<u>Διάλεξη 14</u> <u>Μέθοδος Laplace - Μέθοδος Στάσιμης Φάσης</u>

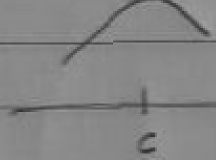
1) <u>Laplace</u>

$$I(x) = \int_a^b f(t)\, e^{x\phi(t)}\, dt \quad , \quad x \to +\infty$$

$\underset{[a,b]}{\text{Max}}\ \phi = \phi(c)$

(i) $c \in \{a, b\}$ $(\phi'(c) \neq 0)$ $\Longrightarrow \phi(t) = \phi(c) + (t-c)\phi'(c)$ — $c = a$: $\phi'(c) < 0$; $c = b$: $\phi'(c) > 0$

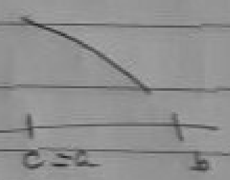
(ii) $\phi'(c) = 0, \ \phi''(c) < 0 \Longrightarrow \phi(t) = \phi(c) + \frac{1}{2}(t-c)^2 \phi''(c)$



Γενικά: $\phi'(c) = \dots = \phi^{(p-1)}(c) = 0 \ , \ \phi^{(p)}(c) \neq 0$

$$\Longrightarrow \phi(t) = \phi(c) + \frac{1}{p!}(t-c)^p \phi^{(p)}(c)$$

A. <u>Περίπτωση $f(c) \neq 0$, $c = a$</u>



$$I(x;\varepsilon) \sim \int_a^{a+\varepsilon} f(a)\, e^{x[\phi(a) + (t-a)\phi'(a)]}\, dt \quad , \quad x \to +\infty$$

$$\sim f(a)\, e^{x\phi(a)} \int_a^{\infty} e^{x(t-a)\phi'(a)}\, dt \qquad (\phi'(a) < 0)$$

$$I(x) \sim -\frac{f(a)\, e^{x\phi(a)}}{x\phi'(a)}$$

B. <u>Περίπτωση $f(c) \neq 0$, $c = b$</u>

$$I(x) \sim \frac{f(b)\, e^{x\phi(b)}}{x\phi'(b)} \quad , \quad x \to +\infty$$

i) Laplace

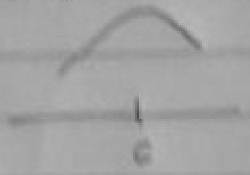
$$I(x) = \int_a^b f(t)\, e^{x\varphi(t)}\, dt \qquad , \quad x \to +\infty$$

$\underset{[a,b]}{\mathrm{Max}}\ \varphi = \varphi(c)$

(i) $c \in \{a, b\} \longrightarrow \varphi(t) = \varphi(c) + (t-c)\varphi'(c)$ — $c = a$: $\varphi'(c) < 0$; $c = b$: $\varphi'(c) > 0$

$(\varphi'(c) \neq 0)$

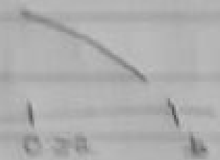
(ii) $\varphi'(c) = 0,\ \varphi''(c) < 0 \longrightarrow \varphi(t) = \varphi(c) + \frac{1}{2}(t-c)^2 \varphi''(c)$



Γενικά: $\varphi'(c) = \dots = \varphi^{(p-1)}(c) = 0 \ ,\ \varphi^{(p)}(c) \neq 0$

$$\longrightarrow \varphi(t) = \varphi(c) + \frac{1}{p!}(t-c)^p \varphi^{(p)}(c)$$

