,p}$ norm of u on the smaller ball of radius $R-r$ is estimated by u & f on $B(R)$, times a constant $\frac{1}{r^2}$.

<u>Proof</u> Choose a cutoff function $\zeta \in C_0^\infty(B(R)) \ni$

$$(50)\quad \left\{\begin{aligned}&0 \le \zeta \le 1,\ \zeta \equiv 1 \text{ on } B(R-r),\\ &\zeta \equiv 0 \text{ on } \mathbb{R}^n \setminus B(R-r/2)),\\ &|\zeta_{x_i}| \le \frac{C}{r},\ |\zeta_{x_ix_j}| \le \frac{C}{r^2}\quad (i,j=1,2,\cdots n)\end{aligned}\right.$$

Let $w \equiv \zeta u$; (139)

then
$$Lw = -a_{ij}(\zeta u)_{x_ix_j} + b_i(\zeta u)_{x_i} + c\zeta u$$
$$= \underbrace{\zeta \underbrace{Lu}_{=f} - a_{ij}\zeta_{x_ix_j}u - 2a_{ij}\zeta_{x_i}u_{x_i} + b_i\zeta_{x_i}u}_{\equiv f'}$$

Hence Lemma 6.8 & (50) imply

$$\sum_{i,j=1}^{n} \|u_{x_ix_j}\|_{L^p(B(R-r))}$$
$$\leq \sum_{i,j=1}^{n} \|w_{x_ix_j}\|_{L^p(B(R))}$$
$$\leq C\|f'\|_{L^p(B(R))} + C\|w\|_{W^{1,p}(B(R))} \quad (51)$$
$$\leq \frac{C}{r^2}\left(\|f\|_{L^p(B(R))} + \|u\|_{L^p(B(R))}\right)$$
$$+ \frac{C}{r}\|\nabla u\|_{L^p(B(R-r/2))}.$$

Now define (140)

$$\phi(r) \equiv \sum_{i,j=1}^{n} \|u_{x_ix_j}\|_{L^p(B(R-r))};$$

then (51) & Lemma 6.12 give

$$\phi(r) \leq \frac{C}{r^2}\left(\|f\|_{L^p} + \|u\|_{L^p}\right)$$
$$+ \frac{C}{r}\left(\varepsilon\phi(r/2) + \frac{C}{\varepsilon}\|u\|_{L^p}\right)$$

Let $\varepsilon = \frac{r}{8C}$ to obtain

$$\phi(r) \leq \frac{1}{8}\phi\left(\frac{r}{2}\right) + \frac{C}{r^2}\left[\|f\|_{L^p} + \|u\|_{L^p}\right]. \quad (52)$$
$$\forall\, 0<r<R$$

The following Lemma applied to inequality (52) completes the proof. ||

Lemma 6.14 Let $\phi(r)$ be a nonnegative bounded function defined for $0<r\le R_0$. If

(53) $$\boxed{\phi(r) \le \frac{1}{8}\phi\left(\frac{r}{2}\right) + \frac{C}{r^2}} \quad \forall\, 0<r\le R_0,$$

then

(54) $$\boxed{\phi(r) \le \frac{2C}{r^2}} \quad \forall\, 0<r\le R_0.$$

Proof Since ϕ is bounded, (54) is clearly true for all r small enough. Hence were (54) false, there would exist some pt $0<r^*\le R_0 \ni$

$$\phi(r^*) > \frac{2C}{(r^*)^2}, \qquad \text{but}$$

$$\phi\left(\frac{r^*}{2}\right) \le \frac{2C}{\left(\frac{r^*}{2}\right)^2}.$$

But then

$$\frac{2C}{(r^*)^2} < \phi(r^*) \le \frac{1}{8}\phi\left(\frac{r^*}{2}\right) + \frac{C}{(r^*)^2} \quad \text{by (53)}$$

$$\le \frac{1}{8}\,\frac{2C}{\left(\frac{r^*}{2}\right)^2} + \frac{C}{(r^*)^2}$$

$$= \frac{2C}{(r^*)^2}, \text{ a contradiction.}$$

||

Theorem 6.15 Assume that u is a smooth solution of

$$(*)\ \left\{ -a_{ij}u_{x_ix_j} + b_iu_{x_i} + cu = f \quad \text{in } \Omega. \right.$$

Suppose the coefficients of L satisfy the hypotheses listed on p. 120.

Then for each domain $\Omega' \subset\subset \Omega$, $\exists$ constants C_1 & C_2, depending only on the bounds on the coefficients, the modulus of continuity of the a_{ij}, n, θ, Θ, p, and $\operatorname{dist}(\Omega', \partial\Omega)$ $\ni$

$$\boxed{\|u\|_{W^{2,p}(\Omega')} \le C_1\|f\|_{L^p(\Omega)} + C_2\|u\|_{L^p(\Omega)}}$$

Proof We may cover Ω' by a finite number of balls $B_k(x_k, r_k)$ of radius

$$r_k = \frac{1}{2}\min\left(R_0;\ \operatorname{dist}(\Omega', \partial\Omega)\right),$$

where R_0 is the number mentioned in Lemma 6.8. Then we apply Proposition 6.13 to the balls $B_k(x_k, 2r_k) \subset \Omega$ to estimate

$$\sum_{i,j=1}^{n} \|u_{x_ix_j}\|_{L^p(B(x_k, r_k))}$$

in terms of r_k & the L^p norm of f & u.

We may add the resulting inequalities to find

$$\sum_{i,j=1}^{n} \|u_{x_ix_j}\|_{L^p(\Omega')} \leq C(\Omega')\left(\|f\|_{L^p} + \|u\|_{L^p}\right);$$

then Lemma 6.12 in turn provides a bound for $\|u\|_{W^{1,p}(\Omega')}$ & hence completes the proof. ||

VII Schauder estimates

7.1 In this chapter we'll prove that if u solves

$$(*)\begin{cases} -a_{ij}u_{x_ix_j} + b_i u_{x_i} + cu = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases}$$

& if f & the coefficients of the elliptic operator are Hölder continuous, then the second derivatives of u are Hölder continuous as well. As in Chapter VI this will follow from the corresponding statement for Δ by a perturbation argument. Our reference is [G-T, Chapters 4 & 6]

A. The Newtonian Potential

7.2 We recall some results from § C of the last chapter:

Definition
$$\boxed{\Gamma(x) = \frac{-1}{n(n-2)\omega_n}\frac{1}{|x|^{n-2}}} \qquad x \neq 0,\ n \geq 3$$

(ω_n = measure of unit ball in $\mathbb{R}^n$)

Note: (a) $\Gamma_{x_i}(x) = \dfrac{1}{n\omega_n}\dfrac{x_i}{|x|^n}$ $\quad x \neq 0$

(b) $\Gamma_{x_ix_j}(x) = \dfrac{1}{n\omega_n}\left(\dfrac{\delta_{ij}}{|x|^n} - \dfrac{n x_i x_j}{|x|^{n+2}}\right)$ $\quad x \neq 0$

(c) $|\Gamma_{x_i}(x)| \leq \frac{C}{|x|^{n-1}} \qquad x \neq 0$

(d) $|\Gamma_{x_ix_j}(x)| \leq \frac{C}{|x|^{n}} \qquad x \neq 0$

Definition Let $f \in L^1(\Omega)$. Then

$$\boxed{w(x) = \int_\Omega \Gamma(x-y) f(y)\,dy}$$

is the Newtonian potential of f (on Ω)

Lemma 7.1

If $f \in L^\infty(\Omega)$, then $w \in C^1(\bar\Omega)$ &

$$(1) \quad \boxed{w_{x_i}(x) = \int_\Omega \Gamma_{x_i}(x-y) f(y)\,dy} \qquad \begin{array}{l} i=1,2,\dots n \\ x \in \Omega \end{array}$$

Proof See [G-T, p. 53]. ||

Definitions Let $0<\alpha\leq 1$

(a) $[u]_{\alpha,\Omega} = [u]_\alpha = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\alpha}$

(b) $\|u\|_{C^{0,\alpha}(\bar\Omega)} = \sup_\Omega |u| + [u]_\alpha$

We'll usually write "C^α" for $C^{0,\alpha}$, when $\alpha<1$.

(c) $\|u\|_{C^{1,\alpha}(\bar\Omega)} = \sup_\Omega |u| + \sum_{i=1}^n \left(\sup_\Omega |u_{x_i}| + [u_{x_i}]_\alpha\right)$

(d) $\|u\|_{C^{2,\alpha}(\bar\Omega)} = \sup_\Omega |u| + \sum_{i=1}^n \sup_\Omega |u_{x_i}|$

$$+ \sum_{i,j=1}^n \left(\sup_\Omega |u_{x_ix_j}| + [u_{x_ix_j}]_\alpha\right)$$

[7.3] Lemma 7.2 If $f \in C^\alpha(\bar\Omega)$ for some $0<\alpha\leq 1$, then $w \in C^2(\Omega)$ &

$$(2) \quad \boxed{\begin{aligned} w_{x_ix_j}(x) &= \int_{\Omega_0} \Gamma_{x_ix_j}(x-y)(f(y)-f(x))\,dx \\ &\quad - f(x)\int_{\partial\Omega_0} \Gamma_{x_i}(x-y)\, n_j(y)\,ds \end{aligned}} \qquad \begin{array}{l} x\in\Omega \\ i,j=1,\dots n \end{array}$$

Here Ω_0 is any smooth domain containing Ω & f is extended to be $\equiv 0$ on $\Omega_0 \setminus \Omega$.
($n = (n_1, \dots n_n)$ = outward unit normal on $\partial\Omega_0$)

Justification of Remark on p (115)

$$W_{x_ix_j} = \int_\Omega \Gamma_{x_ix_j}(x-y)(f(y)-f(x))\,dy - f(x)\int_{\partial\Omega}\Gamma_{x_i}(x-y)n_j\,ds$$

$$= \lim_{\varepsilon\to 0}\int_{\Omega\setminus B(x,\varepsilon)} \Gamma_{x_ix_j}(x-y)\,f(y) - f(x))\,dy - f(x)\int_{\partial\Omega}\Gamma_{x_i}(x-y)n_j\,ds$$

$$= \lim_{\varepsilon\to 0}\int_{\Omega\setminus B(x,\varepsilon)} \Gamma_{x_ix_j}(x-y)f(y) \pm \underbrace{\int_{\partial B(x,\varepsilon)}\Gamma_{x_i}(x-y)n_j\,ds\,f(x)}_{\text{in limit } = +\frac{\delta_{ij}f(x)}{n}}$$

Spse $\Omega \supset$ supt f

$$= \lim_{\varepsilon\to 0}\int_{|x-y|\geq\varepsilon}\Gamma_{x_ix_j}(x-y)\,f(y)\,dy + \frac{\delta_{ij}f(x)}{n} \quad //$$

