ΣΤΟΙΧΕΙΑ ΘΕΩΡΙΑΣ ΠΑΙΓΝΙΩΝ ΚΑΙ ΛΗΨΗΣ ΑΠΟΦΑΣΕΩΝ

ΔΙΑΛΕΞΗ 8: ΕΞΕΛΙΚΤΙΚΕΣ ΔΥΝΑΜΙΚΕΣ

Παναγιώτης Μερτικόπουλος

Εθνικό και Καποδιστριακό Πανεπιστήμιο Αθηνών

Τμήμα Μαθηματικών

Χειμερινό Εξάμηνο, 2023–2024

Outline

1 Population games

2 Exponential weights and the replicator dynamics

3 Asymptotic analysis and rationality

Population games, I: Symmetric models

Definition (Single-population games)

A ***single-population game*** is a collection of the following primitives:

- A continuous ***population of players*** modeled by $\mathcal{N} = [0,1]$
- A finite set of ***actions / pure strategies*** $\mathcal{A} = \{1, \dots, m\}$, common for all players in the population
- An ensemble of ***payoff functions*** $v_\alpha \colon \mathcal{X} \equiv \Delta(\mathcal{A}) \to \mathbb{R}$, one per $\alpha \in \mathcal{A}$

A population game with primitives as above will be denoted by $\mathcal{G} \equiv \mathcal{G}(\mathcal{A}, v)$.

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A population game with primitives as above will be denoted by $\mathcal{G} \equiv \mathcal{G}(\mathcal{A}, v)$.

Setup of the game:

- ***Action selection*** given by some $i \mapsto \chi(i) \in \mathcal{A}$ — # $\chi : \mathcal{N} \to \mathcal{A}$ assumed measurable
- ***Population state*** $x \in \mathcal{X} \equiv \Delta(\mathcal{A})$ defined as — # as a measure: $x = \lambda \circ \chi^{-1}$

$$x_\alpha = \lambda(\chi^{-1}(\alpha)) = \text{mass of players playing } \alpha \in \mathcal{A}$$

- ***Anonymity:*** payoffs determined by the *state* of the population, not *individual* player choices

$$v_\alpha(x) = \text{payoff to } \alpha\text{-players when the population is at state } x \in \mathcal{X}$$

Example I: Symmetric random matching

Example (Symmetric / Single-population random matching)

- ***Given:*** $m \times m$ payoff matrix M — # symmetric two-player finite game
- ***Matching:*** Two players are drawn randomly to play M — # independent draws from $x \in \mathcal{X}$
- If the population is at state $x \in \mathcal{X}$:

$$\mathbb{P}(\text{matching } \alpha \text{ against } \beta) = x_\alpha x_\beta$$

- Mean payoff to an α-strategist:

$$v_\alpha(x) = \mathbb{E}_{\beta \sim x}[M_{\alpha\beta}] = \sum_{\beta \in \mathcal{A}} M_{\alpha\beta} x_\beta = (Mx)_\alpha$$

- Mean population payoff:

$$u(x) = \mathbb{E}_{\alpha,\beta \sim x}[M_{\alpha\beta}] = \sum_{\alpha,\beta \in \mathcal{A}} M_{\alpha\beta} x_\alpha x_\beta = x^\top M x$$

NB:

- Mean population payoff is **quadratic** in x — # symmetric matching

Population games, II: Asymmetric models

Definition (Multi-population games)

A ***multi-population game*** is a collection of the following primitives:

- N distinct ***populations of players:*** $\mathcal{N} = \coprod_{i=1}^{N} [0, \rho_i]$ # ρ_i = total mass of i-th population
- A finite set of ***actions / pure strategies*** $\mathcal{A}_i = \{1, \dots, m_i\}$ per population
- An ensemble of ***payoff functions*** $v_{i\alpha_i} \colon \mathcal{X} \equiv \prod_j \Delta(\mathcal{A}_j) \to \mathbb{R}$, one per $\alpha_i \in \mathcal{A}_i$, $i = 1, \dots, N$

A population game with primitives as above will be denoted by $\mathcal{G} \equiv \mathcal{G}(\mathcal{N}, \mathcal{A}, v)$.

