Διάλεξη 22 24.4.23

## Πίνακας Γραμμικής Συνάρτησης

$V \xrightarrow{f} W$, $f$ γραμμική. Γνωρίζουμε το $f(v_i)$ $\forall v_i \in \overset{\text{βάση}}{B} = \{v_1, \dots, v_n\}$

$$f(v) = f\left(\sum_{i=1}^{n} \lambda v_i\right) = \sum_{i=1}^{n} \lambda_i f(v_i) \quad , \quad v \in V, \ v = \sum_{i=1}^{n}$$

$\boxed{\text{SOS}}$ : θα φτιάξουμε έναν πίνακα ο οποίος θα γνωρίζει τα πάντα για τη γραμμική συνάρτηση

**Βήμα 1**: Σταθεροποιούμε διατεταγμένες βάσεις $B = \{v_1, \dots, v_n\}$ του $V$ και $B' = \{w_1, \dots, w_m\}$ του $W$

[Ο πίνακας που θα κατασκευάσουμε εξαρτάται από τα $B$ και $B'$]

$$W \ni f(v_1) = a_{11}w_1 + a_{21}w_2 + \dots + a_{m1}w_m = \underbrace{\sum_{v=1}^{m} a_{v1} w_v}_{} \to f(v_1)$$

$$f(v_2) = a_{12}w_1 + a_{22}w_2 + \dots + a_{m2}w_m = \sum_{v=1}^{m} a_{v2} w_v \to f(v_2)$$

$$\vdots$$

$$f(v_n) = a_{1n}w_1 + a_{2n}w_2 + \dots + a_{mn}w_m = \sum_{v=1}^{m} a_{vn} w_v \to f(v_n)$$

$\Downarrow$ έχουμε πίνακα:

$$\boxed{(f, B, B') = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \\ & & \dots & a_{mn} \end{pmatrix}} = A$$

↓ οι γραμμές γίναν στήλες

$$\begin{tikzcd} V \ni v \arrow[r, "f"] \arrow[d] & W \ni f(v) \arrow[d] \\ \mathbb{R}^n \ni [v]_B \arrow[r, "A = (f,B,B')"] & {[f(v)]_{B'}} \in \mathbb{R}^m \end{tikzcd}$$

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} = [v]_B \text{ συντεταγ.} \qquad [f(v)]_{B'} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_m \end{pmatrix}$$

$$\Rightarrow \boxed{[f(v)]_{B'} = A \cdot [v]_B}$$

ΑΠΟΔΕΙΞΗ:

$$f(v) = f\left(\sum_{i=1}^{n} \lambda_i v_i\right) = \sum_{i=1}^{n} \lambda_i f(v_i) = \sum_{i=1}^{n} \lambda_i \underbrace{\sum_{j=1}^{m} a_{ji} w_j}_{f(v_i)} = \sum_{j=1}^{m} \Big( \underbrace{\sum_{i=1}^{n} a_{ji} \lambda_i}_{A} \Big) w_i \quad \blacksquare$$

$$\begin{tikzcd} \overset{B_1}{V_1} \ni v \arrow[r, "f"] \arrow[d] & \overset{B_2}{V_2} \ni v' \arrow[r, "g"] \arrow[d] & \overset{B_3}{V_3} \ni v'' \arrow[d] \\ \mathbb{R}^{n_1} \ni [v]_{B_1} \arrow[r, "{(f,B_1,B_2) = A_1}"] & \mathbb{R}^{n_2} \ni [v']_{B_2} \arrow[r, "{(g,B_2,B_3) = A_2}"] & \mathbb{R}^{n_3} \ni [v'']_{B_3} \end{tikzcd}$$

$$\Rightarrow \boxed{(g \circ f, B_1, B_3) = (g, B_1, B_2) \cdot (f, B_1, B_2)}$$

Παράδειγμα $\longrightarrow$ ΣΕΛ. 2

$\boxed{\text{Παράδειγμα}}$: $\mathbb{R}^4 \longrightarrow \mathbb{R}^4$ 24.4.23

- $f(x_1,x_2,x_3,x_4) = (2x_1 - x_2, x_1 + x_3 - x_4, x_1 + 3x_2 - x_3 - x_4, x_2 + x_3)$
- $B = B' = (e_1, e_2, e_3, e_4)$ $\left[ \begin{matrix} e_1 = (1,0,0,0), & e_3 = (0,0,1,0) \\ e_2 = (0,1,0,0), & e_4 = (0,0,0,1) \end{matrix} \right]$

ΛΥΣΗ

$$\left.\begin{matrix} f(e_1) = (2,1,1,0) \\ f(e_2) = (-1,0,3,1) \\ f(e_3) = (0,1,-1,1) \\ f(e_4) = (0,-1,-1,0) \end{matrix}\right\} \Rightarrow \begin{pmatrix} 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & -1 \\ 1 & 3 & -1 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = (f, B, B')$$

παρατηρώ: $(f, B, B') \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 \\ x_2 - x_3 - x_4 \\ x_1 + 3x_2 - x_3 - x_4 \\ x_2 + x_3 \end{pmatrix}$

