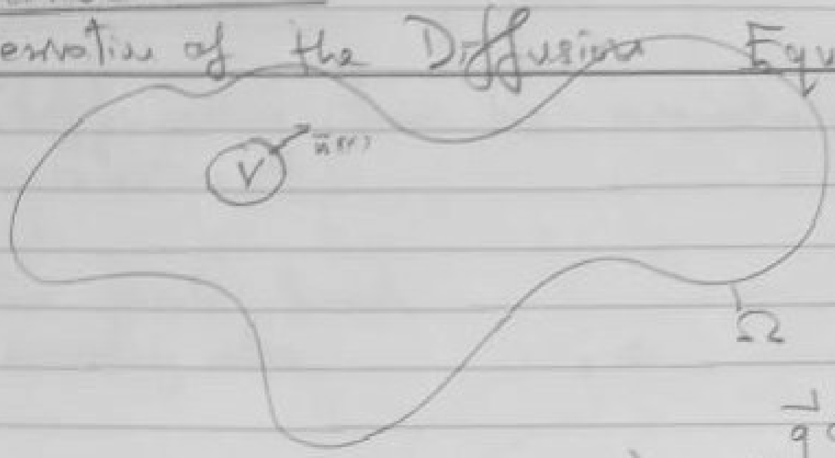
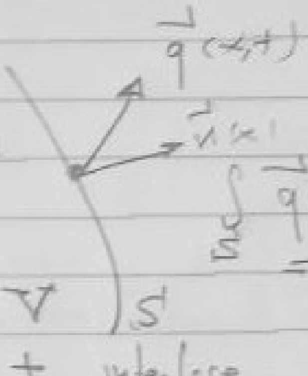
Lecture 2

§2 Derivation of the Diffusion Equation



$\vec{n}(x)$ = outward unit vector



$u(x,t)$ = density of a substance

$\vec{q}(x,t)$ = flux density vector

$\int_S \vec{q}\cdot\vec{n}\, dS$ = amount of mass through S

$M(t) = \int_V u(x,t)\,dx$ = Mass in V at time t

Conservation Law

production of substance

(1) $$\frac{d}{dt} M(t) = -\int_{\partial V} \vec{q}(x,t)\cdot\vec{n}(x)\, dS + \int_V F(x,t)\,dx$$

$\vec{q}(x,t)$, $F(x,t)$, $u(x,t)$ smooth

(1) $\Rightarrow$

(2) $$\int_V \frac{\partial u}{\partial t}\,dx = -\int_{\partial V} \vec{q}(x,t)\cdot\vec{n}(x)\,dS + \int_V F(x,t)\,dx$$

Constitutive Relationship

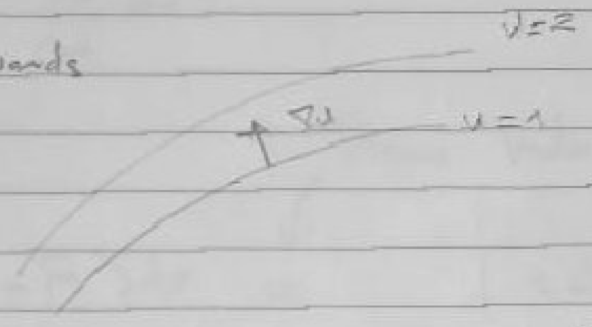
(3) $$\vec{q} = -c\,\nabla u \qquad \text{(Fourier's Law)}$$

(experimental fact)

Physical Meaning of (3)

Stuff moving from high concentration to low concentration.