A. Περίπτωση $f(c) \neq 0$, $c = a$



$$I(x;\varepsilon) \sim \int_a^{a+\varepsilon} f(a)\, e^{x[\varphi(a) + (t-a)\varphi'(a)]}\, dt \qquad , \quad x \to +\infty$$

$$\sim f(a)\, e^{x\varphi(a)} \int_a^{\infty} e^{x(t-a)\varphi'(a)}\, dt \qquad (\varphi'(a) < 0)$$

$$I(x) \sim -\frac{f(a)\, e^{x\varphi(a)}}{x\varphi'(a)}$$

B. Περίπτωση $f(c) \neq 0$, $c = b$

$$I(x) \sim \frac{f(b)\, e^{x\varphi(b)}}{x\varphi'(b)} \qquad , \quad x \to +\infty$$

c. Περίπτωση $f(c)\neq 0$, $c\in(a,b)$

$$I(x,\varepsilon)\sim\int_{c-\varepsilon}^{c+\varepsilon} f(c)\,e^{x[\phi(c)+(t-c)^2\phi''(c)/2]}\,dt, \quad x\to+\infty$$

$$\sim f(c)\,e^{x\phi(c)}\int_{-\infty}^{\infty} e^{x(t-c)^2\phi''(c)/2}\,dt, \quad -''-$$

$$=\frac{\sqrt{2}\,f(c)\,e^{x\phi(c)}}{\sqrt{-x\phi''(c)}}\int_{-\infty}^{\infty} e^{-s^2}\,ds$$

$$s^2=-x(t-c)^2\phi''(c) \longrightarrow \int_{-\infty}^{\infty} e^{-s^2}\,ds=\Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}$$

$$I(x)\sim\frac{\sqrt{2\pi}\,f(c)\,e^{x\phi(c)}}{\sqrt{-x\phi''(c)}}, \quad x\to+\infty$$

Άσκηση 1 $(x\to\infty)$

Δείξτε ασ. συμπεριφορά

$$I(x)=\int_{-\pi/2}^{\pi/2}(t+2)\,e^{-x\cos t}\,dt \sim \frac{4}{x}$$

(Προσοχή! Max και στα δύο άκρα $\pm\frac{\pi}{2}$).

Άσκηση 2 $(x\to+\infty)$

$$\int_0^1 \sin t\, e^{-x\sinh^4 t}\,dt\sim\int_0^1 t e^{-xt^4}\,dt\sim\int_0^\infty t e^{-xt^4}\,dt=\frac{1}{4x^{1/2}}\int_0^\infty s^{-1/2}e^{-s}\,ds$$

$$=\frac{1}{4}\sqrt{\frac{\pi}{x}}$$ (max ϕ εκεί που η f μηδενίζεται!)

Άσκηση 3 $(x \to +\infty)$

$$\int_0^\infty \frac{e^{-x\cosh t}}{\sqrt{\sinh t}}\,dt \sim \int_0^\varepsilon \frac{e^{-x(1+t^2/2)}}{\sqrt{t}}\,dt$$

$$\sim e^{-x}\int_0^\infty \frac{e^{-xt^2/2}}{\sqrt{t}}\,dt$$

$$= (8x)^{-\frac{1}{4}} e^{-x}\int_0^\infty s^{-3/4} e^{-s}\,ds$$

$$= \Gamma\left(\frac{1}{4}\right)(8x)^{-1/4} e^{-x}$$

($f(t) = \infty$ εκεί που t max !)

Άσκηση 4 $(x \to +\infty)$

$$K_\nu(x) = \int_0^\infty e^{-x\cosh t}\cosh(\nu t)\,dt \quad , \ x>0$$

$\Longrightarrow$

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}}\, e^{-x} .$$

\#

<u>...σίμης φάσης</u> $(x \to +\infty)$

$$\ldots(x) = \int_a^b f(t) e^{ix\phi(t)} dt, \qquad f(t) \in \mathbb{R}$$

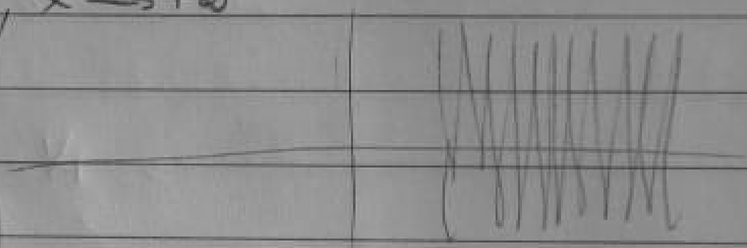
...ωτερα ολοκληρωματα Fourier.