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Setup of the game:

- ***Population state*** $x \in \mathcal{X} \equiv \prod_j \Delta(\mathcal{A}_j)$: — # state of i-th population: $x_i \in \mathcal{X}_i \equiv \Delta(\mathcal{A}_i)$

$$x_{i\alpha_i} = \text{mass of players of population } i \text{ playing } \alpha_i \in \mathcal{A}_i$$

- ***Anonymity:*** payoffs determined by the *state* of the population, not *individual* player choices

$$v_{i\alpha_i}(x) = \text{payoff to players of population } i \text{ playing } \alpha_i \in \mathcal{A}_i \text{ when the population is at state } x \in \mathcal{X}$$

Example II: Asymmetric random matching

Example (Asymmetric / Multi-population random matching)

- ***Given:*** finite game $\Gamma \equiv \Gamma(\mathcal{N}, \mathcal{A}, u)$; N unit mass populations
- ***Matching:*** N players are drawn randomly to play Γ, one per population   # independent draws from $x \in \mathcal{X}$
- If the population is at state $x \in \mathcal{X}$:

$$\mathbb{P}(\text{matching } \alpha_i \text{ against } \alpha_{-i}) = x_{i\alpha_i} \cdot x_{-i,\alpha_{-i}}$$

- Mean payoff to an α-strategist of population i:

$$v_{i\alpha_i}(x) = \mathbb{E}_{\alpha_{-i} \sim x_{-i}}[u_\alpha(\alpha_i; \alpha_{-i})] = u_i(\alpha_i; x_{-i})$$

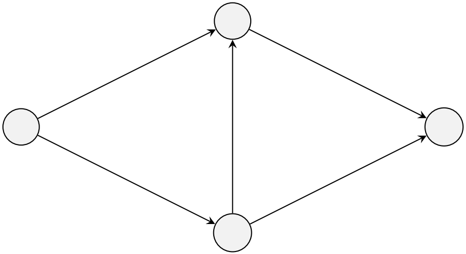
- Mean payoff of population i:

$$u_i(x) = \mathbb{E}_{\alpha \sim x}[u_i(\alpha)] = \sum_{\alpha_1 \in \mathcal{A}_1} \cdots \sum_{\alpha_N \in \mathcal{A}_N} x_{1,\alpha_1} \cdots x_{N,\alpha_N}\, u_i(\alpha_1, \dots, \alpha_N)$$

NB:

- Mean population payoff is **multilinear** in x # asymmetric matching

Example III: Nonatomic congestion games



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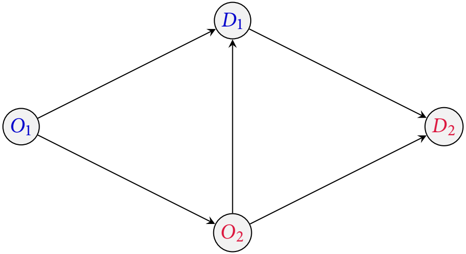

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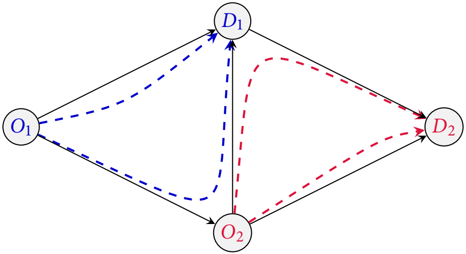

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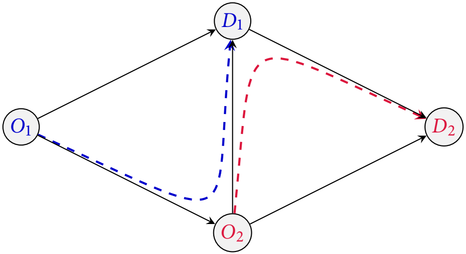

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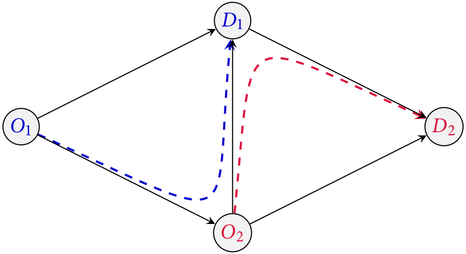

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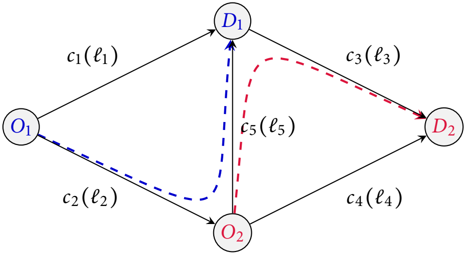

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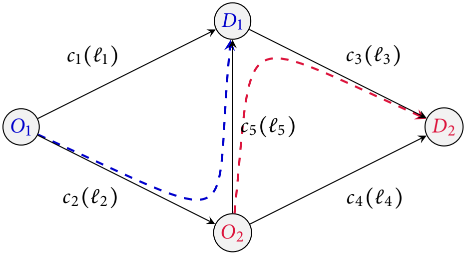