Proof See [G-T, p. 54]. Note that (2) is clearly true formally if we twice differentiate under the integral sign, add & subtract $f(x)\int \Gamma_{x_ix_j}(x-y)\,ds$, & integrate by parts. Note also that

$$|\Gamma_{x_ix_j}(x-y)(f(y)-f(x))| \leq \frac{C|x-y|^\alpha}{|x-y|^n} \quad \&$$

this expression is integrable. //

<u>Proposition</u> 7.3 Let $f\in C^\alpha(\overline{\Omega})$ for some $0<\alpha\leq 1$ & assume w is the Newtonian potential of f in Ω. Then $w\in C^2(\Omega)$ &

$$\boxed{\Delta w = f \quad \text{in } \Omega.}$$

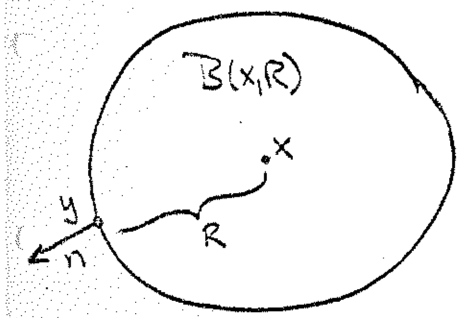
Proof By (2), if we set $\Omega_0 = B(x,R)\supset\Omega$, we have

$$\Delta w(x) = \int_{B(x,R)}\Delta\Gamma(x-y)(f(y)-f(x))\,dy - f(x)\int_{\partial B(x,R)}\Gamma_{x_i}(x-y)\,n_i(y)\,ds$$

$$= -f(x)\int_{\partial B(x,R)}\Gamma_{x_i}(x-y)n_i(y)\,ds$$

Since $\Delta\Gamma(x) = 0$ for $x \neq 0$

$$= \frac{-f(x)}{n\,\omega_n R^n} \int_{\partial B(x,R)} (x_i - y_i)\, n_i \, ds$$



n = outward unit normal at $y \in \partial B(x,R)$

$$= \frac{y-x}{|y-x|}$$

$$= \left(\frac{y_1 - x_1}{R}, \cdots \frac{y_n - x_n}{R} \right)$$

$$= (n_1, \cdots n_n)$$

Hence

$$\Delta w(x) = \frac{f(x)}{n\,\omega_n R^{n-1}} \int_{\partial B(x,R)} \underbrace{\sum_{i=1}^{n} n_i^2}_{=1} \, ds$$

$$= \frac{f(x)}{n\,\omega_n R^{n-1}} \operatorname{mes}(\partial B(x,R))$$

$$= f(x)$$

||

By Lemma 7.2 & Proposition 7.3 we have an explicit representation ((2)) for the second derivatives of a solution w of $\Delta w = f$ in Ω. In the next section we use this to make

estimates on $w_{x_i x_j}$

B. $C^{2,\alpha}$ estimates on the Newtonian potential

$\boxed{7.4}$ **Lemma 7.4** Let $B_1 = B(x_0, R)$ & $B_2 = B(x_0, 2R)$ be 2 concentric balls. Assume

$$f \in C^\alpha(B_2) \text{ for some } \boxed{0 < \alpha < 1}$$

& let w be the Newtonian potential of f in B_2. Then

$$w \in C^{2,\alpha}(B_1)$$

and

$$\boxed{\sup_{B_1} |w_{x_i x_j}| + R^\alpha [w_{x_i x_j}]_{\alpha, B_1} \leq C\left(\sup_{B_2} |f| + R^\alpha [f]_{\alpha, B_2} \right)} \quad (3) \qquad i,j = 1, \cdots n$$

Remark This result &, indeed, the entire collection of Schauder estimates fails for $\alpha = 1$.

Proof Step 1 Estimate on $|w_{x_i x_j}|$

Let $x \in B_1$; then by (2)

$$W_{x_ix_j}(x) = \int_{B_2} \Gamma_{x_ix_j}(x-y)(f(y)-f(x))\,dy \tag{4}$$
$$- f(x)\int_{\partial B_2} \Gamma_{x_i}(x-y)\,n_j\,ds$$

& so

$$|W_{x_ix_j}(x)| \le C[f]_{\alpha,B_2}\int_{B_2}|x-y|^{\alpha-n}\,dy$$
$$+ C\frac{|f(x)|}{R^{n-1}}\int_{\partial B_2} ds \tag{5}$$
$$\le CR^{\alpha}[f]_{\alpha,B_2} + C\sup_{B_2}|f| \quad \forall x \in B_1$$

Step 2: Estimate on $[W_{x_ix_j}]_{\alpha,B_1}$

This is harder. Choose any other point $\bar{x} \in B_1$, so that, by (2) again,

$$W_{x_ix_j}(\bar{x}) = \int_{B_2} \Gamma_{x_ix_j}(\bar{x}-y)(f(y)-f(\bar{x}))\,dy \tag{6}$$
$$- f(\bar{x})\int_{\partial B_2} \Gamma_{x_i}(\bar{x}-y)\,n_j\,ds$$

Subtract (4) from (6) to get

$$W_{x_ix_j}(\bar{x}) - W_{x_ix_j}(x)$$
$$= f(x)I_1 + (f(x)-f(\bar{x}))I_2 + I_3 + I_4 \tag{7}$$
$$+ (f(x)-f(\bar{x}))I_5 + I_6,$$

where

$$I_1 = \int_{\partial B_2}\left(\Gamma_{x_i}(x-y) - \Gamma_{x_i}(\bar{x}-y)\right)n_j\,ds$$

$$I_2 = \int_{\partial B_2} \Gamma_{x_i}(\bar{x}-y)\,n_j\,ds$$

$$\xi = \frac{x+\bar{x}}{2}$$
$$\delta = |x-\bar{x}|$$

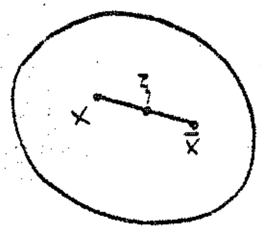
$$I_3 = \int_{B(\xi,\delta)} \Gamma_{x_ix_j}(x-y)(f(x)-f(y))\,dy$$

$$I_4 = \int_{B(\xi,\delta)} \Gamma_{x_ix_j}(\bar{x}-y)(f(x)-f(y))\,dy$$

$$I_5 = \int_{B_2\setminus B(\xi,\delta)} \Gamma_{x_ix_j}(x-y)\,dy$$

$$I_6 = \int_{B_2\setminus B(\xi,\delta)} \left(\Gamma_{x_ix_j}(x-y) - \Gamma_{x_ix_j}(\bar{x}-y)\right)(f(\bar{x})-f(y))\,dy$$

Here $\xi = \dfrac{x+\bar{x}}{2}$ & $\delta = |x-\bar{x}|$



$\longleftarrow B(\xi,\delta)$

Estimate of I_1 By the mean value thm,

$$|I_1| \leq C|x-\bar{x}| \int_{\partial B_2} |\nabla \Gamma_{x_i}(\hat{x}-y)|\, ds \quad \text{for some } \hat{x} \text{ between } x \,\&\, \bar{x} \; (\hat{x} \text{ depends on } y)$$

$$\leq C\frac{|x-\bar{x}|}{R} \quad \text{since } |\hat{x}-y| \geq R \text{ for } \forall y \in \partial B_2$$

$$\leq C\left(\frac{\delta}{R}\right)^{\alpha} \quad \text{since } \delta = |x-\bar{x}| < 2R$$

Estimate of I_2

$$|I_2| \leq \frac{C}{R^{n-1}} \int_{\partial B_2} ds \leq C$$

Estimate of I_3

$$|I_3| \leq C[f]_{\alpha,B_2} \int_{B(\xi,\delta)} |x-y|^{\alpha-n}\, dy$$

$$\leq C[f]_{\alpha,B_2} \int_{B(x,\frac{3\delta}{2})} |x-y|^{\alpha-n}\, dy$$

$$= C\delta^{\alpha}[f]_{\alpha,B_2}$$

$B(\xi,\delta) \subset B(x,\frac{3\delta}{2})$

Estimate of I_4

$|I_4| \leq C\delta^{\alpha}[f]_{\alpha,B_2}$, as in the estimate of I_3.

Estimate of I_5 Integrate by parts to obtain

$$|I_5| = \left| \int_{\partial(B_2 \setminus B(\xi,\delta))} \Gamma_{x_i}(x-y)\, n_j\, ds \right|$$

$$\leq \int_{\partial B_2} |\Gamma_{x_i}(x-y)|\, ds + \int_{\partial B(\xi,\delta)} |\Gamma_{x_i}(x-y)|\, ds$$

$$\leq \frac{C}{R^{n-1}} \int_{\partial B_2} ds + \frac{C}{\delta^{n-1}} \int_{\partial B(\xi,\delta)} ds$$

$$\leq C$$

Estimate of I_6. By the mean value thm,

$$|I_6| \leq C|x-\bar{x}| \int_{B_2 \setminus B(\xi,\delta)} |\nabla \Gamma_{x_ix_j}(\hat{x}-y)|\,|f(\bar{x})-f(y)|\,dy$$

for some $\hat{x}$ between x & $\bar{x}$ ($\hat{x}$ depends on y)

$$\leq C\delta \int_{|y-\xi|\geq\delta} \frac{|f(\bar{x})-f(y)|}{|\hat{x}-y|^{n+1}}\,dy$$

$$\leq C\,\delta\,[f]_{\alpha,B_2} \int_{|y-\xi|\geq\delta} \frac{|\bar{x}-y|^{\alpha}}{|\hat{x}-y|^{n+1}}\,dy$$

Now if $|y-\xi|\geq\delta$, we have

(8) $|\bar{x}-y| \leq |\bar{x}-\xi|+|\xi-y| = \frac{1}{2}\delta+|\xi-y| \leq \frac{3}{2}|\xi-y|$

and

$$|\xi-y| \leq |\hat{x}-y|+|\xi-\hat{x}| = |\hat{x}-y|+\frac{1}{2}\delta$$
$$\leq |\hat{x}-y|+\frac{1}{2}|\xi-y|, \text{ so that}$$

(9) $|\xi-y| \leq 2|\hat{x}-y|$.

Plug (8) & (9) into the estimate on $|I_6|$ above to get

$$|I_6| \leq C\,\delta\,[f]_{\alpha,B_2} \int_{|y-\xi|\geq\delta} |\xi-y|^{\alpha-n-1}\,dy$$
$$= C\delta^{\alpha}[f]_{\alpha,B_2}.$$

Now collect all the estimates on $I_1,\dots I_6$ & plug into (7) to obtain

(10) $$|w_{x_ix_j}(\bar{x})-w_{x_ix_j}(x)| \leq C\left(\frac{\sup_{B_2}|f|}{R^{\alpha}}+[f]_{\alpha,B_2}\right)\delta^{\alpha}$$

for $\delta=|x-\bar{x}|$

Estimates (5) & (10) together prove the lemma. //

$\boxed{7.5}$ **Lemma 7.5** Let $B_1^+=B(x_0,R)\cap\{x_n\geq 0\}$ & $B_2^+=B(x_0,2R)\cap\{x_n\geq 0\}$ for some $x_0\in\{x_n\geq 0\}$. Assume

$$f\in C^{\alpha}(B_2^+) \text{ for some } \boxed{0<\alpha<1}$$

Let w be the Newtonian potential of f in B_2^+. Then

$$w\in C^{2,\alpha}(B_1^+)$$

$$\sup_{B_1^+}|W_{x_ix_j}| + R^\alpha [W_{x_ix_j}]_{\alpha, B_1^+} \le C\left(\sup_{B_2^+}|f| + R^\alpha [f]_{\alpha, B_2^+}\right)$$

Proof We have the representation (2) with $\Omega_0 = B_2^+$. We first estimate $W_{x_ix_j}$ for the case that either i or $j \neq n$. In this situation the part of the boundary integral

$$\int_{\partial B_2^+ \cap \{x_n = 0\}} \Gamma_{x_i}(x-y)\, n_j \, ds = 0 \qquad i \text{ or } j \neq n$$

& then the methods of the last lemma can be employed (with $B(\xi,\delta) \cap B_2^+$ in place of $B(\xi,\delta)$ & $\partial B_2^+ \setminus \{x_n = 0\}$ in place of ∂B_2). This gives the required estimate for $W_{x_ix_j}$ if i or $j \neq n$. The estimate for $W_{x_nx_n}$ follows from this & the equation

$$\Delta w = f \text{ in } B_2^+.$$

$$W_{x_nx_n} + \sum_{i=1}^{n-1} W_{x_ix_i}$$

If $j \neq n$ clear since $n_j = 0$. Let $j = n$. Then $n_n = -1$

$\Gamma_{x_i}(x-y) = k(x_i - y_i)|x-y|^{-n}$

So, for example, in 2 dim $\xi(y_1) = k(x_1 - y_1)|x-y|^{-n}$; ξ is odd $\Rightarrow$ $\int \xi = 0$.