∇u points towards increasing values

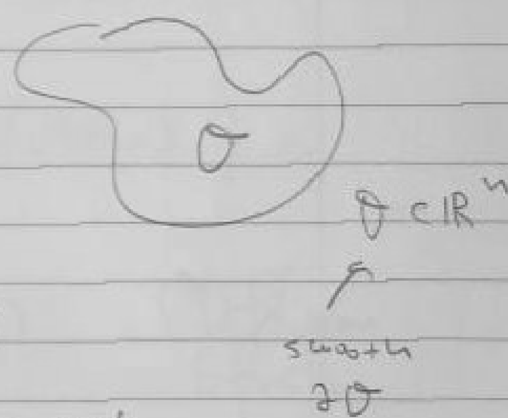


Divergence Theorem

outward normal

(4) $$\int_{O} \operatorname{div} \vec{\Phi}(x)\, dx = \int_{\partial O} \vec{\Phi}(x) \cdot \vec{n}(x)\, dS$$

$\vec{\Phi}(x) = (\Phi_1(x), \dots, \Phi_n(x))$, $\operatorname{div} \vec{\Phi}(x) := \sum_{i=1}^{n} \frac{\partial \Phi_i}{\partial x_i}$



Utilizing (3) in (2):

(5) $$\int_V \frac{\partial u}{\partial t}\, dx = +\int_{\partial V} c\, \nabla u \cdot \vec{n}\, dS + \int_V F(x,t)\, dx$$

(6) $$c\int_{\partial V} \nabla u \cdot \vec{n}\, dS \overset{(4)}{=} c \int_V (\operatorname{div} \nabla u)\, dx$$

$$\operatorname{div} \nabla u = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\frac{\partial u}{\partial x_i}\right) = \Delta u$$

(5), (6) $\Longrightarrow$

(7) $$\int_V \frac{\partial u}{\partial t}\, dx = \int_V (c\, \Delta u + F)\, dx$$

Shrink V at x_0

$|V|$ = Lebesgue measure of V

$$\frac{1}{|V|}\int_V \frac{\partial u}{\partial t}\,dx$$

Mean Value Th

$$= \frac{1}{|V|}\int_V (c\Delta u + F)\,dx = [c\Delta u(\hat{x}_0) + F(\hat{\hat{x}}_0)]$$

$\hat{x}_0, \tilde{x}_0 \in V$

$$\frac{1}{|V|}\int_V \frac{\partial u}{\partial t}\,dx = \frac{\partial u}{\partial t}(\tilde{x}_0, t)$$

Take $V \longrightarrow x_0$ (shrink)

$\therefore$

(8) $$\frac{\partial u}{\partial t} = c\Delta u + F(x,t)$$ (Diffusion Equation)

Differential form of conservation law assuming smoothness.

If $F(x,t)$ Independent of t, $F(x,t) = F(x)$

then time independent solutions of (8),

$u(x,t) = u(x)$, satisfy

(9) $$c\Delta u + F(x) = 0$$ (Poisson's Eq)

If $F \equiv 0$ then

(10) $$\Delta u = 0 \qquad \text{(Laplace's Equation)}$$

§3 The fundamental Solutions $\square$

Look for radial solutions of (10):

$$u(x) = v(|x|), \quad v = v(r), \quad r = |x| = (x_1^2 + \dots + x_n^2)^{1/2}$$

$$\frac{\partial u}{\partial x_i} = \frac{\partial}{\partial x_i}(v(r)) = \frac{\partial v(r)}{\partial r}\frac{\partial r}{\partial x_i} = v'(r)\frac{x_i}{r}$$

$$\left(\frac{\partial r}{\partial x_i} = \frac{\partial}{\partial x_i}\left[(x_1^2 + \dots + x_n^2)^{1/2}\right] = \frac{1}{2}(x_1^2 + \dots + x_n^2)^{-\frac{1}{2}} \cancel{2} x_i = \frac{x_i}{|x|} \right)$$

$$\frac{\partial^2 u}{\partial x_i^2} = \frac{\partial}{\partial x_i}\left(v'(r)\frac{x_i}{r}\right) = \frac{\partial}{\partial x_i}\left(\frac{v'(r)}{r}\right)x_i + \frac{v'(r)}{r}$$

$$\frac{\partial}{\partial x_i}\left(\frac{v'(r)}{r}\right) = \frac{\partial}{\partial r}\left(\frac{v'(r)}{r}\right)\frac{\partial r}{\partial x_i}$$