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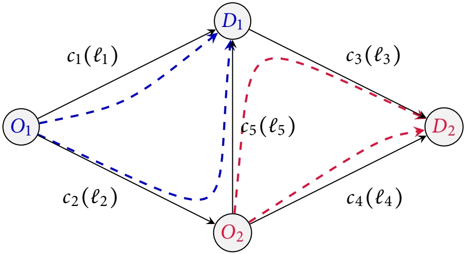

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- **Nonatomic congestion game:** $\mathcal{G} \equiv \mathcal{G}(\mathcal{N}, \mathcal{A}, -c)$

Mixing versus matching

⚠ **Symmetric Random matching $\neq$ Mixed extension**

Population matched against itself $\Longrightarrow$ ***symmetric interactions***

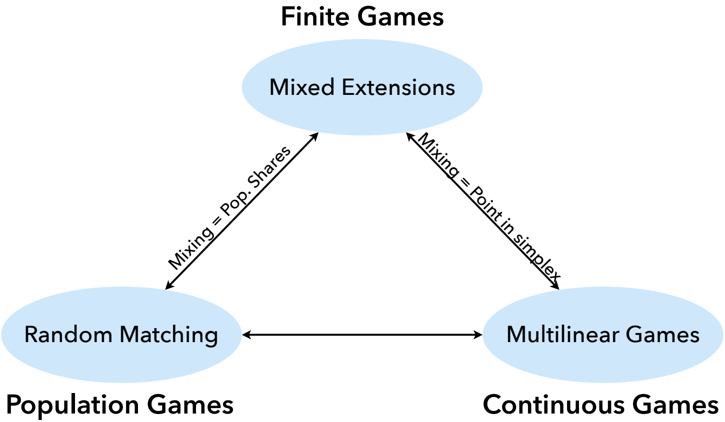
⚠ ***Asymmetric*** **random matching = Mixed Extension**

Populations matched against each other $\Longrightarrow$ ***asymmetric interactions***

⚠ **Multi-population games $\supsetneq$ Mixed Extensions**

Nonatomic congestion games, ...

Relations between classes



Nash equilibrium

Nash equilibrium (Nash, 1950, 1951)

"No player has an incentive to deviate from their chosen strategy if other players don't"

- **In finite games** (mixed extension formulation):

$$u_i(x_i^*; x_{-i}^*) \geq u_i(x_i; x_{-i}^*) \quad \text{for all } x_i \in \mathcal{X}_i,\ i \in \mathcal{N}$$

- **In population games:**

$$v_{i\alpha_i}(x^*) \geq v_{i\beta_i}(x^*) \quad \text{whenever } \alpha_i \in \operatorname{supp}(x^*)$$

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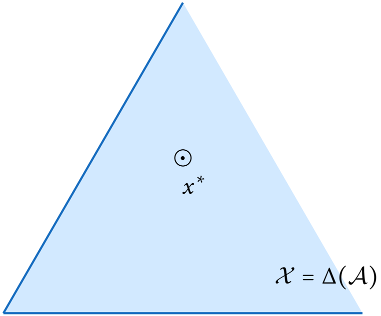
Variational formulation (Stampacchia, 1964)

$$\langle v(x^*), x - x^* \rangle \leq 0 \quad \text{for all } x \in \mathcal{X}$$

where $v(x) = (v_1(x), \dots, v_N(x))$ is the ***payoff field*** of the game

Geometric characterization

At Nash equilibrium, payoff vectors are outward-pointing



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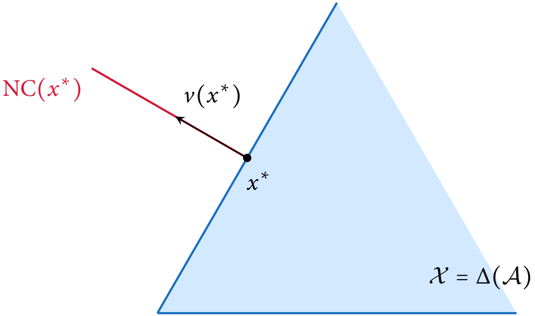

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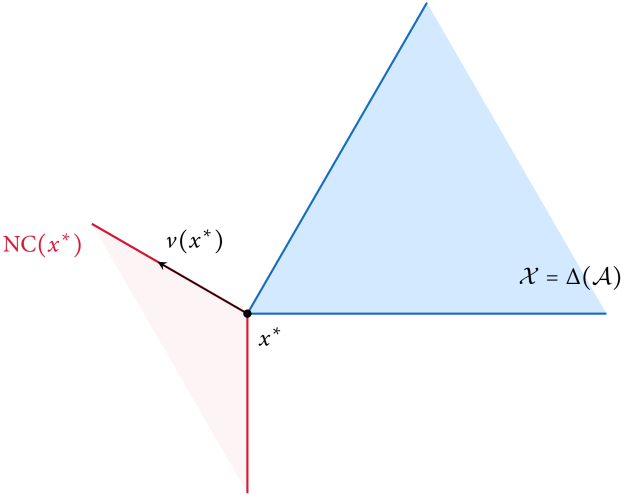