C. Global $C^{2,\alpha}$ estimates

7.6 From Lemma 7.4 & the representation of a smooth function with compact support in terms of the Newtonian potential of its Laplacian we obtain

Theorem 7.6 Let $u \in C_0^\infty(\mathbb{R}^n)$ be a smooth function with compact support solving

$$\Delta u = f \quad \text{in } \mathbb{R}^n.$$

If $B = B(x_0, R)$ is any ball containing the support of u, then

$$\sup_{\mathbb{R}^n}|u| \le CR^2 \sup_{\mathbb{R}^n}|f|$$

$$\sup_{\mathbb{R}^n}|\nabla u| \le CR \sup_{\mathbb{R}^n}|f| \tag{11}$$

$$\sup_{\mathbb{R}^n}|u_{x_ix_j}| + R^\alpha [u_{x_ix_j}]_\alpha \le C\left(\sup_{\mathbb{R}^n}|f| + R^\alpha |f|_\alpha\right)$$

$0 < \alpha < 1$, $C = C(n,\alpha)$

Proof Since u has compact support we have

$$u(x) = \int_{\mathbb{R}^n} \Gamma(x-y) f(y)\, dy = \int_B \Gamma(x-y) f(y)\, dy$$

The first two estimates follow from this representation, Lemma 7.1, & the facts

$$|\Gamma(x)| \leq \frac{C}{|x|^{n-2}}, \quad |\nabla\Gamma(x)| \leq \frac{C}{|x|^{n-1}}$$ (158)

The last estimate comes from Lemma 7.4: note that $f = \Delta u \equiv 0$ on $\underbrace{B(x_1 2R) \setminus B(x_0,R)}_{B}$

|| Liou's Lemma (Lee Berger pg 35)

Now consider the general elliptic equation

$$(*) \begin{cases} -a_{ij} u_{x_i x_j} + b_i u_{x_i} + cu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Assumptions on the coefficients

(a) $a_{ij}, b_i, c \in C^\alpha(\overline{\Omega})$ for some $0<\alpha<1$

(b) the a_{ij} satisfy the ellipticity condition (E)

Lemma 7.7 (a)

$$\boxed{\|fg\|_{C^\alpha(\overline{\Omega})} \leq \|f\|_{C^\alpha(\overline{\Omega})} \|g\|_{L^\infty(\Omega)} + \|g\|_{C^\alpha(\overline{\Omega})} \|f\|_{L^\infty(\Omega)}}$$

$X_1 \subset X_2 \subset X_3$ compact

(159)

(b) $\forall \varepsilon > 0 \; \exists C(\varepsilon) > 0 \ni$

$$\boxed{\|u\|_{C^{1,1}(\overline{\Omega})} \leq \varepsilon \|u\|_{C^{2,\alpha}(\overline{\Omega})} + C(\varepsilon) \sup_\Omega |u|}$$

$X_3 = \cancel{\sup |\cdot|}\; C^0(\Omega)$, $X_2 = C^{1,1}(\overline{\Omega})$, $X_1 = C^{2,\alpha}(\Omega)$

Proof (a)

$$[fg]_{\alpha,\Omega} = \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x)g(x) - f(y)g(y)|}{|x-y|^\alpha}$$

$$\leq \sup_{\substack{x,y\in\Omega \\ x\neq y}} \left(|f(x)| \frac{|g(x)-g(y)|}{|x-y|^\alpha} + |g(y)| \frac{|f(x)-f(y)|}{|x-y|^\alpha} \right)$$ //

(b) This follows as in the proof of Lemma 6.11: the space $C^{2,\alpha}(\overline{\Omega})$ is compactly embedded in $C^{1,1}(\overline{\Omega})$ (Ascoli Thm). //

As in Chapter VI, § D we first derive the required estimates for the cases of a ball & a half ball:

Lemma 7.8 Assume that the coefficients a_{ij}, b_i & c satisfy hypotheses (a) & (b) (with Ω replaced by some ball centered at the origin).

Then $\exists$ constants $R_0 > 0$, $C_1, C_2 > 0$, depending only on the C^α norms of the coefficients, θ, Θ, n & $0<\alpha<1$ $\ni$ if

$$\frac{R_0}{4} < R \leq R_0$$

& u solves $(*)$ in $\Omega = B(R)$, with $u \equiv 0$ near $\partial B(R)$, then

is needed because of the "compact support" in Th. 7.6

$$(12) \quad \boxed{\|u\|_{C^{2,\alpha}(B(R))} \leq C_1 \|f\|_{C^\alpha(B(R))} + C_2 \|u\|_{C^{1,1}(B(R))}}$$

<u>Proof</u> We mimic the proof of Lemma 6.8 & so, changing to new variables z, rewrite $(*)$ as

$$\begin{cases} \Delta u = f' & \text{in } B' \\ u \equiv 0 & \text{near } \partial B', \end{cases}$$

where B' is an ellipsoid (= image of $B(R)$ under the change to the z-variables) &

$$(13) \quad f'(z) = (a_{ij}''(0) - a_{ij}''(z))\, u_{z_i z_j} + b_i'' u_{z_i} + cu - f$$

By Theorem 7.6 we have

$$\|u\|_{C^{2,\alpha}(B')} \leq \frac{C}{R^\alpha} \sup_{B'} |f| + C\,[f]_{\alpha, B'}$$

$$(14) \quad \leq C(R)\left(\|f\|_{C^\alpha(B')} + \|u\|_{C^{1,1}(B')}\right) + C \sup_{\substack{1\leq i,j\leq n \\ z\in B'}} |a_{ij}''(0) - a_{ij}''(z)| \sum_{i,j=1}^{n} [u_{z_i z_j}]_{\alpha, B'}$$

by (13) & Lemma 7.7(a).

Next choose R_0 so small that if $R_0/2 \leq R \leq R_0$, then

$$(15) \quad C \sup_{\substack{1\leq i,j\leq n \\ z\in B'}} |a_{ij}''(0) - a_{ij}''(z)| \leq 1/2.$$

We plug this into the calculations above to get the desired estimate in B'; converting back to the original variables x, we have proved estimate (12) ||

<u>Remark</u> Note that the <u>Hölder</u> continuity of the a_{ij} is used not for (15) (this requires only continuity), but rather in obtaining the last part of (14) from Lemma 7.7 (a) ||

17.2 Lemma 7.9 Assume the coefficients a_{ij}, b_i & c satisfy hypotheses (a) & (b) on p. (158).

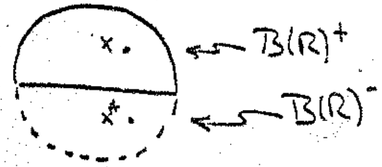
Then $\exists$ constants $R_0 > 0$, $C_1, C_2 > 0$, depending only on the C^α norms of the coefficients, θ, Θ, and $0 < \alpha < 1$ $\ni$ if

$$\frac{R_0}{4} \le R \le R_0$$

& u solves $(*)$ in $\Omega = B(R)^+$, with $u \equiv 0$ near $\partial B(R)^+ \setminus \{x_n = 0\}$ & $u = 0$ on $\{x_n = 0\}$, then

$$(16) \quad \boxed{\|u\|_{C^{2,\alpha}(B(R)^+)} \le C_1 \|f\|_{C^\alpha(B(R)^+)} + C_2 \|u\|_{C^{1,1}(B(R)^+)}}$$

Proof Define $x^* = (x_1, \cdots x_{n-1}, -x_n) \in B(R)^-$ for $x = (x_1, \cdots x_{n-1}, x_n) \in B(R)^+$



Set $f^*(x) = \begin{cases} f(x) & \text{if } x \in B(R)^+ \\ f(x_1, \cdots x_{n-1}, -x_n) & \text{if } x \in B(R)^- \end{cases}$

(Note this is not the "extension by reflection" used in Lemma 6.9)

Then

$$(17) \quad \|f^*\|_{C(B(R))} \le 2\|f\|_{C^\alpha(B(R)^+)}.$$

Now as in the proof of Lemma 6.9 we convert to new coordinates $z \ni (*)$ becomes

$$\begin{cases} \Delta u = f' & \text{in } (B')^+ \\ u \equiv 0 & \text{near } \partial B' \setminus \{z_n = 0\} \\ u = 0 & \text{on } \{z_n = 0\} \end{cases}$$

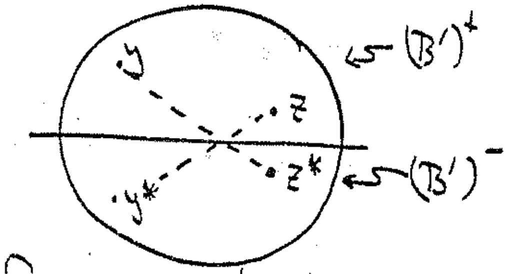
where $(B')^+$ is the image of $B(R)^+$ under the change to the new variables z & f' is defined by (13).

We have the representation

$$(18) \quad u(z) = \int_{(B')^+} \left(\Gamma(z-y) - \Gamma(z^*-y)\right) f'(y)\, dy$$

(We may check (as in Proposition 7.3) that the expression w on the right hand side of (18) solves $\Delta w = f'$ in $\{z_n > 0\}$, $w = 0$ on $\{z_n = 0\}$. If we extend u to be zero on $\{z_n > 0\} \setminus (B')^+$ we see that w solves the same problem in $\{z_n > 0\}$ & so, by uniqueness, $w = u$)

Now $|z^* - y| = |z - y^*|$ (see picture)
& so $\Gamma(z^* - y) = \Gamma(z - y^*)$



Hence

$$\int_{(B')^+} \Gamma(z^* - y) f'(y)\, dy = \int_{(B')^+} \Gamma(z - y^*) f'(y)\, dy$$

$$= \int_{(B')^-} \Gamma(z - y) f'^*(y)\, dy.$$

We plug this into (18) to obtain

$$(19)\ u(z) = 2\int_{(B')^+} \Gamma(z-y) f'(y)\, dy - \int_{B'} \Gamma(z-y) f'^*(y)\, dy.$$

Then by Lemma 7.4, Lemma 7.5, (17), & (19)

$$\|u\|_{C^{2,\alpha}((B')^+)} \leq C \|f'\|_{C^\alpha((B')^+)}.$$

This estimate & the definition of f' lead (as in the proof of Lemma 6.9) to the proof of the lemma. ||

7.8 Theorem 7.10 (Global Schauder Estimates)

Assume that u is a smooth solution of

$$(*) \begin{cases} -a_{ij} u_{x_i x_j} + b_i u_{x_i} + cu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where the coefficients satisfy hypotheses (a) & (b), and $\partial\Omega$ is smooth.

