Εξισώσεις Εξέλιξης

χώρος x
χρόνος t

Εξίσωση Διάχυσης

Α.Σ. $\boxed{t=0}$

ανοικτό

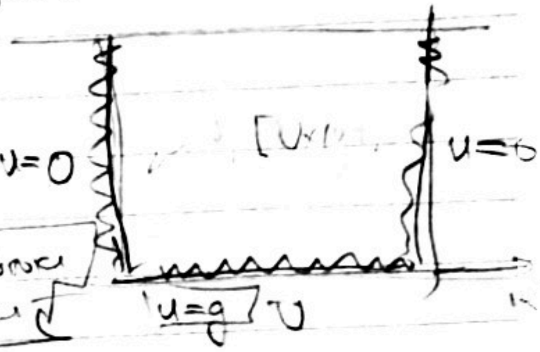
κλειστό

1. Dirichlet (D)

$$\begin{cases} u_t = \Delta u, & (x,t) \in U \times (0,T] \\ u(x,0) = g(x), & x \in U \\ u = 0, & \partial U \times (0,T] \end{cases}$$

Σ.Σ.

Α.Σ. δεν ικανοποιεί
Σ.Σ. εν γένει



Neumann

$$\begin{cases} u_t = \Delta u, & (x,t) \in U \times (0,T] \\ u(x,0) = g(x), & x \in U \\ \dfrac{\partial u}{\partial \nu} = 0, & \partial U \times (0,T] \end{cases}$$

$= \partial_p$

Cauchy

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u, & (x,t) \in \mathbb{R}^n \times (0,T] \\ u(x,0) = g(x), & x \in \mathbb{R}^n, \quad g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \end{cases}$$

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2. Κλίμακες – [illegible] παράμετρος $\boxed{\lambda > 0}$

$(x,t) \to (\lambda x, \lambda^2 t)$ u,

$u(x,t)$ λύση

$u^\lambda(x,t) = u(\lambda x, \lambda^2 t)$

$\Delta u = 0$
αναλλοίωτη
$y = \lambda x$

$\dfrac{1}{|x|^n}$

Ομοιότητα

$|x|$

αναλλ.

$\dfrac{|x|^2}{t}$ $\boxed{\text{αναλλοίωτη}}$

Αναζητούμε λύσεις $\boxed{u(x,t) = v\left(\dfrac{|x|^2}{t}\right)}$

υπογραφτικα στο αρχ.

Μη Αυτοομοιότ.

$u(x,t)$

Αρχικά $u^{\lambda}(x,t) = \lambda^{\alpha} u(\lambda^{\beta}x, \lambda t)$ λύση

$$\frac{\partial}{\partial t} u^{\lambda} = \Delta u^{\lambda} \iff$$

$$\lambda^{\alpha+1} u_t \cancel{(\lambda^{\beta}x, \lambda t)} = \lambda^{\alpha} \left(u(\lambda^{\beta}x, \lambda t) \right)_{,ii}$$

$$= \lambda^{\alpha+2\beta} u_{,ii} \cancel{(\lambda^{\beta}x, \lambda t)}$$

$$\alpha+1 = \alpha+2\beta \iff \boxed{\beta = \frac{1}{2}}$$

$$u(x,t) \text{ λύση} \Rightarrow \lambda^{\alpha} u(\lambda^{1/2}x, \lambda t) \text{ λύση.}$$

$$\boxed{\lambda t = 1} \Rightarrow \lambda = \frac{1}{t}$$

v_{y_i}

Απόσταση (Δεν ...)

$$\left(\frac{1}{t}\right)^{\alpha} u\left(\frac{x}{\sqrt{t}}, 1\right) \quad \boxed{\text{λύση}}$$

$$\frac{x_i}{t^{1/2}} = y_i$$

$$\tilde{u}(x,t) = \left(\frac{1}{t}\right)^{\alpha} v\left(\frac{x}{\sqrt{t}}\right) \text{ λύση}, \quad v = v(y)$$

$$y_i = \frac{x_i}{\sqrt{t}}$$

$$\tilde{u}_t = (-\alpha) t^{-\alpha-1} v\left(\frac{x}{\sqrt{t}}\right) + t^{-\alpha} v_{y_i} \cancel{x_i} \left(-\frac{1}{2}\right) t^{-3/2}$$

$$= (-\alpha) t^{-\alpha-1} v\left(\frac{x}{\sqrt{t}}\right) - t^{-\alpha-1} v_{y_i} \cdot y_i$$

$$\tilde{u}_{x_i} = t^{-\alpha} v_{y_i} \frac{1}{\sqrt{t}} = t^{-\alpha-\frac{1}{2}} v_{y_i}$$

$$\tilde{u}_{x_i x_i} = \boxed{t^{-\alpha-1} v_{y_i y_i}}$$

$$\tilde{u}_t = \tilde{u}_{x_i x_i} \iff$$

$$\Delta_y v + \frac{1}{2} y \cdot \nabla v = -\alpha v$$

$$v(y) = w(|y|)$$

$$\Delta v = \frac{\partial^2 w}{\partial r^2} + \frac{n-1}{r}\frac{\partial w}{\partial r}$$

$$y \cdot \nabla w(|y|) = y_i \frac{\partial}{\partial y_i} w(r) = y_i\, w'(r) \frac{y_i}{r}$$

$$= \underline{\underline{w'(r) r}}$$

$$\frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1} w'\right) + \frac{1}{2} r w' + \alpha w = 0$$

