$$\operatorname{supp} f(x-y) \cap \{|x-y|<R\} \subset \{|y|<|x|+R\}$$

Lecture 3

Clarification on p 14 :

$$M\int_{B(x;R)} |\Phi(y)|\,dy = M\int_{|y-x|<R} |\Phi(y)|\,dy$$

$$\leq M\int_{|y|<R+|x|} |\Phi(y)|\,dy \qquad (|y-x|>|y|-|x|)$$

$$= M\int_{S^{n-1}}\left(\int_0^{R+|x|}\frac{C_n}{r^{n-2}}\,r^{n-1}\,dr\right)d\sigma$$

$$= \frac{MC_n}{2}\left(R+|x|\right)^2|S^{n-1}|$$

2. • $u_{x_i}(x) = \int_{\mathbb{R}^n} \Phi(y)\, f_{x_i}(x-y)\,dy \qquad (1)$

• $u_{x_i x_j}(x) = \int_{\mathbb{R}^n} \Phi(y)\, f_{x_i x_j}(x-y)\,dy \qquad (2)$

(Method of difference quotients :

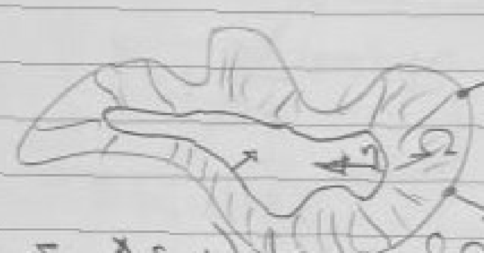
$$\frac{u(x+he_i)-u(x)}{h} = \int_{\mathbb{R}^n}\Phi(y)\left[\frac{f(x+he_i-y)-f(x-y)}{h}\right]dy$$

Ασκηση (Ταυτοτητα Green)

Κανοντας χρηση του θεωρηματος της αποκλισης δειξτε την ταυτοτητα για $u, v \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^n$, φραγμενο με συνορο $\partial\Omega$:

α) $$\int_\Omega \Delta u \, v \, dx = -\int_\Omega \nabla u \, \nabla v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, dS$$

$|\vec{n}| = 1$

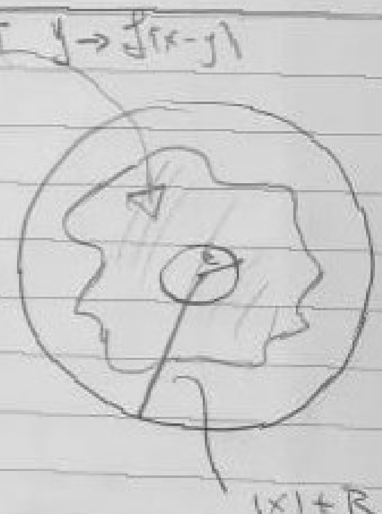


$\vec{n}$ = προς τα εξω μοναδιαιο καθετος

β) $$\int_\Omega (\Delta u \, v - \Delta v \, u) \, dx = \int_{\partial\Omega} \left(\frac{\partial u}{\partial n} v - \frac{\partial v}{\partial n} u \right) dS$$

---

(5) $$J_\varepsilon = \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y) \Delta_x f(x-y) \, dy$$



$$= \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \Phi(y) \Delta_y f(x-y) \, dy$$

support $y \longrightarrow f(x-y)$ εντος

$$\{|x-y| < R\} \subset \{|y| < |x| + R\}.$$

$$= \int_{B(0,|x|+R) \setminus B(0,\varepsilon)} \Phi(y) \Delta_y f(x-y) \, dy$$

(2) is established similarly.

Also from (2) the continuity of $u_{x_i x_j}(x)$ follows via the continuity of $f_{x_i x_j}$ (Check!)

$\therefore \quad u \in C^2(\mathbb{R}^n)$.

Άσκηση 1 (Σφαιρικές Συντεταγμένες)

(i) Υπολογίστε

$$\int_{a<|x|<b} |x|^\lambda dx \quad , \quad x \in \mathbb{R}^n$$

(ii) Δείξτε ότι $\displaystyle\int_{\mathbb{R}^n} e^{-\pi|x|^2} dx = 1$.

---

3. $$\Delta u(x) \overset{(2)}{=} \int_{\mathbb{R}^n} \Phi(y)\, \Delta_x f(x-y)\, dy \qquad (3)$$

$$= \int_{B(0,\varepsilon)} \text{———} \, " \, \text{——} \, dy + \int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \text{———} \, " \, \text{——} \, dy$$

$$=: I_\varepsilon + J_\varepsilon$$

(4) $$|I_\varepsilon| \leq \max_{z \in \mathbb{R}^n} |\Delta f(z)| \int_{B(0,\varepsilon)} |\Phi(y)|\, dy$$

$$\leq C \int_{S^{n-1}} \int_0^\varepsilon \frac{1}{r^{n-2}} r^{n-1}\, dr\, d\sigma = \hat{C}\varepsilon^2 \qquad (n \geq 3)$$

