Liouville II

1) Έστω ότι $\Delta u = 0$, $u \geq 0$, $u : \mathbb{R}^n \to \mathbb{R}$

$\Rightarrow$

$u \equiv$ σταθερά

Liouville:

$\Delta u = 0$, $|u(x)| \le K$, $u : \mathbb{R}^n \to$

$\Rightarrow u \equiv$ σταθ.

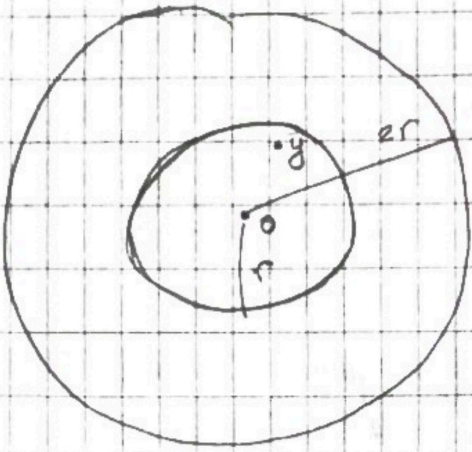
Απ I

$\Delta u_{x_i} = 0$

$$|u_{x_i}(0)| = \left|\frac{1}{cr^n}\int_{B_r} u_{x_i}\, dx\right| = \frac{1}{cr^n}\int_{\partial B_r} u \nu_i\, dS \le \frac{1}{cr^n}\int_{\partial B_r} |u|\, dS$$

$r \to \infty$ $\to 0$

Απ (Άγγελος)



Έστω y αυθαίρετο, $|y| \le r$

$$u(0) = ⨍_{B(x,2r)} u\,dz \geq \frac{1}{|B(x,2r)|}\int_{B(y,r)} u\,dz = \frac{1}{2^n} ⨍_{B(y,r)} u\,dz = \frac{1}{2^n} u(y)$$

$\Rightarrow \quad 2^n u(0) \geq u(y)$

$\Rightarrow \quad |u(y)| = u(y) \le 2^n u(0)$

Άρα $|u(y)|$ σταθερά μέσω Liouville I.

□

Πόρισμα

Έστω $\Delta u = 0$, $u \le K$ $(u \geq K)$, $u : \mathbb{R}^n \to$

Απ

$v = K - u \geq 0$, $\Delta v = 0$

$(u - K \geq 0 \Rightarrow v = u - K)$

□

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2) Υποθέσεις

$$\Delta u \geq 0 \quad , u = \mathbb{R}^n \to \mathbb{R} \quad , u \in C^2$$

$$u \in K$$

$$\not\Rightarrow u = \text{σταθερά}, \quad \boxed{n=3} \quad — \ \underline{\text{Αντιπαράδειγμα}}$$

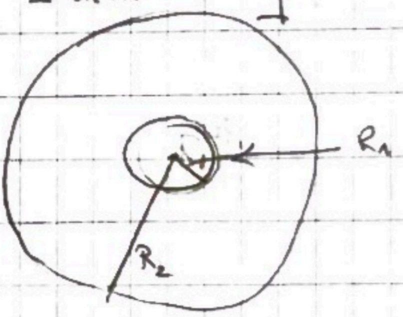
$$u(r) = \begin{cases} -\frac{1}{8}(15-10r^2+3r^4) \\ -\frac{1}{r}, & r \geq 1 \end{cases}$$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \Delta_{S_r^{n-1}}$$

$$\Rightarrow u \equiv \text{σταθερά} \quad \boxed{n=2}$$

Απ $(n=2)$ $\left[\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \ - \text{αρχικά}\right]$

$$M(r) = \max_{x^2+y^2=r^2} u$$



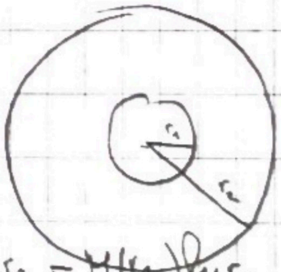
$$\phi(r) = a + b\ln r \quad - \boxed{\text{Αρμονική}}$$

$$R_1 < r_1 \leq r_2 < R_2$$

$$\phi(r_1) = M(r_1) \Leftrightarrow a + b\ln r_1 = M(r_1) \quad (i)$$
$$\phi(r_2) = M(r_2) \Leftrightarrow a + b\ln r_2 = M(r_2) \quad (ii)$$

Επιλύω a, b

$$b(\ln r_1 - \ln r_2) = M(r_1) - M(r_2)$$
$$b\ln\left(\frac{r_1}{r_2}\right) = M(r_1) - M(r_2)$$



$$\boxed{b = \frac{M(r_1) - M(r_2)}{\ln\left(\frac{r_1}{r_2}\right)}}$$

$$a\ln r_2 + b\ln r_1 \ln r_2 = \cancel{M(r_1)}\ln r_2$$
$$a\ln r_1 + b\ln r_2 \ln r_1 = M(r_2)\ln r_1$$

$$\Rightarrow \boxed{a \cancel{\ln\frac{r_2}{r_1}} = \frac{M(r_1)\ln r_2 - M(r_2)\ln r_1}{\ln\frac{r_2}{r_1}}}$$

$$\phi(r) = \frac{M(r_1)\ln r_2 - M(r_2)\ln r_1}{\ln\frac{r_2}{r_1}} + \frac{M(r_1) - M(r_2)}{\ln\left(\frac{r_2}{r_1}\right)}\ln r$$