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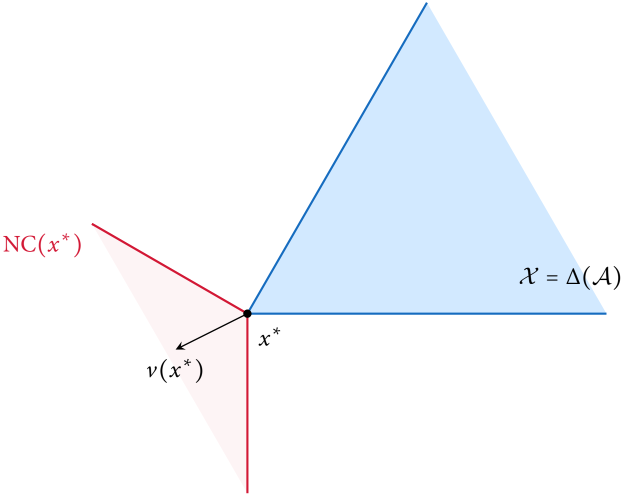

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1 Population games

2 Exponential weights and the replicator dynamics

3 Asymptotic analysis and rationality

Basic questions

How do players learn from the history of play?

Do players end up playing a Nash equilibrium?

Learning, evolution and dynamics

What is “learning” in games?

Learning, evolution and dynamics

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The basic process:

- Players choose strategies and receive corresponding payoffs
- Depending on outcome and information revealed, they choose new strategies and they play again
- Rinse, repeat

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The basic questions:

- *How do populations evolve over time?* — # Population biology
- *How do people learn in a game?* — # Economics
- *What algorithms should we use to learn in a game?* — # Computer science
- Given a dynamical system on $\mathcal{X}$, what is its long-term behavior? — # Mathematics

Age the First (1970's-1990's): Population Biology

- Strategies are ***phenotypes*** in a given species

$$z_\alpha = \text{absolute population mass of type } \alpha \in \mathcal{A}$$
$$z = \sum_\alpha z_\alpha = \text{absolute population mass}$$

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$$\dot{z}_\alpha = z_\alpha v_\alpha$$

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- Evolution of population shares ($x_\alpha = z_\alpha / z$):

$$\dot{x}_\alpha = \frac{d}{dt}\frac{z_\alpha}{z} = \frac{\dot{z}_\alpha z - z_\alpha \sum_\beta \dot{z}_\beta}{z^2} = \frac{z_\alpha}{z} v_\alpha - \frac{z_\alpha}{z} \sum_\beta \frac{z_\beta}{z} v_\beta$$

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Replicator dynamics (Taylor & Jonker, 1978)

$$\dot{x}_\alpha = x_\alpha [v_\alpha(x) - u(x)] \tag{RD}$$

Age the Second (1990's-2010's): Economics

- Agents receive **revision opportunities** to switch strategies

$$\rho_{\alpha\beta}(x) = \text{conditional switch rate from } \alpha \text{ to } \beta$$

NB: dropping player index for simplicity

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Imitate with probability proportional to excess payoff (Helbing, 1992; Schlag, 1998)

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- Inflow/outflow:

$$\textit{Incoming toward } \alpha = \sum_\beta \text{mass}(\beta \rightsquigarrow \alpha) = \sum_{\beta\in\mathcal{A}} x_\beta \rho_{\beta\alpha}(x)$$

$$\textit{Outgoing from } \alpha = \sum_\beta \text{mass}(\alpha \rightsquigarrow \beta) = x_\alpha \sum_{\beta\in\mathcal{A}} \rho_{\alpha\beta}(x)$$

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$$\textit{Outgoing from } \alpha = \sum\nolimits_\beta \text{mass}(\alpha \rightsquigarrow \beta) = x_\alpha \sum\nolimits_{\beta\in\mathcal{A}} \rho_{\alpha\beta}(x)$$

- Detailed balance:

$$\dot{x}_\alpha = \text{inflow}_\alpha(x) - \text{outflow}_\alpha(x) = \cdots = x_\alpha [v_\alpha(x) - u(x)] \tag{RD}$$

Age the Third (2000's-present): Computer Science

Learning in finite games

Require: finite game $\Gamma \equiv \Gamma(\mathcal{N}, \mathcal{A}, u)$

repeat

At each epoch $t \geq 0$ **do simultaneously** for all players $i \in \mathcal{N}$ — # continuous time