$$= \frac{\partial}{\partial r}\left(\frac{v'(r)}{r}\right)\frac{x_i}{r}$$

$$\therefore \sum \frac{\partial^2 u}{\partial x_i^2} = \sum_{i=1}^{n}\left[\frac{\partial}{\partial r}\left(\frac{v'(r)}{r}\right)\frac{x_i}{r}\right]x_i$$

$$+ n\frac{v'(r)}{r}$$

$$= \frac{\partial}{\partial r}\left(\frac{v'(r)}{r}\right)r + n\frac{v'(r)}{r}$$

$$= v'' + \frac{(n-1)}{r} v' = \frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1} v'\right)$$

- $\frac{\partial}{\partial x_i}|x| = \frac{x_i}{r} = \cos\theta_i$ $\left(\begin{array}{l} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{array}\right)$

- $\frac{1}{r}$ Mean curvature of

(11) $$\frac{1}{r^{n-1}} \frac{\partial}{\partial r}\left(r^{n-1} v'\right) = 0$$

$\Rightarrow$

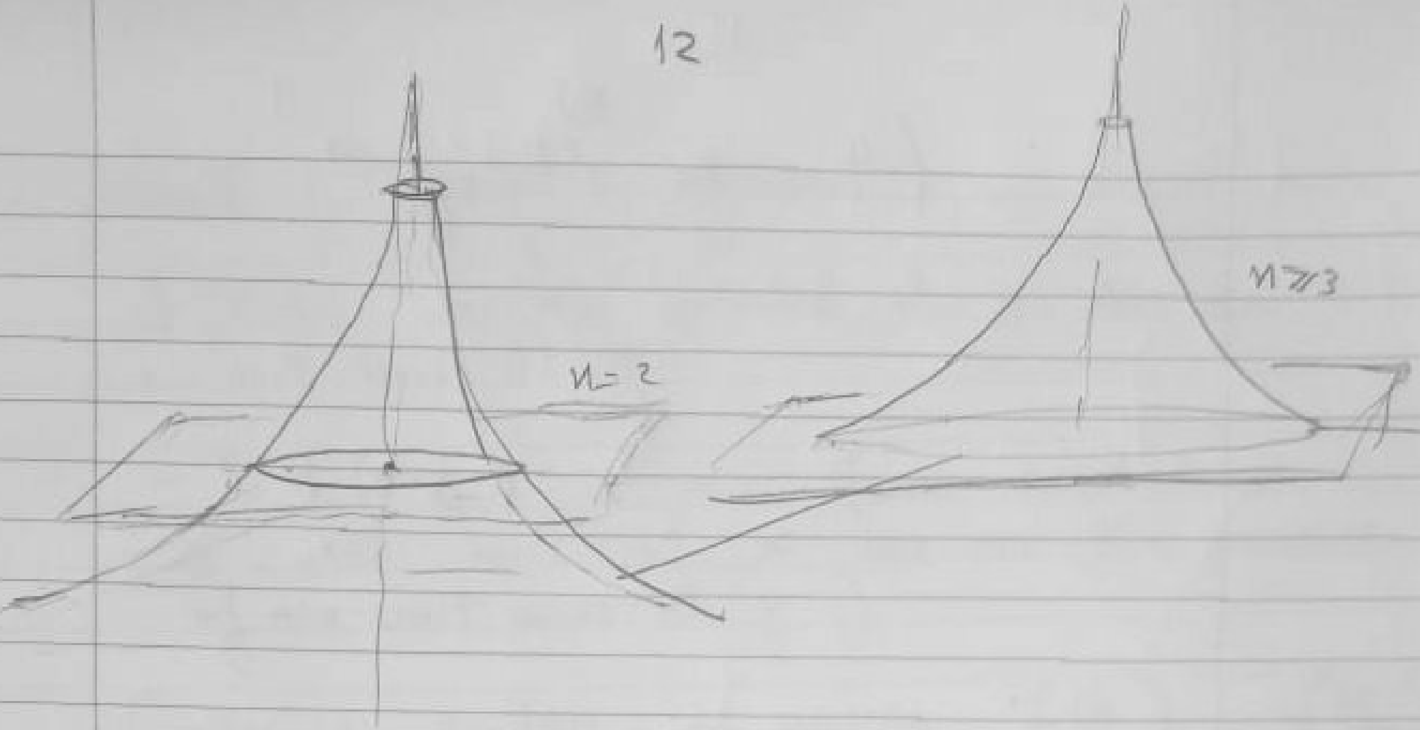
$$r^{n-1} v' = C_1$$

(12) $$v(r) = \begin{cases} c_1 \ln r + c_2 & , n = 2 \\ \frac{c_1}{r^{n-2}} + c_2 & , n \geq 3 \\ c_1 r + c_2 & , n = 1. \end{cases}$$

Physical Meaning (Electrostatics).

$$\Phi(x) = \begin{cases} \frac{1}{2}|x| , & n = 1 \\ -\frac{1}{2}\ln|x| , & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} , & n \geq 3. \end{cases}$$

$\alpha(n) = |B_1|$



- Singularity at $r = 0$

- $\Phi(x)$ harmonic for $x \neq 0$

- $\Phi(x-\xi)$ harmonic for $x \neq \xi$.

Th1 (Solution of $-\Delta u = f$ on $\mathbb{R}^n$) convolution

Define

(13) $$u(x) = \int_{\mathbb{R}^n} \Phi(x-y)\, f(y)\, dy \quad \left(= \Phi * f \right)$$