Then $\exists$ constants C_1 & C_2, depending only on the C^α norms of the coefficients, Ω, n, α, and θ $\ni$

$$(20)\quad \boxed{\|u\|_{C^{2,\alpha}(\bar\Omega)} \leq C_1 \|f\|_{C^\alpha(\bar\Omega)} + C_2 \sup_\Omega |u|}$$

If $c \leq 0$ in Ω, we may take $C_2 = 0$.

Proof The proof of estimate (20) follows from Lemmas 7.8 & 7.9, exactly as Theorem 6.10 followed from Lemmas 6.8 & 6.9: We cover Ω with finitely many balls B_k of radius $\frac{R_0}{4} \leq r_k \leq \frac{R_0}{2}$ & consider the 2 cases that (i) $B_k \subset \Omega$ or (ii) $B_k \cap \partial\Omega \neq \phi$.

If $c \leq 0$, then by Theorem 3.1

$$\sup_\Omega |u| \leq C \sup_\Omega |f|.$$

||

Remarks (a) A careful examination of the proof shows that the global Schauder estimates require that $\partial\Omega$ be of class $C^{2,\alpha}$ (i.e. $\partial\Omega$ can be written locally as the graph of a $C^{2,\alpha}$ function w; cf. p. 29 & 30)

(b) The estimates extend to the case that u is not zero on $\partial\Omega$, but rather takes on smooth boundary values ϕ. In this case we use the Remark on p. 19 to reduce to the case $u = 0$ on $\partial\Omega$. This gives

$$(\circ)' \quad \|u\|_{C^{2,\alpha}(\bar\Omega)} \le C\Big(\|f\|_{C^{\alpha}(\bar\Omega)} + \sup_{\Omega}|u| + \|\phi\|_{C^{2,\alpha}(\bar\Omega)}\Big),$$

where we have assumed that ϕ is the restriction to $\partial\Omega$ of a $C^{2,\alpha}$ function (also denoted by ϕ) defined on all of $\bar\Omega$. See [G-T, p. 131] a proof that such an extension exists. //

D. Interior $C^{2,\alpha}$ estimates

$\boxed{7.9}$ Definition Let $x, y \in \Omega$.

$$d_x = \operatorname{dist}(x, \partial\Omega)$$
$$d_y = \operatorname{dist}(y, \partial\Omega)$$
$$d_{xy} = \min(d_x, d_y) \qquad (\text{Cf. p. } 78)$$

Definitions For $0 < \alpha \le 1$ define:

(a) $$\|u\|^{*}_{C^{\alpha}(\Omega)} = \sup_{\Omega}|u| + \sup_{\substack{x,y\in\Omega \\ x\ne y}} d_{xy}^{\alpha}\,\frac{|u(x)-u(y)|}{|x-y|^{\alpha}}$$

(b) $$\|u\|^{*}_{C^{1,\alpha}(\Omega)} = \|u\|^{*}_{C^{\alpha}(\Omega)} + \sum_{i=1}^{n}\Big(\sup_{x\in\Omega} d_x|u_{x_i}(x)| + \sup_{\substack{x,y\in\Omega \\ x\ne y}} d_{xy}^{1+\alpha}\,\frac{|u_{x_i}(x)-u_{x_i}(y)|}{|x-y|^{\alpha}}\Big)$$

(c) $$\|u\|^{*}_{C^{2,\alpha}(\Omega)} = \|u\|^{*}_{C^{1,\alpha}(\Omega)} + \sum_{i,j=1}^{n}\Big(\sup_{x\in\Omega} d_x^2|u_{x_ix_j}(x)| + \sup_{\substack{x,y\in\Omega \\ x\ne y}} d_{xy}^{2+\alpha}\,\frac{|u_{x_ix_j}(x)-u_{x_ix_j}(y)|}{|x-y|^{\alpha}}\Big)$$

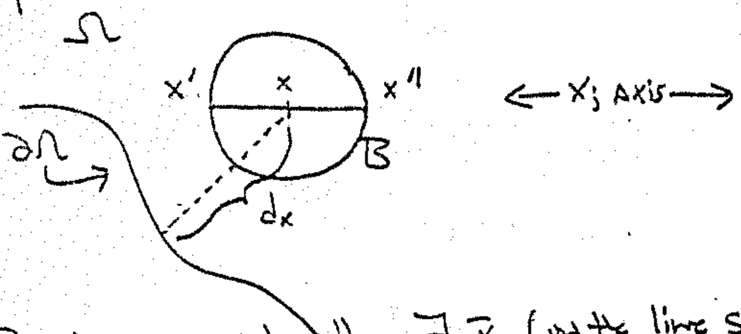
These are "weighted" versions of the ordinary C^{α}, $C^{1,\alpha}$ etc norms defined on p. 146 and (since d_x & d_{xy} are small near $\partial\Omega$) allow for bad behavior of u & its derivatives near $\partial\Omega$.

Lemma 7.11 $\forall \varepsilon > 0 \; \exists \; C(\varepsilon) = C(\varepsilon,\alpha) > 0 \ni$

$$(21) \quad \boxed{\|u\|^*_{C^{1,1}(\Omega)} \le \varepsilon \|u\|^*_{C^{2,\alpha}(\Omega)} + C(\varepsilon) \sup_\Omega |u|}$$

Proof Pick any $x \in \Omega$ & set $d = \mu d_x$ (where $\mu \le 1/2$ will be selected later).

Let $B = B(x,d)$. Choose x' & x'' to be the endpoints of a line segment with center x, parallel to the x_j axis. (See picture)



By the mean value thm, $\exists \; \bar{x}$ (on the line segment between x' & x'') $\ni$

$$|u_{x_ix_j}(\bar{x})| = \frac{|u_{x_i}(x') - u_{x_i}(x'')|}{2d} \le \frac{1}{d} \sup_B |\nabla u|$$

Hence

$$|u_{x_ix_j}(x)| \le |u_{x_ix_j}(\bar{x})| + |u_{x_ix_j}(\bar{x}) - u_{x_ix_j}(x)|$$

$$\le \frac{1}{d} \sup_B |\nabla u| + \Big(\sup_{\substack{z,y \in \Omega \\ z \ne y}} \frac{|u_{x_ix_j}(y) - u_{x_ix_j}(z)|}{|y-z|^\alpha}\Big) d^\alpha$$

$$\le \frac{1}{d}\Big(\sup_{y\in B} \frac{1}{d_y}\Big)\Big(\sup_{y \in B} d_y |\nabla u(y)|\Big) + d^\alpha \Big(\sup_{y,z \in B} \frac{1}{d_y^{2+\alpha}}\Big)\Big(\sup_{\substack{y,z\in B \\ y \ne z}} d_{yz}^{2+\alpha} \frac{|u_{x_ix_j}(y) - u_{x_ix_j}(z)|}{|y-z|^\alpha}\Big)$$

Now $d = \mu d_x$ & $d_y, d_{yz} \ge \frac{1}{2} d_x \; \forall y,z \in B$; therefore the inequality above gives

$$d_x^2 |u_{x_ix_j}(x)| \le \frac{C}{\mu} \|u\|^*_{C^{0,1}(\Omega)} + C\mu^\alpha \|u\|^*_{C^{2,\alpha}(\Omega)}$$

We now choose μ so small that $C\mu^\alpha = \varepsilon$; this yields

$$(22) \quad \|u\|^*_{C^{1,1}(\Omega)} \le \varepsilon \|u\|^*_{C^{2,\alpha}(\Omega)} + C(\varepsilon)\|u\|^*_{C^{0,1}(\Omega)}$$

The same argument with u in place of u_{x_i} gives

$$(23) \quad \|u\|^*_{C^{0,1}(\Omega)} \le \varepsilon \|u\|^*_{C^{1,\alpha}(\Omega)} + C(\varepsilon) \sup_\Omega |u|.$$

Estimates (22) & (23) together imply (21). $\|$

7.10 **Theorem 7.12** (Interior Schauder Estimates)

Assume that u is a smooth solution of

$$(*)\ \sum -a_{ij}u_{x_ix_j} + b_i u_{x_i} + cu = f$$

in Ω & that the coefficients of the elliptic operator satisfy hypotheses (a) & (b) (on p. 158)

Then $\exists$ constants C_1 & C_2, depending only on the C^α norms of the coefficients, n, α, and θ $\ni$

$$(24)\quad \boxed{\|u\|^*_{C^{2,\alpha}(\Omega)} \le C_1 \|f\|^*_{C^\alpha(\Omega)} + C_2 \sup_\Omega |u|}$$

Corollary 7.13 Under the same hypotheses

$$\boxed{\|u\|_{C^{2,\alpha}(\Omega')} \le C_1 \|f\|_{C^\alpha(\Omega)} + C_2 \sup_\Omega |u|}$$

$\forall\ \Omega' \subset\subset \Omega$. (Here C_1 & C_2 depend also on $\mathrm{dist}(\Omega', \partial\Omega) > 0$)

Proof Define

$$(25)\quad M(u) \equiv \sup_{\substack{x,y\in\Omega \\ x\neq y}} d_{xy}^{2+\alpha} \sum_{i,j=1}^n \frac{|u_{x_ix_j}(x) - u_{x_ix_j}(y)|}{|x-y|^\alpha}$$

& pick 2 points $x, y \in \Omega$ $\ni$

$$(26)\quad \frac{1}{2} M(u) \le d_{xy}^{2+\alpha} \sum_{i,j=1}^n \frac{|u_{x_ix_j}(x) - u_{x_ix_j}(y)|}{|x-y|^\alpha}$$

Case 1: $|x-y| \ge \frac{1}{4} d_{xy}$ In this situation,

$$\frac{1}{2} M(u) \le C d_{xy}^2 \sum_{i,j=1}^n \left(|u_{x_ix_j}(x)| + |u_{x_ix_j}(y)|\right)$$

$$(27)\quad \le C \|u\|^*_{C^{1,1}(\Omega)}$$

Case 2: $|x-y| < \frac{1}{4} d_{xy}$ Let us assume WLOG that $d_{xy} = d_x$.