Choose ***mixed strategy*** $x_i(t) \in \mathcal{X}_i := \Delta(\mathcal{A}_i)$ — # mixing

Encounter ***mixed payoff vector*** $v_i(x(t))$ and get **mixed payoff** $u_i(x(t)) = \langle v_i(t), x(t) \rangle$ — # feedback phase

until end

Defining elements

- ***Time:*** continuous
- ***Players:*** finite
- ***Actions:*** finite
- ***Mixing:*** yes
- ***Feedback:*** mixed payoff vectors

Exponential weights

Exponential reinforcement mechanism:

- Score each action based on its cumulative payoff over time:

$$y_{i\alpha_i}(t) = \int_0^t v_{i\alpha_i}(x(s))\, ds$$

- Play an action with probability exponentially proportional to its score

$$x_{i\alpha_i}(t) \propto \exp(y_{i\alpha_i}(t))$$

Exponential weights in continuous time

$$\begin{aligned} \dot{y}_{i\alpha_i} &= v_{i\alpha_i}(x) \\ x_{i\alpha_i} &= \frac{\exp(y_{i\alpha_i})}{\sum_{\beta_i \in \mathcal{A}_i} \exp(y_{i\beta_i})} \end{aligned} \tag{EW}$$

Replicator dynamics

How do mixed strategies evolve under (EW)?

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Replicator dynamics (Taylor & Jonker, 1978)

$$\begin{aligned} \dot{x}_{i\alpha_i} &= x_{i\alpha_i}\left[v_{i\alpha_i}(x) - \textstyle\sum_{\beta_i \in \mathcal{A}_i} x_{i\beta_i} v_{i\beta_i}(x)\right] \\ &= x_{i\alpha_i}\left[u_i(\alpha_i; x_{-i}) - u_i(x)\right] \end{aligned} \tag{RD}$$

"The per capita growth rate of a strategy is proportional to its payoff excess"

➻ Hofbauer & Sigmund (1998); Weibull (1995); Hofbauer & Sigmund (2003); Sandholm (2010)

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Proposition

Solution orbits of (EW) $\Longleftrightarrow$ *Interior orbits of* (RD)

Basic properties

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Structural properties

Weibull, 1995; Hofbauer & Sigmund, 1998

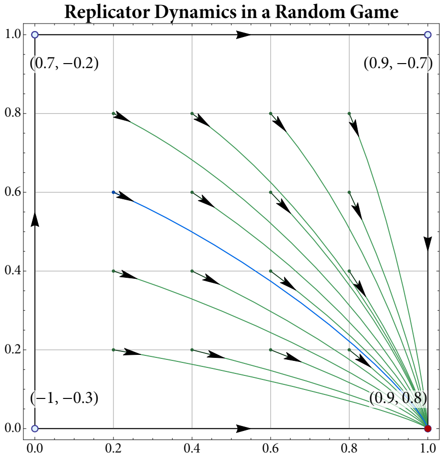
- **Well-posed:** every initial condition $x \in \mathcal{X}$ admits unique solution trajectory $x(t)$ that exists for all time
  # Assuming v Lipschitz
- **Consistent:** $x(t) \in \mathcal{X}$ for all $t \geq 0$
  # Assuming $x(0) \in \mathcal{X}$
- **Faces are forward invariant** (“strategies breed true”):

$$x_{i\alpha_i}(0) > 0 \iff x_{i\alpha_i}(t) > 0 \quad \text{for all } t \geq 0$$
$$x_{i\alpha_i}(0) = 0 \iff x_{i\alpha_i}(t) = 0 \quad \text{for all } t \geq 0$$

Evolution of mixed strategies I: 2×2 *games*

What do the dynamics look like?

phase portraits



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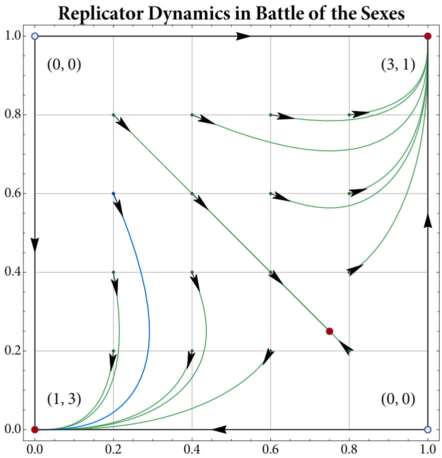