(Show that for $n=2$ $\quad |I_\varepsilon| \leq \hat{C}\varepsilon^2 |\ln\varepsilon|$)

$$= \int_{|x-y|<R+1} \Phi(y) \left[\frac{f(x+he_i-y)-f(x-y)}{h} \right] dy$$

$$\left(\text{supp } y \to f(x+he_i-y) \subset |x-y|<R+1 \right.$$

$$\left. \text{supp } y \to f(x-y) \subset |x-y|<R+1 \right) \Big)$$

We will utilise Lebesgue's dominated convergence theorem:

$$\lim_{h\to 0} \frac{f(x+he_i-y)-f(x-y)}{h} = \frac{\partial f}{\partial x_i}(x-y)$$

$$\left| \frac{f(x+he_i-y)-f(x-y)}{h} \right| = \left| \frac{\partial f}{\partial x_i}(x-\hat{y}) \right| < M$$

$$|\Phi(y)| \left| \frac{f(x+he_i-y)-f(x-y)}{h} \right| < M |\Phi(y)|$$

$$\int_{|y|<|x|+R+1} M |\Phi(y)| dy \le M \int_{S^{n-1}} \int_0^{|x|+R+1} \frac{1}{r^{n-2}} r^{n-1} \, dr \, d\sigma$$

$$< \infty.$$

<u>Conclusion:</u>

$$\lim_{h\to 0} \frac{u(x+he_j)-u(x)}{h} = \int_{\mathbb{R}^n} \lim_{h\to 0} \left[\Phi(y) \left(\frac{f(x+he_i-y)-f(x-y)}{h} \right) \right]$$

and (1) is established.

In the annulus $B(0,|x|+R)\setminus B(0,\varepsilon)$

$y \to \Phi(y)$, $y \to f(x-y)$ are C^2, so we can apply Annexe 2 a)

$$(6) \int_{B(0,|x|+R)\setminus B(0,\varepsilon)} \Phi(y)\,\Delta_y f(x-y)\,dy = \overbrace{-\int_{B(0,|x|+R)\setminus B(0,\varepsilon)} \left(\nabla \Phi(y)\,\nabla_y f(x-y)\right)dy}^{K_\varepsilon}$$

$$+ \int_{\partial B(0,|x|+R)} \Phi(y)\,\cancelto{0}{\frac{\partial f}{\partial n}(x-y)}\,dS_y + \underbrace{\int_{\partial B(0,\varepsilon)} \Phi(y)\,\frac{\partial f(x-y)}{\partial \nu_y}\,dS_y}_{L_\varepsilon}$$

support $\left(f\big|_{\partial B(0,|x|+R)} \equiv 0\right)$

$\left(\nu_y = -n_y,\ \text{inward on } \partial B(0,\varepsilon)\right)$

$$= K_\varepsilon + L_\varepsilon$$

$$(7)\ |L_\varepsilon| = \|\nabla f\|_{L^\infty(\mathbb{R}^n)} \int_{\partial B(0,\varepsilon)} |\Phi(y)|\,dS_y$$

$$\leq \|\nabla f\|_{L^\infty(\mathbb{R}^n)} \int_{\partial B(0,\varepsilon)} \frac{C(n)}{\varepsilon^{n-2}}\,dS_y = C\varepsilon^{n-1}\frac{1}{\varepsilon^{n-2}} = C\varepsilon$$

$(n \geq 3)$

(what about $n=2$?)

Only term that contributes is K_ε.

K_ε

Applying Green's Identity in (6)

(8) $$-\int_{B(0,|x|+R)\setminus B(0,\varepsilon)} \nabla\Phi(y)\,\nabla_y f(x-y)\,dy$$

$$= \int_{B(0,|x|+R)\setminus B(0,\varepsilon)} \overbrace{\cancel{\Delta\Phi(y)}}^{0\ \text{(harmonicity)}} f(x-y)\,dy - \overbrace{\cancel{\int_{\partial B(0,|x|+R)} \frac{\partial\Phi(y)}{\partial\nu_y} f(x-y)\,dS}}^{0\ \text{(support)}}$$

$$-\int_{\partial B(0,\varepsilon)} \frac{\partial\Phi}{\partial\nu_y}(y)\, f(x-y)\,dS_y =$$

· Calculate:

(9) $$\left.\frac{\partial\Phi(y)}{\partial\nu_y}\right|_{r=\varepsilon} = -\frac{\partial}{\partial r}\left[\frac{1}{n(n-2)\alpha(n)}\,\frac{1}{\varepsilon^{n-2}}\right]\Bigg|_{r=\varepsilon}$$

$$= \frac{1}{n\alpha(n)}\,\frac{1}{\varepsilon^{n-1}}$$

∴

(10) $$-\int_{\partial B(0,\varepsilon)} \frac{\partial\Phi}{\partial\nu_y}(y)\, f(x-y)\,dS_y$$

$$= \frac{1}{\varepsilon^{n-1}\, n\alpha(n)} \int_{\partial B(0,\varepsilon)} f(x-y)\,dS_y$$

$$= \frac{1}{|\partial B(0,\varepsilon)|} \int_{\partial B(0,\varepsilon)} f(x-y)\, dS_y$$

Area of sphere

$z = x - y$

$$= \frac{1}{|\partial B(x,\varepsilon)|} \int_{\partial B(x,\varepsilon)} f(z)\, dS_z$$

f continuous $\Rightarrow$

$$\lim \frac{1}{|\partial B(x,\varepsilon)|} \int_{\partial B(x,\varepsilon)} f(z)\, dS_z = f(x).$$

$\square$