$$\frac{\partial}{\partial r}\left(r^{n-1} w'\right) + \underbrace{\frac{1}{2} r^n w' + \alpha r^{n-1} w}_{\text{ολικό διαφορικό ώστε να ολοκληρωθεί}} = 0$$

$$\frac{1}{2}\frac{\partial}{\partial r}\left(r^n w\right) = \frac{1}{2}\left[\underline{n r^{n-1}} w + r^n w'\right]$$

$$\boxed{\frac{n}{2} = \alpha}$$

$$\frac{\partial}{\partial r}\left[\left(r^{n-1} w'\right) + \frac{1}{2}\left(r^n w\right)\right] = 0$$

$$r^{n-1} w' + \frac{1}{2} r^n w = C$$

Υποθέτουμε $w' r^{n-1} \to 0$, $w r^n \to 0$, $r \to +\infty$

$\boxed{C = 0}$

$$\boxed{w' + \frac{r}{2} w = 0}$$

$$W = b e^{-r/4}$$

$$\tilde{u}(x,t) = \frac{1}{t^{m/2}} e^{-\frac{|x|^2}{4t}}$$

Ορ: Θεμελιώδης Λύση

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, & t>0 \\ 0, & t<0 \end{cases}$$

$$\boxed{\begin{cases} \frac{\partial \Phi}{\partial t} = \Delta \Phi \\ \Phi(x,0) = \delta_0(x) \end{cases}}$$

$\boxed{\text{Άσκηση 1}}$ (Μη γραμμική)

$u_t = (u^m)_{xx}$, $m>1$, $u(x,t)\geq 0$

1. $u^\lambda(x,t) = u(\lambda x, \lambda^\sigma t)$ λύση

2. $u^\lambda(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$ λύση

3. $u^\lambda(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$ λύση για $\lambda t = 1$

Συμπέρασμα (Θεμελιώδης Λύση)

$$\tilde{u}(x,t) = \frac{1}{t^{\alpha m+1}} \left(c - \frac{(m-1)\alpha}{2m} \frac{x^2}{t^{2\alpha}} \right)_+^{\frac{1}{m-1}}, \quad \text{λύση}$$

Επιλογή c τ.ω.

$$\int_{\mathbb{R}} \tilde{u}(x,t)\,dx = 1 \Leftrightarrow \frac{1}{\sqrt{a\gamma}} \int_{-\sqrt{c^*}}^{\sqrt{c^*}} (c^* - \xi^2)^{\frac{1}{m-1}} d\xi = 1, \quad \gamma = \frac{m-1}{2m}$$

Παρατηρήσεις

$x=(x_1,\dots,x_n)$

1) $$\int_{\mathbb{R}^n} \frac{1}{(4\pi)^{n/2}} e^{-\frac{|x|^2}{4}} dx = 1$$

$$\left(\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4}} dx = \prod_{i=1}^{n} \int_{\mathbb{R}} e^{-\frac{x_i^2}{4}} dx_i = \left(\sqrt{4\pi}\right)^n \right)$$

2) $$\Phi(x,t) = \frac{1}{(\sqrt{t})^n} \Phi\left(\frac{x}{(\sqrt{t})^n}, 1\right) = \frac{1}{\varepsilon^n} \Phi\left(\frac{x}{\varepsilon}, 1\right)$$

$$\therefore \quad \int_{\mathbb{R}^n} \Phi(x,t)\,dx = 1 \quad \forall t>0.$$

3) Εναλλακτική (γραφτική) εξαγωγή της Θεμελιώδους μέσω Μετασχηματισμών Fourier

$$(Fu)(y) = \hat{u}(y) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i x\cdot y} u(x)\,dx \quad , \quad x=(x_1,\dots,x_n)$$
$$y=(y_1,\dots,y_n)$$

Υπενθυμίζουμε

$$F\left(e^{-\frac{|x|^2}{4}}\right) = e^{-|y|^2} \qquad (\text{"σταθερό" σημείο})$$

$$F(\delta) = 1 \qquad (\text{"μονάδα" ως προς συνέλιξη})$$

$$F(\partial^\alpha u) = (iy)^\alpha F(u) \, ,$$

$$\begin{cases} \Phi_t = \Delta \Phi & , \ t>0, \ x \in \mathbb{R}^n \\ \Phi(x,0) = \delta_0(x) \end{cases}$$

$$\left(\frac{\partial}{\partial t}\Phi(x,t)\right)^{\wedge} = (\Delta \Phi)^{\wedge} = -|y|^2 \hat{\Phi}(y,t)$$

$$\| $$

$$\boxed{\frac{\partial \hat{\Phi}}{\partial t} + |y|^2 \hat{\Phi} = 0}$$

$$\boxed{\hat{\Phi}(\cdot,0) = 1}$$

$$\Rightarrow \quad \hat{\Phi}(y,t) = e^{-|y|^2 t}$$

$$\Rightarrow \quad \Phi(x,t) = \frac{1}{(\sqrt{4\pi t})^n} \, e^{-\frac{|x|^2}{4t}} .$$

<u>Άσκηση 2</u>

Να βρεθεί η "Θεμελιώδης Λύση" για την

$$\begin{cases} u_t = \Delta u^m & , \ m>1, \ u(x,t) \geq 0 \\ u(x,0) = \delta(x) \end{cases}$$

ακολουθώντας τα βήματα της Άσκησης 1.

$\square$