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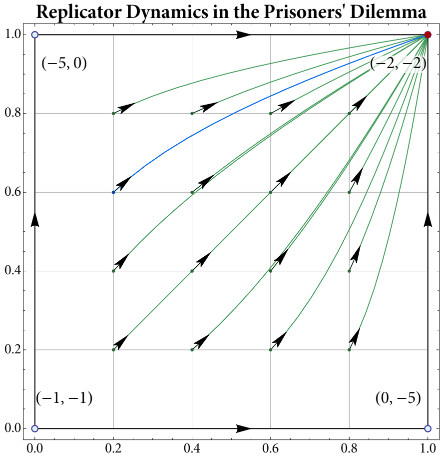

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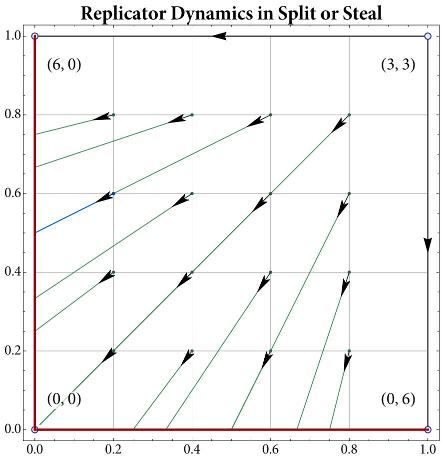

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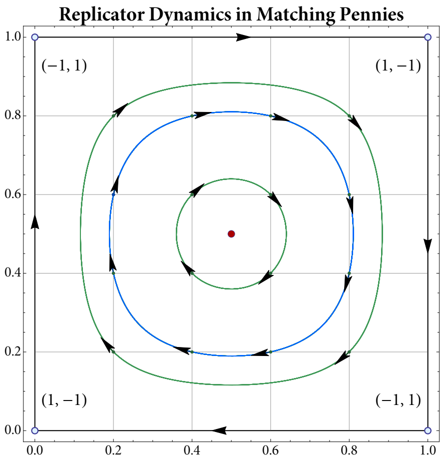

Evolution of mixed strategies II: $2 \times 2 \times 2$ *games*

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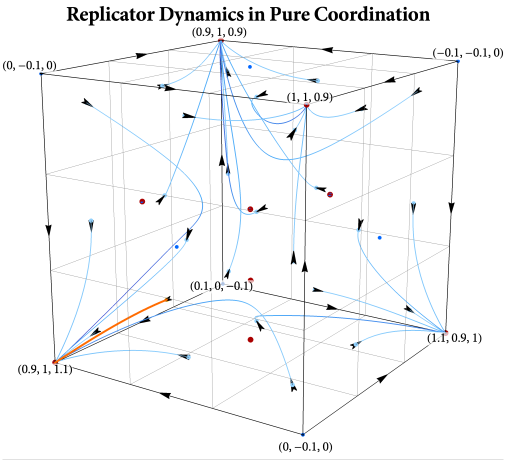

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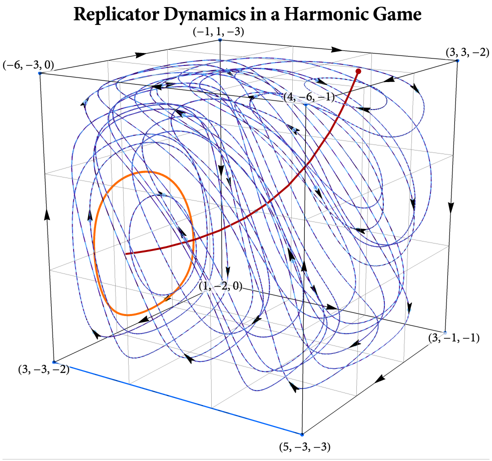

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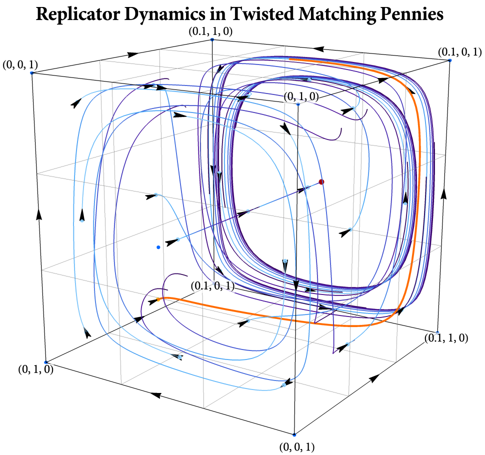

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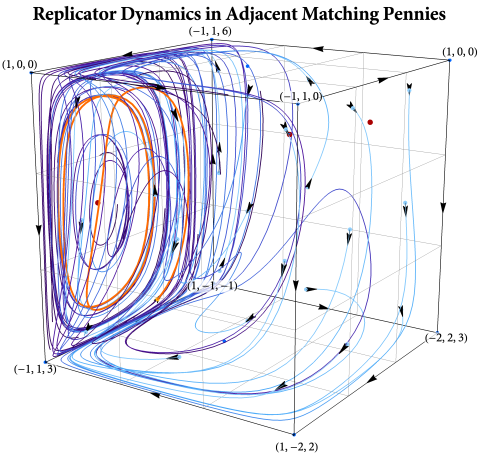