Assume $f \in C_c^2(\mathbb{R}^n)$. Then $u \in C^2(\mathbb{R}^n)$

and

(14) $$-\Delta u = f \qquad , x \in \mathbb{R}^n$$

Remark (Physist's argument)

$\Phi(x-\xi)$ is the potential at x due to a unit mass at ξ.

($-\nabla \Phi(x-\xi)$ is the force exerted on a unit mass at x due to the presence of the unit mass at ξ.)

Consider a system of masses $f(\xi_1), \dots, f(\xi_k)$ at $\xi_1, \dots, \xi_k$. By superposition they should generate at x the potential

$$\sum_{i=1}^{k} \Phi(x-\xi_i) f(\xi_i).$$

Thus the continuous distribution of masses $f(\xi)$ should generate

$$\lim_{k\to\infty} \sum_{i=1}^{k} \Phi(x-\xi_i) f(\xi_i) = \int_{\mathbb{R}^n} \Phi(x-\xi) f(\xi) d\xi$$

$$-\Delta_x \Phi(x-\xi) = \delta_\xi(x) \qquad (\text{Dirac})$$

$\Longrightarrow$

$$-\Delta u(x) = f(x).$$

$\square$

Proof of Theorem 1 ($n \geq 3$)

1. $u(x) = \int_{\mathbb{R}^n} \Phi(x-y) f(y) dy \overset{z=x-y}{=} \int_{\mathbb{R}^n} \Phi(z) f(x-z) dy$

$$= \int_{\mathbb{R}^n} \Phi(y) f(x-y) dy$$

• $\int_{\mathbb{R}^n} | \cdot | dy < \infty$

$$\left(\int_{\mathbb{R}^n} \Phi(y) f(x-y) dy = \int_{B(x;R)} \Phi(y) f(x-y) dy \right.$$

since $\operatorname{supp} f \subset B(0;R)$)

$$\int_{B(x;R)} |\Phi(y) f(x-y)| dy \leq \int_{B(x;R)} |\Phi(y)| \max_{B} |f(x-y)|$$

$$\leq M \int_{B(x;R)} |\Phi(y)| dy$$

spherical coordinates

$$\leq M \int_{S^{n-1}} \left(\int_0^R \frac{C_n}{r^{n-2}} r^{n-1} dr \, d\sigma \right.$$

$C_n = \frac{1}{n(n-2)\alpha(n)}$

$$= \frac{M C_n}{2} R^2 |S^{n-1}| \Big)$$