Define $B = B(x, \frac{3d_x}{4})$ & choose ζ be be a smooth cutoff function $\ni$

$$(28)\quad \begin{cases} \zeta \equiv 1 \text{ on } B(x, \frac{d_x}{2}) \\ \zeta \equiv 0 \text{ near } \partial B \\ |\zeta_{x_i}| \le \frac{C}{d_x}, \quad |\zeta|_{C^\alpha} \le \frac{C}{(d_x)^\alpha}, \quad |\zeta|_{C^{1,\alpha}} \le \frac{C}{(d_x)^{1+\alpha}} \\ |\zeta_{x_ix_j}| \le \frac{C}{(d_x)^2}, \quad |\zeta|_{C^{2,\alpha}} \le \frac{C}{(d_x)^{2+\alpha}} \end{cases}$$

In the ball B, the function

$$v \equiv \zeta u$$

solves
$$\begin{cases} Lv = f' \text{ in } B, & v \equiv 0 \text{ near } \partial B \end{cases}$$

where $Lv = -a_{ij}v_{x_ix_j} + b_i v_{x_i} + cv$

and
$$f' = f\zeta - a_{ij}(2u_{x_i}\zeta_{x_j} + u\zeta_{x_ix_j}) + b_i u \zeta_{x_i}.$$

We extend v to be $\equiv 0$ on $\Omega \setminus B$ & apply the global Schauder estimate (Theorem 7.10). This yields

$$\sum_{i,j}^{n} \frac{|u_{x_ix_j}(x) - u_{x_ix_j}(y)|}{|x-y|^{\alpha}} \leq \|v\|_{C^{2,\alpha}(\bar{\Omega})}$$

(29)
$$\leq C\|f'\|_{C^{\alpha}(\bar{\Omega})} + C \sup_{\Omega} |u|$$

Now the definition of f' & the estimates on ζ & its derivatives from (28) give:

$$\|f'\|_{C^{\alpha}(\Omega)} \leq C\left(\frac{\|f\|^*_{C^{\alpha}}}{d_x^{\alpha}} + \frac{1}{(d_x)^{2+\alpha}} \|u\|^*_{C^{1,1}(\Omega)} \right)$$

We plug this estimate into (29), multiply by $(d_x)^{2+\alpha} = (d_{xy})^{2+\alpha}$ & recall (26):

(30) $$M(u) \leq C\|u\|^*_{C^{1,1}(\Omega)} + C\|f\|^*_{C^{\alpha}(\Omega)}$$

Thus both cases give the same inequality (30). By the definition of $M(u)$ & of $\|u\|^*_{C^{2,\alpha}(\Omega)}$ we therefore have

$$\|u\|^*_{C^{2,\alpha}(\Omega)} \leq C\|u\|^*_{C^{1,1}(\Omega)} + C\|f\|^*_{C^{\alpha}(\Omega)}$$
$$\leq \varepsilon \|u\|^*_{C^{2,\alpha}(\Omega)} + C(\varepsilon)\sup_{\Omega}|u| + C\|f\|^*_{C^{\alpha}(\Omega)}$$

by Lemma 7.11. ||

Hölder Continuity of Solutions of Parabolic Equations in Divergence Form

A. Introduction

We'll assume u is a smooth solution of

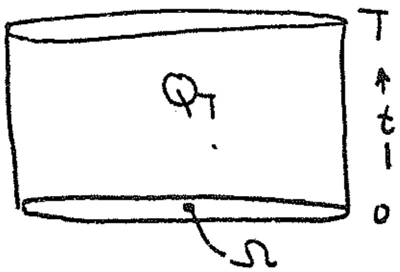
$$(*)\quad \begin{cases} u_t - (a_{ij} u_{x_i})_{x_j} = (f_i)_{x_i} & \text{in } Q_T \subset \mathbb{R}^{n+1} \\ u = 0 & \text{on } \Gamma_T\,, \end{cases}$$

where

(a) the a_{ij} satisfy the ellipticity condition

$$(E)\quad \begin{cases} \exists\ \Theta \geq \theta > 0 \ni \\ \theta|\xi|^2 \leq a_{ij}(x,t)\xi_i\xi_j \leq \Theta|\xi|^2 \quad \forall (x,t)\in Q_T \\ \qquad \& \ \forall \xi \in \mathbb{R}^n \end{cases}$$

(b) $Q_T \equiv \Omega \times (0,T)$



(c) $\Gamma_T \equiv \partial\Omega \times [0,T] \cup \{(x,0) \mid x \in \Omega\}$

= boundary of Q_T (except for the "top")

= parabolic boundary of Q_T

Definition Suppose $u: Q_T \to \mathbb{R}$. Define the norm

$$\boxed{|u|_{Q_T} \equiv \left[\sup_{0\leq t\leq T} \int_\Omega u(x,t)^2\,dx + \int_0^T\!\!\int_\Omega |\nabla u(x,t)|^2\,dx\,dt\right]^{1/2}}$$

Here & afterwards $\nabla u(x,t) = (u_{x_1}(x,t), \cdots u_{x_n}(x,t))$; this is the gradient with respect to the x-variables only

Lemma 1 Suppose $u=0$ on Γ_T. Then $\exists\ C$, depending only on n, $\ni$

$$\boxed{\|u\|_{L^{2(1+2/n)}(Q_T)} \leq C|u|_{Q_T}}$$

Proof Set $s = 2(1+2/n)$; then

$$2 < s < 2^* \left(= \tfrac{2n}{n-2}\right).$$

Hence for each fixed $0 \leq t \leq T$

$$(1)\quad \|u\|_{L^s(\Omega)} \leq \|u\|^{\alpha}_{L^{2^*}(\Omega)} \|u\|^{1-\alpha}_{L^2(\Omega)}$$

③

for $\frac{1}{s} = \frac{\alpha}{2^*} + \frac{(1-\alpha)}{2}$

We recall the definition of 2^* & solve for α to find

(2) $\qquad \alpha = \frac{n}{n+2}$

Then (1) & Sobolev's inequality imply

$$\int_0^T\int_\Omega |u(x,t)|^s \,dx\,dt = \int_0^T \|u(\cdot,t)\|^s_{L^s(\Omega)}\,dt$$

$$\leq C\int_0^T \|\nabla u(\cdot,t)\|^{\alpha s}_{L^2(\Omega)} \|u(\cdot,t)\|^{(1-\alpha)s}_{L^2(\Omega)}\,dt$$

$$\leq C \sup_{0\leq t\leq T} \|u(\cdot,t)\|^{(1-\alpha)s}_{L^2(\Omega)} \int_0^T \|\nabla u(\cdot,t)\|^{\alpha s}_{L^2(\Omega)}\,dt$$

But by (2): $\alpha s = \frac{n}{n+2}\cdot 2\left(\frac{n+2}{n}\right) = 2$ & so

$$\left(\int_0^T\int_\Omega |u(x,t)|^s\,dx\,dt\right)^{1/s} \leq C\left(\sup_{0\leq t\leq T}\|u(\cdot,t)\|_{L^2(\Omega)}^{\frac{s-2}{s}}\right)\cdot\left(\int_0^T\int_\Omega |\nabla u|^2\,dx\,dt\right)^{1/s}$$

$$\leq C\left(\sup_{0\leq t\leq T}\|u(\cdot,t)\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(Q_T)}\right)$$

(by Young's inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, with $p = \frac{s}{s-2}$, $q = \frac{s}{2}$, $\frac{1}{p}+\frac{1}{q} = 1$)

$$\leq C\,|u|_{Q_T} \qquad //$$

Recall

Lemma 2 Suppose $\phi : [0,\infty) \to [0,\infty)$ is nonincreasing & $\exists$ constants $C \geq 0, \alpha > 0, \boxed{\beta > 1} \ni$

$$\boxed{\phi(h) \leq \frac{C}{(h-k)^\alpha}[\phi(k)]^\beta} \quad \forall h > k \geq 0$$

Then $\qquad \phi(d) = 0,$

for $\quad d = \left(C\,\phi(0)^{\beta-1}\,2^{\alpha\beta/(\beta-1)}\right)^{1/\alpha}$

(See p. 45 of class notes)

B. G[illegible] L^∞ estimates

Theorem 3 Let $u : \overline{Q}_T \to \mathbb{R}$ be a smooth solution of

$$(*) \quad \begin{cases} u_t - (a_{ij} u_{x_i})_{x_j} = (f_i)_{x_i} & \text{in } Q_T \\ u = 0 & \text{on } \Gamma_T, \end{cases}$$

where the a_{ij} satisfy (E) &

$$(3) \qquad f_i \in L^p(Q_T) \quad \text{for some } \boxed{p > n+2}.$$

Then $\exists\, C \ni$

$$(4) \qquad \boxed{\|u\|_{L^\infty(Q_T)} \le C \sum_{i=1}^n \|f_i\|_{L^p(Q_T)} \left[\mathrm{mes}(Q_T)\right]^{\frac{1}{n+2} - \frac{1}{p}}}$$

Proof Define

$$A_k(t) \equiv \{x \in \Omega \mid u(x,t) > k\} \qquad 0 \le t \le T,\ k \ge 0$$

&

$$\phi(k) = \int_0^T \mathrm{mes}\, A_k(t)\, dt$$

$$= \mathrm{mes}\{(x,t) \in Q_T \mid u(x,t) > k\}$$

Now fix any $k \ge 0$ & calculate

$$\int_0^T \frac{d}{dt} \int_\Omega (u-k)^{+2}\, dx\, dt = 2 \int_0^T \int_\Omega (u-k)^+ u_t\, dx\, dt$$

$$= 2 \int_0^T \int_\Omega (u-k)^+ (a_{ij} u_{x_i})_{x_j}\, dx\, dt$$

$$+ 2 \int_0^T \int_\Omega (u-k)^+ (f_i)_{x_i}\, dx\, dt \qquad \text{by } (*)$$

$$= -2 \int_0^T \int_{A_k(t)} a_{ij} u_{x_i} u_{x_j}\, dx\, dt$$

$$- 2 \int_0^T \int_{A_k(t)} u_{x_i} f_i\, dx\, dt$$

The upper limit of integration T could have been replaced by any $0 \le T^* \le T$. Hence

$$\sup_{0 \le t \le T} \int_\Omega (u(x,t) - k)^{+2}\, dx + 2 \int_0^T \int_{A_k(t)} a_{ij} u_{x_i} u_{x_j}\, dx\, dt$$

$$\le 4 \int_0^T \int_{A_k(t)} |u_{x_i}|\, |f_i|\, dx\, dt.$$

$$\sup_{0\le t\le T}\int_\Omega (u-k)^{+2}\,dx + \int_0^T\!\!\int_{A_k(t)} |\nabla u|^2\,dx\,dt$$

(5)

$$\le C\sum_{i=1}^{n}\int_0^T\!\!\int_{A_k(t)} |f_i|^2\,dx\,dt$$

The right hand side of (5) is

$$\le C\sum_{i=1}^{n}\int_0^T \left(\int_\Omega |f_i|^{2q}dx\right)^{1/q} \operatorname{mes} A_k(t)^{\frac{q-1}{q}}\,dt$$

$$\le C\sum_{i=1}^{n}\left(\int_0^T\!\!\int_\Omega |f_i|^{2q}\,dx\,dt\right)^{1/q}\cdot\left(\int_0^T \operatorname{mes} A_k(t)\,dt\right)^{\frac{q-1}{q}}$$

for $q=\frac{p}{2}$ (p from (3))

$$\le C\left(\sum_{i=1}^{n}\|f_i\|^2_{L^p(Q_T)}\right)\phi(k)^{1-\frac{2}{p}}$$

The left hand side of (5) is

$$|(u-k)^+|^2_{Q_T} \ge C\,\|(u-k)^+\|^2_{L^{2(1+2/n)}(Q_T)}$$

by Lemma 1

We combine these estimates of the left- & right hand sides of (5) to obtain

$$\left(\int_0^T\!\!\int_{A_k(t)} (u-k)^{+}\,dx\,dt\right) \le C\left(\sum_{i=1}^{n}\|f_i\|^2_{L^p(Q_T)}\right)\phi(k)^{1-2/p} \qquad (6)$$

But for $h>k$ we have

$$(h-k)^2\,\phi(h)^{n/n+2} = (h-k)^2\left(\int_0^T \operatorname{mes} A_h(t)\,dt\right)^{n/n+2}$$

$$\le \left(\int_0^T\!\!\int_{A_h(t)} (u-k)^{+\,2(1+2/n)}\,dx\,dt\right)^{n/n+2}$$

$$\le \left(\int_0^T\!\!\int_{A_k(t)} (u-k)^{+\,2(1+2/n)}\,dx\,dt\right)^{n/n+2}$$

This estimate & (6) yield

$$(h-k)^2\,\phi(h)^{n/n+2} \le C\left(\sum_{i=1}^{n}\|f_i\|^2_{L^p}\right)\phi(k)^{1-2/p}$$

& so

$$\phi(h) \le C\,\frac{\left(\sum_{i=1}^{n}\|f_i\|_{L^p}^{\frac{2(n+2)}{n}}\right)\phi(k)^\beta}{(h-k)^\alpha},$$

for

$$\beta = \left(1-\frac{2}{p}\right)\frac{n+2}{n}$$

& $\alpha = \dfrac{2(n+2)}{n}$.

$$\geq |\nabla (v^{p/2})|^2$$

& so (8) gives

$$\sup_{-R^2 \le t \le 0} \int_{B(R)} \zeta^2 v^p \, dx + \iint_{Q(R)} |\nabla (v^{p/2} \zeta)|^2 \, dx\,dt$$

(9) $$\le C \iint_{Q(R)} v^p (\zeta |\zeta_t| + |\nabla \zeta|^2) \, dx\,dt.$$

The expression on the left hand side is $|\zeta v^{p/2}|^2_{Q_T}$ & so by Lemma 1:

(10) $$\left(\iint_{Q(R)} (\zeta v^{p/2})^{\frac{2(n+2)}{n}} \, dx\,dt \right)^{n/n+2}$$

$$\le C \iint_{Q(R)} v^p (\zeta |\zeta_t| + |\nabla \zeta|^2) \, dx\,dt;$$

the constant C does not depend on ζ, p, or R.