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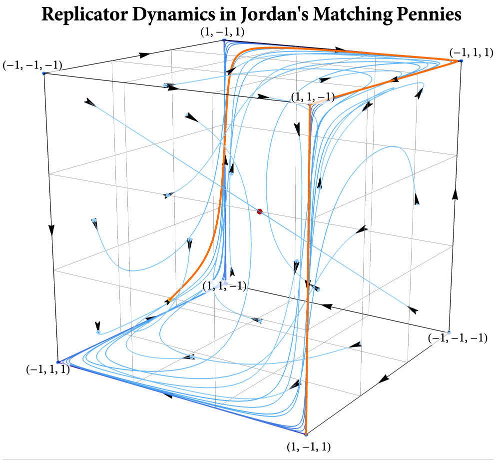

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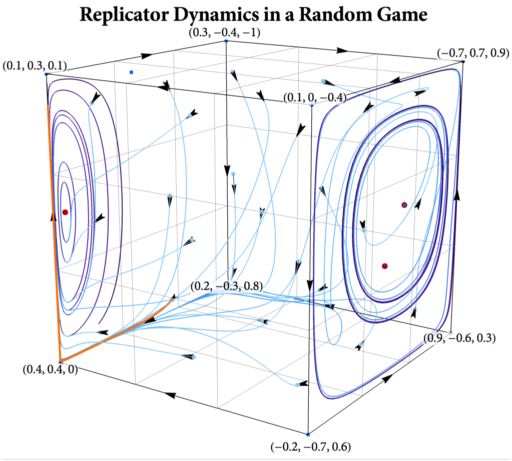

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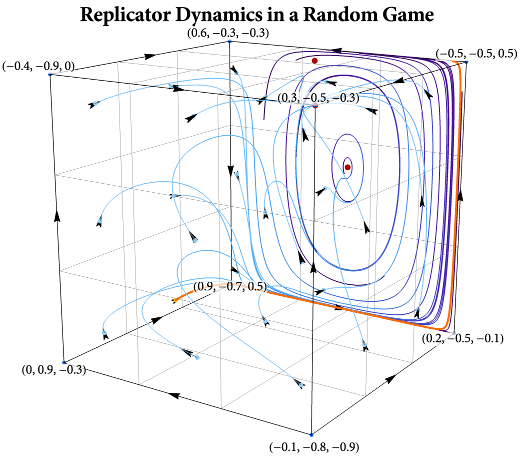

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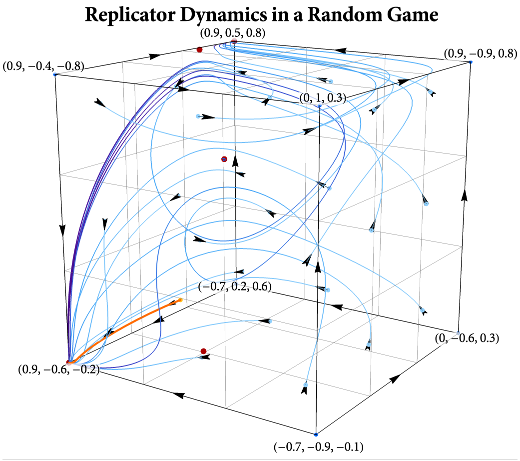

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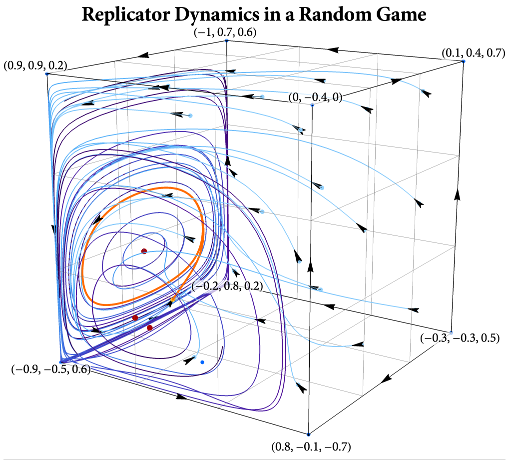

Outline

1. Population games
2. Exponential weights and the replicator dynamics
3. Asymptotic analysis and rationality

Dynamics and rationality

Are game-theoretic solution concepts consistent with the players' dynamics?