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$\boxed{\text{Παράδειγμα:}}$

- $f(x_1,x_2,x_3,x_4) = (2x_1 - x_2, x_1 + x_3 - x_4, x_1 + 3x_2 - x_3 - x_4, x_2 + x_3)$
- $B = \{\overset{\varepsilon_1}{(1,1,1,1)}, \overset{\varepsilon_2}{(0,1,1,1)}, \overset{\varepsilon_3}{(0,0,1,1)}, \overset{\varepsilon_4}{(0,0,0,1)}\} \rightsquigarrow$ γ.α. (κάτω τριγ. πίνακας)
- $B' = \{e_1, e_2, e_3, e_4\}$

ΛΥΣΗ

$$\left.\begin{matrix} f(\varepsilon_1) = (1,1,2,2) \\ f(\varepsilon_2) = (-1,0,1,2) \\ f(\varepsilon_3) = (0,0,-2,1) \\ f(\varepsilon_4) = (0,-1,-1,0) \end{matrix}\right\} \Rightarrow \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 2 & 1 & -2 & -1 \\ 2 & 2 & 1 & 0 \end{pmatrix} = (f, B', B)$$

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$\boxed{\text{Παράδειγμα}}$: $\mathbb{R}_{\leq 3}[x] = \{f(x) \in \mathbb{R}[x], \deg f \leq 3\}$

$\left.\begin{matrix} \mathbb{R}_{\leq 3}[x] \xrightarrow{F} \mathbb{R}_{\leq 3}[x] \\ f \longrightarrow 2f + f' \end{matrix}\right\}$ Να βρεθεί ο πίνακας της $F$ ως προς $B = B' = \{\underset{v_1}{1}, \underset{v_2}{x}, \underset{v_3}{x^2}, \underset{v_4}{x^3}\}$

ΛΥΣΗ

$$\left.\begin{matrix} f(v_1) = 2\cdot 1 + 1' = 2 + 0x + 0x^2 + 0x^3 \\ f(v_2) = 2\cdot x + x' = 2x + 1 = 1 + 2x + 0x^2 + 0x^3 \\ f(v_3) = 0 + 2x + 2x^2 + 0x^3 \\ f(v_4) = 0 + 0x + 3x^2 + 2x^3 \end{matrix}\right\} \Rightarrow \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix} = (F, B, B')$$

$F(a_0 + a_1x + a_2x^2 + a_3x^3) = 2(a_0 + a_1x + a_2x^2 + a_3x^3) + a_1 + 2a_2x + 3a_3x^2 =$
$= \underline{2(a_0 + a_1)} + \underline{(2a_1 + 2a_2)}x + \underline{(2a_2 + 3a_3)}x^2 + \underline{2a_3}x^3$

Συνέχεια

σελ 3

Οπότε: 24.4.23

$$\mathbb{R}_{\leq 3}[X] \xrightarrow{F} \mathbb{R}_{\leq 3}[X]$$

$$V = a_0 + a_1 X + a_2 X^2 + a_3 X^3 \longmapsto F(V)$$

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = [V]_B \in \mathbb{R}^4 \qquad \begin{pmatrix} 2(a_0 + a_1) \\ 2a_1 + 2a_2 \\ 2a_2 + 3a_3 \\ 2a_3 \end{pmatrix}$$

Επίσης, $(F, B, B') = \underbrace{\begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{pmatrix}}_{A} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}$

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$\boxed{\text{Παράδειγμα}}$: $\mathbb{R}_{\leq 2}[X] \longrightarrow \mathbb{R}_{\leq 3}[X]$

$I: f \longmapsto \int_0^x f(t)dt$ $\Big\} \Rightarrow (I, \{1, X, X^2\}, \{1, X, X^2, X^3\}) = ;$

ΛΥΣΗ:

$$\left.\begin{aligned} I(1) &= X \\ I(X) &= X^2/2 \\ I(X^2) &= X^3/3 \end{aligned}\right\} \Rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}$$

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$\boxed{\text{Παράδειγμα}}$ $V \xrightarrow{F} W$ $(V < W)$, $V = \langle e_1, e_2, e_3 \rangle$, $W = \langle e_1, e_2, e_3, e_4, e_5 \rangle$

$B = \{e_1, e_2, e_3\}$, $B' = \{e_1, e_2, e_3, e_4, e_5\}$

$$\left.\begin{aligned} f(e_1) &= e_1 \\ f(e_2) &= e_2 \\ f(e_3) &= e_3 \end{aligned}\right\} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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$\rightsquigarrow W \xrightarrow{G} V$

$$\sum_{i=1}^{5} \lambda_i e_i \longrightarrow \sum_{i=1}^{3} \lambda_i e_i \quad \left.\begin{aligned} g(e_1) &= e_1 \\ g(e_2) &= e_2 \\ g(e_3) &= e_3 \\ g(e_4) &= g(e_5) = 0 \end{aligned}\right\} \Rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

↳ Άρα $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix}$