We now iterate inequality (10) for various choices of p & ζ:

$$\boxed{R_k = \frac{R}{2}\left(1 + \frac{1}{2^k}\right)} \qquad k = 0, 1, 2, \ldots$$

& choose $\zeta = \zeta_k \ni$

$$\begin{cases} \zeta \equiv 1 \text{ on } Q(R_{k+1}), \ \zeta \equiv 0 \text{ on } Q_T \setminus Q(R_k), \\ |\nabla \zeta| \le \dfrac{2}{R_k - R_{k+1}} \le \dfrac{2^{k+3}}{R} \\ |\zeta_t| \le \dfrac{2}{R_k^2 - R_{k+1}^2} \le \dfrac{2^{2k+5}}{R^2} \end{cases}$$

Plug this choice of ζ in (10) to get

$$\left(\iint_{Q(R_{k+1})} v^{p\left(\frac{n+2}{n}\right)} \, dx\,dt \right)^{n/n+2} \le \frac{C 4^k}{R^2} \iint_{Q(R_k)} v^p \, dx\,dt$$

Now take p^{th} roots of both sides & set $p = p_k$, where

$$\boxed{p_k = 2\left(\frac{n+2}{n}\right)^k} \qquad k = 0, 1, 2, \cdots;$$

this gives

(11) $$\|v\|_{L^{p_{k+1}}(Q(R_{k+1}))} \le \frac{C^{1/p_k} 4^{k/p_k}}{R^{2/p_k}} \|v\|_{L^{p_k}(Q(R_k))}$$

Then

(12) $a_{k+1} \le \gamma_k a_k$

for $a_k = \|v\|_{L^{p_k}(Q(R_k))}$

& $\gamma_k = \dfrac{C^{1/p_k}\, 4^{k/p_k}}{R^{2/p_k}}.$

We iterate (12) to find

(13) $$\|v\|_{L^\infty(Q(R/2))} \le \lim_{n\to\infty} (\gamma_0\gamma_1 \cdots \gamma_n) \underbrace{\|v\|_{L^2(Q(R))}}_{=a_0}$$

By the ratio test $\sum_{k=0}^\infty k/p_k < \infty$ & also

$$\sum_{k=0}^\infty \frac{1}{p_k} = \frac{1}{2}\sum_{k=0}^\infty \left(\frac{n}{n+2}\right)^k = \frac{1}{2}\left(\frac{1}{1-\frac{n}{n+2}}\right) = \frac{n+2}{4}$$

Hence (13) & the definition of the γ_k give

$$\|v\|_{L^\infty(Q(R/2))} \le \frac{C}{R^{\frac{n+2}{2}}} \|v\|_{L^2(Q(R))}. \quad //$$

D. Technical lemmas

Definition Pick $(x_0,t_0) \in Q_T$. Then

$$\boxed{\omega(R) = \max_{Q(R)} u - \min_{Q(R)} u}$$

is the (parabolic) oscillation of u (with respect to the point (x_0,t_0)).

Lemma 6 Suppose $\exists$ constants $C_1 \ge 0$, $0 < \alpha \le 1$, & $\boxed{0<\eta<1}$ such that

$$\boxed{\omega(R/4) \le \eta\, \omega(R) + C_1 R^\alpha} \quad \forall\, 0<R\le R_0$$

Then $\exists$ $0<\gamma\le 1$ & $C_2 \ge 0$, depending only on C_1, α, η & $\sup \omega(R)$ $\ni$

$$\boxed{\omega(R) \le C_2 \left(\frac{R}{R_0}\right)^\gamma} \quad \forall\, 0<R\le R_0$$

(See p. 56 of class notes)

Lemma 7 Suppose that $u: B(R) \to \mathbb{R}$ & N is a subset of $B(R)$ of positive measure on which $u \equiv 0$. Assume also that $\zeta(x) = \zeta(|x|)$ is an arbitrary nonincreasing function of $|x|$ $\ni$ $0 \le \zeta \le 1$

and $\zeta(x) = 1$ for $x \in N$.

Then $\exists\, C$, depending only on n $\ni$

(14) $$\boxed{\int_{B(R)} u^2(x)\, \zeta^2(x)\, dx \le C \left(\frac{R^n}{\operatorname{mes}(N)}\right)^2 R^2 \int_{B(R)} |Du(x)|^2 \zeta^2(x)\, dx}$$

For a proof, see pages 89-91 in [L,S,U]

(= Ladyženskaja, Solonnikov, & Ural'ceva, Linear & Quasilinear Equations of Parabolic Type, Amer. Math. Society, 1968)

(Cf also p. 58 in the class notes) //

Lemma 8 Let u be a smooth solution of

$$(*)\left\{\ u_t - (a_{ij} u_{x_i})_{x_j} = 0\right.$$

in Q_T, where the a_{ij} satisfy the ellipticity condition (E). Assume that $B(R) \subset \Omega$, $0 \le u \le 1$ & that

(15) $$\boxed{\operatorname{mes}\{x \in B(R) \mid u(x,0) \le 1/2\} \ge \tfrac{1}{2} \operatorname{mes} B(R)}$$

Then $\exists$ constants $0 < b < 1$ & $\lambda > 0$, depending only on n, θ & Θ $\ni$

(16) $$\boxed{\operatorname{mes}\{x \in B(R) \mid u(x,t) \le 7/8\} \ge b \operatorname{mes} B(R)}$$

for all $\boxed{0 \le t \le \lambda R^2}$

This says that if condition (15) holds at $t=0$, it still holds (in the modified form (16)) for $0 \le t \le \lambda R^2$.

Proof Let $\zeta = \zeta(x)$ be a cutoff function, independent of t. Suppose $0 \le k \le 1$. Then

$$\int_0^t \frac{d}{dt}\int_{B(R)} (u-k)^{+2} \zeta^2\, dx\, d\tau = 2\int_0^t\int_{B(R)} (u-k)^+ u_t \zeta^2\, dx\, d\tau$$

$$= 2\int_0^t\int_{B(R)} (u-k)^+ (a_{ij} u_{x_i})_{x_j} \zeta^2\, dx\, d\tau$$

$$= -2\int_0^t\int_{A_{k,R}^{(\tau)}} a_{ij} u_{x_i} u_{x_j} \zeta^2\, dx\, d\tau$$

$$-4\int_0^t\int_{A_{k,R}^{(\tau)}} (u-k)^+ a_{ij} u_{x_i} \zeta \zeta_{x_j}\, dx\, d\tau,$$

where

$$A_{k,R}^{(\tau)} = \{x \in B(R) \mid u(x,\tau) > k\}$$

Now use (E) & standard tricks to get

$$\int_{B(R)} (u(x,t)-k)^{+2} \zeta^2\,dx \le \int_{B(R)} (u(x,0)-k)^{+2} \zeta^2\,dx$$

(17) $$+ C_1 \int_0^t \int_{B(R)} (u-k)^{+2} |\nabla \zeta|^2\,dx\,dt.$$

Now choose $1 > \sigma > 0$, $\lambda > 0$, & $1 > b > 0$ $\ni$

(18) $$\left(n\sigma + \frac{8}{9} + \frac{64}{9} \frac{C_1 \lambda}{\sigma^2} \right) = 1 - b < 1$$

These choices being made, now select $\zeta \ni$

$$\begin{cases} \zeta \equiv 1 \text{ on } B((1-\sigma)R), \ \zeta \equiv 0 \text{ on } \Omega \setminus B(R), \\ 0 \le \zeta \le 1, \ |\nabla \zeta| \le \frac{2}{\sigma R} \end{cases}$$

Plug this choice of ζ in (17) & set

$$k = 1/2.$$

Now

$$\int_{B(R)} (u(x,0) - 1/2)^{+2} \zeta^2(x)\,dx \le \int_{B(R)} (u(x,0) - 1/2)^{+2}\,dx$$

(19) $$\le \int_{A_{1/2,R}(0)} (1 - 1/2)^2\,dx \quad \text{since } 0 \le u \le 1$$

$$\le \frac{1}{4} \text{ mes } A_{1/2,R}(0) \le \frac{1}{8} \text{ mes } B(R) \text{ by (15)}$$

Also

$$\left(\frac{3}{8}\right)^2 \text{mes } A_{7/8,(1-\sigma)R}(t) \le \int_{B((1-\sigma)R)} (u(x,t) - 1/2)^{+2}\,dx$$

(20) $$\le \int_{B(R)} (u(x,t) - 1/2)^{+2} \zeta^2\,dx$$

We combine (17), (19) & (20) to find that for $\underline{0 \le t \le \lambda R^2}$

(21) $$\left(\frac{3}{8}\right)^2 \text{mes } A_{7/8,(1-\sigma)R}(t) \le \frac{1}{8} \text{ mes } B(R) + \frac{C_1}{\sigma^2} \lambda \text{ mes } B(R)$$

Note that

(22) $$1 - (1-\sigma)^n \le n\sigma \qquad \text{for } 0 < \sigma < 1$$

Hence by (21)

$$\text{mes } A_{7/8,R}(t) \le \text{mes } A_{7/8,(1-\sigma)R}(t) + \text{mes}\,(B(R) \setminus B((1-\sigma)R))$$

$$\le \text{mes } B(R) \cdot \left(\frac{8}{9} + \frac{64}{9} \frac{C_1}{\sigma^2} \lambda \right)$$

(23) $$+ \text{mes } B(R) \cdot (1 - (1-\sigma)^n) \quad \text{by (21)}$$

$$\le \text{mes}\,(B(R)) \left(\frac{8}{9} + \frac{64}{9} \frac{C_1}{\sigma^2} \lambda + n\sigma \right) \text{ by (22)}$$

$$= (1-b) \text{ mes } B(R) \quad \text{by (18)}$$

$$\beta = \left(1-\frac{2}{p}\right)\frac{n+2}{n} > \left(1-\frac{2}{n+2}\right)\frac{n+2}{n} = 1$$

& so Lemma 2 applies. We have

$$\phi(d) = 0$$

for

$$d \le C\,\phi(0)^{\frac{\beta-1}{\alpha}}$$

$$\le C\left(\sum_{i=1}^{n} \|f_i\|_{L^p(Q_T)}\right) \operatorname{mes}(Q_T)^{\frac{1}{n+2}-\frac{1}{p}} \qquad C = \frac{C(\alpha)}{\delta}$$

This provides the stated upper bound for u; the same method applied to $-u$ gives the lower bound. ||

C. Local L^∞ estimates

Definition v is a subsolution of $(*)$ if

$$v_t - (a_{ij} v_{x_i})_{x_j} \le 0 \quad \text{in } Q_T$$

Lemma 4 If u solves

$$(*)\begin{cases} u_t - (a_{ij} u_{x_i})_{x_j} = 0 \text{ in } Q_T \end{cases}$$

& ϕ is convex, then $v = \phi(u)$ is a subsolution.