- Do dominated strategies die out in the long run?
- Are Nash equilibria stationary?
- Are they ***stable?*** Are they ***attracting?***
- Do the replicator dynamics always converge?
- What other behaviors can we observe?
- …

Dominated strategies

Suppose $\alpha_i \in \mathcal{A}_i$ is ***dominated*** by $\beta_i \in \mathcal{A}_i$

- Consistent payoff gap:

$$v_{i\alpha_i}(x) \leq v_{i\beta_i}(x) - \varepsilon \quad \text{for some } \varepsilon > 0$$

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- Consistent difference in choice probabilities

$$\frac{x_{i\alpha_i}(t)}{x_{i\beta_i}(t)} = \frac{\exp(y_{i\alpha_i}(t))}{\exp(y_{i\beta_i}(t))} \leq \exp(-\varepsilon t)$$

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Let $x(t)$ be a solution orbit of (EW)/(RD). *If $\alpha_i \in \mathcal{A}_i$ is dominated, then*

$$x_{i\alpha_i}(t) = \exp(-\Theta(t)) \quad \text{as } t \to \infty$$

In words: under (EW)/(RD)*, dominated strategies become extinct at an exponential rate.*

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Self-check: extend to *iteratively* dominated strategies

Stationarity of equilibria

Nash equilibrium: $v_{i\alpha_i}(x^*) \geq v_{i\beta_i}(x^*)$ for all $\alpha_i, \beta_i \in \mathcal{A}_i$ with $x^*_{i\alpha_i} > 0$

- Supported strategies have equal payoffs:

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✗ **The converse does not hold!**

➭ **Self-check:** All vertices of $\mathcal{X}$ are stationary. General statement?

Stability

Are all stationary points created equal?

Definition (Lyapunov stability)

x^* is ***(Lyapunov) stable*** if, for every neighborhood $\mathcal{U}$ of x^* in $\mathcal{X}$, there exists a neighborhood $\mathcal{U}'$ of x^* such that

$$x(0) \in \mathcal{U}' \implies x(t) \in \mathcal{U} \quad \text{for all } t \geq 0$$

➡ Trajectories that start close to x^* remain close for all time

Stability and equilibrium

Proposition (Folk)

Suppose that x^* is Lyapunov stable under (EW) / (RD)*. Then x^* is a Nash equilibrium.*

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Suppose that x^* is Lyapunov stable under (EW)/(RD). *Then x^* is a Nash equilibrium.*

Proof. Argue by contradiction:

- **Suppose that x^* is not Nash.** Then

$$v_{i\alpha_i^*}(x^*) = u_i(\alpha_i^*; x_{-i}^*) < u_i(\alpha_i; x_{-i}^*) = v_{i\alpha_i}(x^*)$$

for some $\alpha_i^* \in \operatorname{supp}(x_i^*)$, $\alpha_i \in \mathcal{A}_i$, $i \in \mathcal{N}$

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- If $x(t)$ is contained in $\mathcal{U}$ for all $t \geq 0$ (Lyapunov property), then:

$$y_{i\alpha_i^*}(t) - y_{i\alpha_i}(t) = c + \int_0^t [v_{i\alpha_i^*}(x(s)) - v_{i\alpha_i}(x(s))]\, ds < c - \varepsilon t$$

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- We conclude that $x_{i\alpha_i^*}(t) \to 0$, contradicting the Lyapunov stability of x^*. □

Asymptotic stability

Are Nash equilibria attracting?

Definition

- x^* is ***attracting*** if $\lim_{t\to\infty} x(t) = x^*$ whenever $x(0)$ is close enough to x^*
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Proof. Compare scores:

- If $\alpha^* = (\alpha_1^*, \dots, \alpha_N^*)$ is strict Nash $\implies v_{i\alpha_i^*}(x^*) > v_{i\alpha_i}(x^*)$ for all $\alpha_i \in \mathcal{A}_i \setminus \{\alpha_i^*\}$
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- Proof complete by showing Lyapunov stability — Left as self-check exercise □

The "folk theorem" of evolutionary game theory

Theorem ("folk"; Hofbauer & Sigmund, 2003)

Let Γ be a finite game. Then, under (RD), we have:

1. x^* *is a Nash equilibrium* $\implies$ x^* *is stationary*
2. x^* *is the limit of an interior trajectory* $\implies$ x^* *is a Nash equilibrium*
3. x^* *is stable* $\implies$ x^* *is a Nash equilibrium*
4. x^* *is asymptotically stable* $\iff$ x^* *is a strict Nash equilibrium*

Notes:

- Single-population case similar ***except*** $\implies$ of (4)
- ✗ **Converse to (1), (2) and (3) does not hold!**
- ✓ Proof of (2) similar to (3) — Do as self-check
- Proof of "$\impliedby$" in (4): requires different techniques

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