$$= \frac{1}{\ln\frac{r_2}{r_1}}\left[M(r_1)\cancel{\ln r_2} - M(r_2)\ln r_1 - \left(M(\cancel{r_1}) - M(r_2)\right)\ln r\right]$$

$$= \frac{1}{\ln \frac{r_2}{r_1}} \left[M(r_1) \ln\left(\frac{r_2}{r}\right) + M(r_2) \ln\left(\frac{r}{r_1}\right) \right]$$

$$\max_{|x|=r_2} u = \phi(r)$$

$$\| \qquad v := u - \phi(r)$$

$$\begin{cases} \Delta v \geq 0 & r_1 \leq r \leq r_2 \\ v \leq 0 & |x| = r_1, r_2 \end{cases}$$

$$u - \phi(r) \leq \max u - \phi(r) = 0$$

$$\Rightarrow \quad v \leq 0 \quad r_1 \leq |x| \leq r_2$$

$$u(x) \leq \phi(|x|)$$

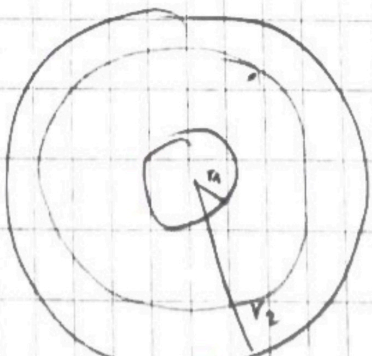
$$\Rightarrow \quad \boxed{u \leq \phi(r),} \quad r_1 \leq r \leq r_2$$

$$\Rightarrow \quad \max_{|x|=r} u \leq \phi(r)$$

$$\Rightarrow \quad (*) \quad \boxed{M(r) \leq \frac{M(r_1)\ln\left(\frac{r_2}{r}\right) + M(r_2)\ln\left(\frac{r}{r_1}\right)}{\ln\left(\frac{r_2}{r_1}\right)}}$$

(Hadamard θεώρημα 3 κύκλων)

$r_1 < r < r_2$



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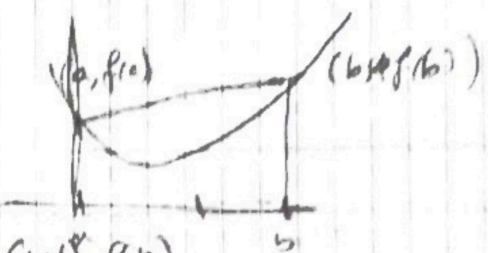

Κυρτότητα

$$f(\underbrace{tb+(1-t)a}_{\xi}) \le t f(b) + (1-t) f(a)$$

$$tb+(1-t)a = \xi \qquad \boxed{f(\xi) \le \frac{\xi-a}{b-a} f(b) + \frac{b-\xi}{b-a} f(a)}$$

$$a+t(b-a) = \xi$$

$$t = \frac{\xi-a}{b-a}, \quad 1-t = 1-\frac{\xi-a}{b-a} = \frac{b-\cancel{a}-\xi+\cancel{a}}{b-a}$$

$$= \frac{b-\xi}{b-a}$$

$$f(\xi)$$

$$\cancel{\ln \xi - a}$$

$$\cancel{\frac{\xi-a}{b-a}}$$

$$f(\xi) \le \frac{\ln\xi - \ln a}{\ln b - \ln a} f(b) + \frac{\ln b - \ln \xi}{\ln b - \ln a} f(a)$$

$$\Leftrightarrow \boxed{f(\xi) \le \frac{\ln\left(\frac{\xi}{a}\right) f(b) + \ln\left(\frac{b}{\xi}\right) f(a)}{\ln\left(\frac{b}{a}\right)}}$$

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Πόρισμα $(n=2)$

[$\Delta u \ge 0$ εκτός ενδεχομένως του $x=0$ $\left(u \in C^2(|x|>0)\right)$

[Έστω $u \in K$.

$\Longrightarrow$ $u =$ σταθερά.

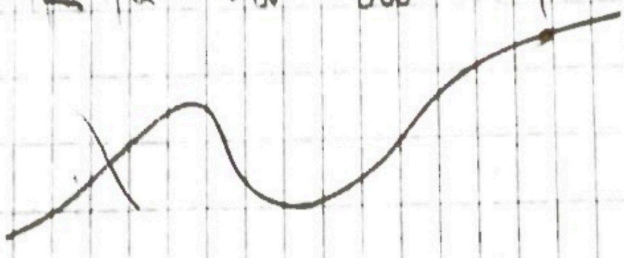
Απ

r σφαιρικές

$(*) \Rightarrow |M(r_2)| \le K, \quad r_2 \to \infty \Rightarrow \boxed{u(r) \le M(r_1)}$

$M(r_1) \le K, \quad r_1 \to 0 \Rightarrow u(r) \le M(r_2)$

$\Rightarrow$ u $\leq$ μια στην δυο ανεξάρτητες τιμές $M(r_1)$, $M(r_2)$



$$M(r) \leq M(r_1) \quad , \quad r \geq r_1$$

$$M(r) \leq M(r_2) \quad , \quad r \leq r_2$$

$$\Rightarrow \boxed{M(r) \equiv \text{σταθερά}}$$

Ισχυρή Αρχ Μεγίστου $\Rightarrow$

$$u \equiv \text{σταθερά} \qquad \square$$