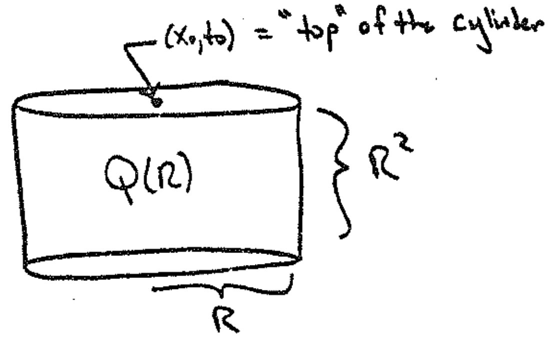
Proof

$$v_t - (a_{ij} v_{x_i})_{x_j} = \phi'(u)u_t - (a_{ij}\phi'(u)u_{x_i})_{x_j}$$

$$= \phi'(u)\underbrace{(u_t - (a_{ij}u_{x_i})_{x_j})}_{=0} - \underbrace{\phi''(u)\, a_{ij} u_{x_i} u_{x_j}}_{\ge 0} \qquad ||$$

Definition Let $(x_0,t_0) \in \mathbb{R}^{n+1}$ & $R>0$. Then $Q(R)$, the parabolic cylinder of radius R & top (x_0,t_0), is the set

$$\boxed{Q(R) = \{(x,t) \mid |x-x_0| \le R,\ t_0 - R^2 \le t \le t_0\}}$$



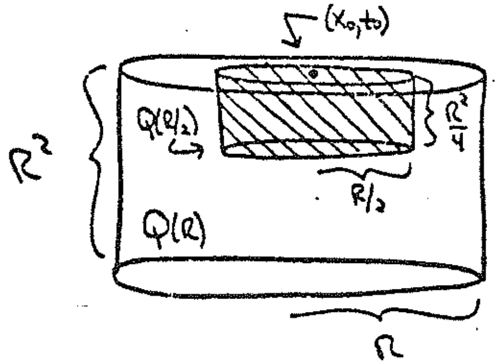
NB. The height is R^2, not R - Hence

$$\operatorname{mes}(Q(R)) = C R^{n+2}$$

of $(*)$. Choose any point $(x_0,t_0) \in Q_T$ & $R>0$ so small that $Q(R) \subset Q_T$.

Then $\exists C$, depending only on n, θ, Θ $\ni$

$$\boxed{\sup_{Q(R/2)} v \leq C\left[\frac{1}{R^{n+2}} \iint_{Q(R)} v^2\,dx\,dt\right]^{1/2}}$$



The L^∞ norm of v in the small cylinder is estimated by the L^2 norm of v in the large cylinder.

<u>Proof</u> (Iteration method) Let $p \geq 2$ & choose any cutoff function $\zeta(x,t) \ni \zeta \equiv 0$ near the parabolic boundary of $Q(R)$ (ie near the bottom & vertical sides)

WLOG set $(x_0,t_0) = (0,0)$, $Q(R) = B(R) \times (-R,0)$

Then we calculate

$$(7) \quad \int_{-R^2}^0 \frac{d}{dt}\left(\int_{B(R)} \zeta^2 v^p\,dx\right) dt =$$

$$= p \int_{-R^2} \int_{B(R)} v^{p-1} v_t \zeta^2\,dx\,dt + 2 \int_{-R^2} \int_{B(R)} \zeta \zeta_t v^p\,dx\,dt$$

$$\leq p \int_{-R^2}^0 \int_{B(R)} v^{p-1} (a_{ij} v_{x_i})_{x_j} \zeta^2\,dx\,dt + 2\int_{-R^2}^0 \int_{B(R)} \zeta\zeta_t v^p\,dx\,dt$$

$$= -p(p-1) \int_{-R^2}^0 \int_{B(R)} v^{p-2} a_{ij} v_{x_i} v_{x_j} \zeta^2\,dx\,dt$$

$$- 2p \int_{-R^2}^0 \int_{B(R)} v^{p-1} a_{ij} v_{x_i} \zeta \zeta_{x_j}\,dx\,dt$$

$$+ 2 \int_{-R^2}^0 \int_{B(R)} \zeta\zeta_t v^p\,dx\,dt.$$

The upper limit of integration 0 in (7) may be replaced by any $-R^2 \leq T^* \leq 0$ & so the calculation just done & standard tricks give

$$\sup_{-R^2 \leq t \leq 0} \int_{B(R)} \zeta^2 v^p\,dx + p(p-1) \iint_{Q(R)} v^{p-2} |Dv|^2 \zeta^2\,dx\,dt$$

$$(8) \qquad \leq C \iint_{Q(R)} v^p \left(\zeta|\zeta_t| + |D\zeta|^2\right) dx\,dt$$

Hence

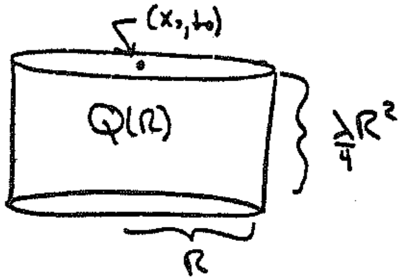
$$\text{mes}\{x \in B(R) \mid u(x,t) \leq 7/8\}$$
$$= \text{mes}\, B(R) - \text{mes}\, A^{(t)}_{7/8,R}$$
$$\geq b\, \text{mes}\, B(R) \quad \text{by (23).} \quad \|$$

In light of Lemma 8 we make the

<u>Redefinition</u> Let $(x_0,t_0) \in \mathbb{R}^{n+1}$ & $R>0$. Then

$$\boxed{Q(R) = \{(x,t) \mid |x-x_0| \leq R,\ t_0 - \tfrac{\lambda R^2}{4} \leq t \leq t_0\}},$$

where $\lambda > 0$ is the number from Lemma 8.



E. Local Hölder continuity

<u>Proposition 9</u> Assume that $0 \leq u \leq 1$ solves

$$(*) \left\{ u_t - (a_{ij} u_{x_i})_{x_j} = 0 \quad \text{in } Q_T, \right.$$

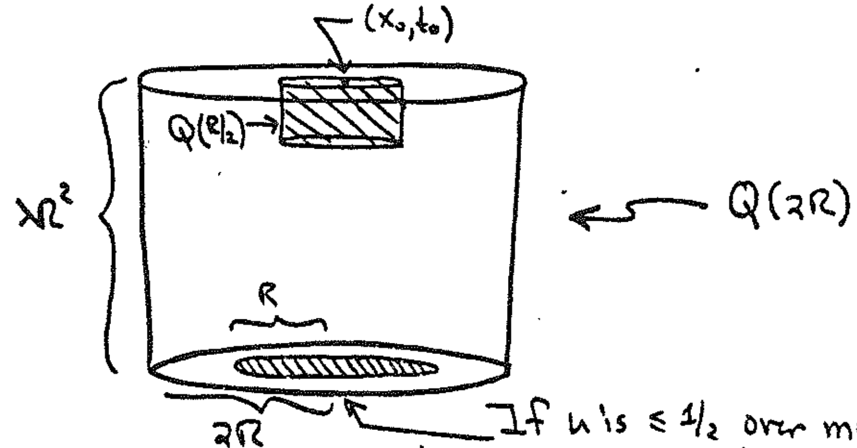
where the a_{ij} satisfy the ellipticity condition (E).

Pick any point $(x_0,t_0) \in Q_T$ & $R>0$ so small that $Q(2R) \subset Q_T$. Assume that

$$(24) \quad \boxed{\text{mes}\{x \in B(x_0,R) \mid u(x, t_0 - \lambda R^2) \leq 1/2\} \geq \tfrac{1}{2}\, \text{mes}\, B(R)}$$

Then $\exists\ 0 \leq C < 1$, depending only on n, θ & Θ, $\ni$

$$(25) \quad \boxed{\sup_{Q(R/2)} u(x,t) \leq C}$$



If u is $\leq 1/2$ over more than half of this set, then $u \leq C < 1$ on the small cylinder $Q(R/2)$ above.

<u>Proof</u> WLOG set

$$x_0 = 0$$
$$t_0 = \lambda R^2$$

Let $\varepsilon > 0$, and set $v = \phi(u) = \log^+\left(\frac{1}{8(1-u+\varepsilon)}\right)$

Note that $\phi'' = (\phi')^2$. Let $\eta(x,t)$ be a cutoff function vanishing near $\partial_p Q(2R)$; ~~and~~

multiply (*) by $\phi'(u)\eta^2$ and integrate over $Q(2R)$.

$$\iint_{Q(2R)} \phi'(u)\, u_t\, \eta^2\, dx\, dt$$

$$+ \iint a_{ij}\, u_{x_j} \left(\phi'(u)\eta^2\right)_{x_i} dx\, dt = 0$$

Thus

$$\iint v_t \eta^2 + \iint a_{ij}\, \eta^2 \phi''(u)\, u_{x_j} u_{x_i}$$

$$= - \iint a_{ij}\, u_{x_j}\, \phi'(u)\, 2\eta\, \eta_{x_i}$$

and so

$$\iint (v\eta^2)_t + \iint a_{ij}\, v_{x_j} v_{x_i}\, \eta^2$$

$$\leq \left| \iint a_{ij}\, v_{x_j}\, 2\eta\, \eta_{x_i} \right|$$

There follows

$$\int v\eta^2(x,t) + \theta \iint |\nabla v|^2 \eta^2$$

$$\leq \varepsilon \iint |\nabla v|^2 \eta^2 + C(\varepsilon) \iint |\nabla \eta|^2$$

$$+ \mu \iint v^2 \eta^2 + C(\mu) \iint (\eta_t)^2$$

and so

$$\iint_{Q(2R)} |\nabla v|^2 \eta^2\, dx\, dt \leq C \iint_{Q(2R)} |\nabla \eta|^2\, dx\, dt$$

$$+ \mu \iint_{Q(2R)} v^2 \eta^2\, dx\, dt$$

$$+ C(\mu) \iint_{Q(2R)} |\eta_t|^2\, dx\, dt \qquad (29)$$

Set (25)

$$\eta(x,t) = \zeta(x)\chi(t),$$

where

$$\begin{cases} \zeta(x) = \zeta(|x|) \text{ is a smooth, radial function of } |x|, \\ 0 \le \zeta \le 1, \ \zeta \equiv 1 \text{ on } B(R), \ \zeta \equiv 0 \text{ near } \partial B(2R) \\ |\nabla \zeta| \le 2/R \end{cases}$$

and

$$\begin{cases} \chi(t) \text{ is a smooth function of } t, \ 0 \le \chi \le 1, \\ \chi(t) \equiv 1 \text{ for } t \ge \frac{3}{4}\lambda R^2, \ \chi \equiv 0 \text{ for } t \text{ near } 0, \\ |\chi'| \le C/R^2 \end{cases}$$