<u>...mann - Lebesgue Λημμα</u>

Re (e^{ixt}), Im$(e^{\ldots})$

$$\int_a^b f(t) e^{ixt} dt \to 0, \quad x \to +\infty$$

$$\int_a^b |f(t)| dt < \infty.$$



<u>...κηση 5</u> $(x \to \infty)$

...ειξτε οτι $\int_a^b f(t) e^{ix\phi(t)} dt \to 0 \quad (x \to +\infty)$

...ν ισχυουν

(H1) $\int_a^b |f(t)| dt < \infty$

(H2) $\phi(t) \in C^1$, $\phi(t)$ οχι σταθερα σε $\forall$ υποδιαστημα του $[a,b]$

(Υποδειξη: Χρηση R-L)

Ασκηση 5 δειχνει οτι $\int_0^{10} t^3 e^{ix\sin^2 t} dt \to 0 \quad (x \to +\infty)$

...λλα <u>οχι</u> οτι

$$\int_0^{10} t^3 e^{2ix} dt \to 0 \qquad (\text{; ισχυει ;})$$

$\#$

$$= \int_a^b f(t) e^{ix\phi(t)} dt$$

τω $\underline{\phi'(t) \neq 0 \quad , t \in [a,b]}$

$$I(x) = \int_a^b f(t) e^{ix\phi(t)} dt = \int_a^b f(t) \frac{1}{ix\phi'(t)} d_t \left(e^{ix\phi(t)}\right)$$

$$= \frac{f(t)}{ix\phi'(t)} \Big|_a^b - \int_a^b e^{ix\phi(t)} \frac{d}{dt}\left(\frac{f(t)}{ix\phi'(t)}\right) dt$$

$$\longrightarrow \frac{f(t)}{ix\phi'(t)} \Big|_{t=a}^{t=b} \qquad (\gamma \times 5).$$

$$I(x) \sim \frac{1}{x} \quad , x \to \infty$$

Έστω ότι $\exists$ στάσιμο σημείο $c \in [a,b]$, $\phi'(c) = 0$

Ιδέα: R-L κάποιου τύπου εφαρμόζεται. Όμως $I(x)$ φθίνει πιο αργά από $\frac{1}{x}$.

(Λιγότερη αναίρεση !)

Θα δείξουμε ότι αν $\phi''(c) \neq 0$ $(f(c) \neq 0)$ $\Rightarrow I(x) \sim \frac{1}{\sqrt{x}}$

— " — $\phi''(c) = 0$, $\phi'''(c) \neq 0$ $(f(c) \neq 0)$ $\Rightarrow I(x) \sim \frac{1}{\sqrt[3]{x}}$

κ.ο.κ.

$\omega \quad \phi'(a)=0 \quad , \phi'(t)\neq 0 \;, t\in (a,b]$

$$I(x)=\int_a^{a+\varepsilon} f(t)\,e^{ix\phi(t)}\,dt+\int_{a+\varepsilon}^{b} f(t)\,e^{ix\phi(t)}\,dt = I + II$$

$$II(x)\sim \frac{\lambda}{x} \quad , \quad x\to\infty \quad , \quad a+\varepsilon\le t\le b$$

$$f(t)\sim f(a) \quad , \quad \phi(t)\sim \phi(a)+\frac{\phi^{(p)}(a)(t-a)}{p!}$$

$$I(x)\sim \int_a^{a+\varepsilon} f(a)\exp\left\{ix\left[\phi(a)+\frac{\lambda}{p!}\phi^{(p)}(a)(t-a)^p\right]\right.$$

$$\sim \int_a^{\infty} f(a) \quad \text{——} '' \quad \text{——} \quad (\,;$$

$$\sim f(a)\,e^{ix\phi(a)}\int_0^{\infty}\exp\left[\frac{ix}{p!}\phi^{(p)}(a)s^p\right]ds \,.$$