We plug this special choice of η in (29) to obtain

$$\int_0^{t_0}\int_{B(2R)} |\nabla v|^2 \zeta^2 \chi^2 \, dx\,dt \le C \int_0^{t_0}\int_{B(2R)} \chi^2 |\nabla\zeta|^2 \, dx\,dt$$

$$(30) \qquad + \mu \int_0^{t_0}\int_{B(2R)} v^2 \zeta^2 \chi^2 \, dx\,dt$$

$$+ \frac{C}{\mu} \int_0^{t_0}\int_{B(2R)} \zeta^2 (\chi')^2 \, dx\,dt$$

Now let

$$\mu = \varepsilon'/R^2$$

; recall that $\operatorname{mes}(Q(2R)) = CR^{n+2}$, $t_0 = \lambda R^2$; then (30) becomes

(26)

$$\int_0^{t_0} \chi^2 \int_{B(2R)} |\nabla v|^2 \zeta^2 \, dx\,dt \le C(\varepsilon) R^n$$

$$(31) \qquad + \frac{\varepsilon'}{R^2} \int_0^{t} \chi^2 \int_{B(2R)} v^2 \zeta^2 \, dx\,dt$$

Now by hypothesis (24) & Lemma 8 we have

$$\operatorname{mes}\{x \in B(R) \mid u(x,t) \le 7/8\} \ge b \operatorname{mes}(B(R)) \quad (b>0)$$

$\forall\ 0 \le t \le \lambda R^2 = t_0$. Hence, by the definition of ϕ, we have that

$$(32) \quad v(x,t) = 0 \text{ for } x \in N_t \equiv \{x \in B(R) \mid u(x,t) \le 7/8\}$$

Therefore <u>for each</u> <u>$0 \le t \le t_0$</u> we may apply Lemma 7 to find

$$(33) \int_{B(2R)} v^2(x,t)\zeta^2(x)\, dx \le CR^2 \int_{B(2R)} |\nabla v(x,t)|^2 \zeta^2(x)\, dx$$

We plug (33) into (31) & then choose ε' small to find

$$\int_0^{t_0}\int_{B(2R)} |\nabla v|^2 \chi^2 \zeta^2 \, dx\,dt \le CR^n.$$

Since $\chi \equiv \zeta \equiv 1$ on $Q(R)$ we thus have proved

(33) $\iint_{Q(R)} |\nabla v|^2 \, dx\, dt \le C R^n$.

Now we again may use (32), this time on $B(R) \times \{t\}$ for each $0 \le t \le t_0$ & with $\delta \equiv 1$, to obtain the estimate

(34) $\iint_{Q(R)} v^2 \, dx\, dt \le C R^2 \iint_{Q(R)} |\nabla v|^2 \, dx\, dt$

Finally by Theorem 5,

(35) $\sup_{Q(R/2)} v^2 \le \frac{C}{R^{n+2}} \iint_{Q(R)} v^2 \, dx\, dt$.

We combine (33)-(35) to find

$$\sup_{Q(R/2)} v^2 \le C, \text{ where } C \text{ does not depend on } R.$$

By the definition of $v = \phi(u)$ we thus have

$$-\log(8(1-u(x,t)+\epsilon)) \le C \quad \forall (x,t) \in Q(R/2)$$

& so

$$8(1-u+\epsilon) \ge e^{-C} \quad \forall (x,t) \in Q(R/2);$$

ie

$$u(x,t) \le 1 - \frac{e^{-C}}{8} < 1 \quad \forall (x,t) \in Q(R/2) \quad //$$

Theorem 10 (Nash-Kružkov) Let u be a smooth solution of

$$(*) \left\{ u_t - (a_{ij} u_{x_i})_{x_j} = 0 \quad \text{in } Q_T, \right.$$

where the a_{ij} satisfy (E). Let $Q' \subset Q_T$ be a subdomain with $\operatorname{dist}(Q', \Gamma_T) > 0$.

Then $\exists$ constants $C \ge 0$ & $0 < \gamma < 1$, depending only on n, θ, Θ, $\operatorname{dist}(Q', \Gamma_T)$, and $\|u\|_{L^2(Q_T)}$ $\ni$

$$\boxed{|u(x,t) - u(x',t')| \le C\left(|x-x'|^{\gamma} + |t-t'|^{\gamma/2}\right)}$$

$\forall (x,t), (x',t') \in Q'$

Remark So u is "twice as Hölder continuous" in x as in t

Proof Pick any point $(x_0,t_0) \in Q_T'$ & set $R_0 = \frac{1}{2}\operatorname{dist}(Q', \Gamma_T)$.
We'll consider only $R \le R_0$.

Let $\omega(R)$ be the (parabolic) oscillation of u with respect to the point (x_0,t_0) (see p. 16)

For a fixed $R \le R_0$ we may assume WLOG that

(36) $\max_{Q(R)} u(x,t) = 1, \quad \min_{Q(R)} u(x,t) = 0.$

(If not we consider $\tilde{u} = a(u+b)$, which also solves (*), & adjust a & b to achieve (36).

Multiplication by a does change the oscillation, but in the main estimate (37) below the effects of this multiplication cancel out)

Now either u or 1-u satisfies hypothesis (24) of Proposition 9 (with the ball $B(R/2)$ replacing $B(R)$):

Case 1 $\text{mes}\{x \in B(x_0, R/2) \mid u(x, t_0 - \lambda \frac{R^2}{4}) \le 1/2\}$
$$\ge \frac{1}{2} \text{mes}(B(R/2))$$

Then, by Proposition 9, $\exists\, C < 1$ (independent of R) $\ni$
$$\max_{Q(R/4)} u \le C < 1 .$$

Hence
$$\omega(R/4) = \max_{Q(R/4)} u - \min_{Q(R/4)} u$$
$$(37) \qquad \le C = C\,\omega(R) \quad \text{by } (36)$$
$$= \eta\, \omega(R) \quad \text{for } \eta = C < 1 .$$

Case 2 $\text{mes}\{x \in B(x_0, R/2) \mid 1 - u(x, t_0 - \lambda \frac{R^2}{4}) \le 1/2\}$
$$\ge \frac{1}{2} \text{mes}(B(R/2))$$

This case is similar & also leads to (37)

Thus $\exists\, \eta < 1 \ni$
$$\omega(R/4) \le \eta\, \omega(R) \qquad \forall\, 0 < R < R_0 .$$

By Lemma 6, therefore, $\exists\, C$ & $0 < \gamma < 1 \ni$
$$(38) \qquad \omega(R) \le C\left(\frac{R}{R_0}\right)^{\gamma} \qquad \forall\, 0 < R \le R_0$$

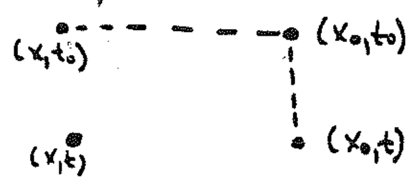
Now fix any $(x_0, t_0) \in Q'$ & chose any point $(x, t_0) \in Q' \ni R = |x - x_0| \le R_0$. Then

$$|u(x_0, t_0) - u(x, t_0)| \le \omega(R) \longleftarrow \text{oscillation calculated wrt the point } (x_0, t_0)$$
$$\le \frac{C R^{\gamma}}{R_0^{\gamma}} \quad \text{by } (38)$$
$$(39) \qquad = C|x - x_0|^{\gamma}$$

Next choose any point $(x_0, t) \in Q'$, with $t \le t_0$ & d

R by
$$(40) \qquad |t - t_0| = \frac{\lambda}{4} R^2$$

Then
$$|u(x_0, t_0) - u(x_0, t)| \le \omega(R) \le \frac{C R^{\gamma}}{R_0^{\gamma}}$$
$$(41) \qquad \le C|t - t_0|^{\gamma/2} \quad \text{by } (40)$$



Finally if (x,t) is any point in $Q' \cap Q(R_0)$ we have

$$|u(x_0,t_0)-u(x,t)| \leq |u(x_0,t_0)-u(x_0,t)| + |u(x_0,t)-u(x,t)|$$

(42) $$\leq C\,|t_0-t|^{\gamma/2} + |u(x_0,t)-u(x,t)| \quad \text{by (41)}$$

$$\leq C\left(|t_0-t|^{\gamma/2} + |x_0-x|^{\gamma}\right) \quad \text{by (39)}$$

(with the roles of (x_0,t_0) & (x,t) interchanged)

This proves the theorem if $(x,t) \in Q' \wedge Q(R_0)$. If $(x,t) \in Q' \setminus Q(R_0)$, then <u>either</u> $|x-x_0| \geq R_0$ <u>or</u> $|t-t_0| \geq \frac{\Delta}{4} R_0^2$. In this case

$$|u(x_0,t_0)-u(x,t)| \leq 2 \max_{Q'} |u|$$

$$\leq \frac{C}{R_0^{\gamma}}\left(|x-x_0|^{\gamma} + |t-t_0|^{\gamma/2}\right). \qquad //$$

Finally we prove local Hölder continuity for the case that $(*)$ has a non-zero right hand side.

<u>Theorem 11</u> Suppose u solves

$$(*)\left\{ \; u_t - (a_{ij} u_{x_i})_{x_j} = (f_i)_{x_i} \quad \text{in } Q_T, \right.$$

where the a_{ij} satisfy (E) &

$$f_i \in L^p(Q_T) \quad \text{for some } \boxed{p > n+2}$$

Let $Q' \subset Q_T$ be a subdomain with $\operatorname{dist}(Q', \Gamma_T) > 0$. Then $\exists$ constants C & $0<\gamma<1$, depending only on $n, p, \theta, \Theta, \operatorname{dist}(Q',\Gamma_T), \|u\|_{L^2(Q_T)}$ & $\|f_i\|_{L^p(Q_T)}$ $\ni$

$$\boxed{|u(x,t) - u(x',t')| \leq C\left(|x-x'|^{\gamma} + |t-t'|^{\gamma/2}\right)}$$

$\forall\, (x,t), (x',t') \in Q'$.

<u>Proof</u> Fix $(x_0,t_0) \in Q_T$ & R_0 as in the last proof. Let $R \leq R_0$. We now write

$$u = v + w,$$

where

$$\begin{cases} v_t - (a_{ij} v_{x_i})_{x_j} = (f_i)_{x_i} & \text{in } Q(2R) \\ v = 0 & \text{on parabolic boundary of } Q(2R) \end{cases}$$

and $\{ v_t - (a_{ij} w_{x_i})_{x_j} = 0$ in $Q(2R)$.

By Theorem 3, we have

$$\|v\|_{L^\infty(Q(2R))} \le C \sum_{i=1}^n \|f_i\|_{L^p} \cdot \text{mes}(Q(2R))^{\frac{1}{n+2} - \frac{1}{p}}$$

(43)
$$\le C R^{1 - \frac{n+2}{p}}$$

Furthermore, by the proof of Theorem 10,

(44) $\omega_w(R/4) \le \eta\, \omega_w(R), \qquad \eta < 1$

where ω_w = oscillation of w. Hence

$$\omega(R/4) \le \omega_w(R/4) + \omega_v(R/4)$$
$$\le \eta\, \omega_w(R) + C R^{1 - \frac{n+2}{p}} \quad \text{by (43) \& (44)}$$
$$\le \eta\, \omega(R) + C R^\alpha \quad \text{for } \alpha = 1 - \frac{n+2}{p} > 0$$

Lemma 6 therefore implies

$$\omega(R) \le C\left(\frac{R}{R_0}\right)^\gamma \quad \forall\ 0 < R \le R_0$$

& the rest is like the proof of Theorem 10. //