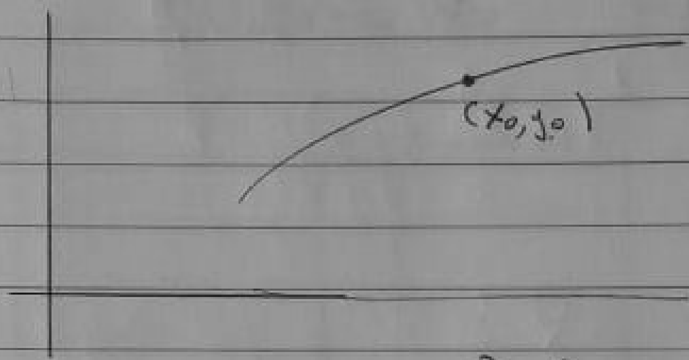
Διάλεξη 1

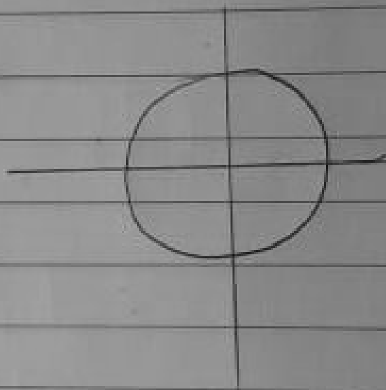
$$F(x,y)=0 \quad \begin{cases} y=y(x)=f(x) \\ x=x(y)=g(y) \end{cases}$$

$$0=\frac{d}{dx}F(x,f(x))=F_x(x,f(x))+F_y(x,f(x))f'(x) \Rightarrow f'(x)=-\frac{F_x(x,f(x))}{F_y(x,f(x))}$$



$F(x_0,y_0)=0$

$F_y(x_0,y_0)\neq 0.$



$$F(x,y)=(y-x^2)(y-x^3) \quad \begin{cases} y=x^2=f_1(x) \\ y=x^3=f_2(x) \end{cases}$$

$F(1,1)=0 \quad , \frac{\partial F}{\partial y}(1,1)=0$

Θεώρημα Πεπλεγμένης Συνάρτησης

Υπ : $F\in C^1$ σε περιοχή του (x_0,y_0)

$F(x_0,y_0)=0 \quad , F_y(x_0,y_0)=0$

Συμ

$\exists\, \delta_1>0, \delta_2>0$ τ.ω. στο $R=\{(x,y) \mid |x-x_0|\leq\delta_1, |y-y_0|\leq\delta_2\}$

(i) $\forall\, x\in(x_0-\delta_1, x_0+\delta_1) \quad \exists!\ (y_0-\delta_2, y_0+\delta_2)$

$F(x,y)=0$

$\therefore\ y=f(x)$ καλώς ορισμένη

$F(x,y)=0 \quad (x,y)\in R \Rightarrow y=f(x)$

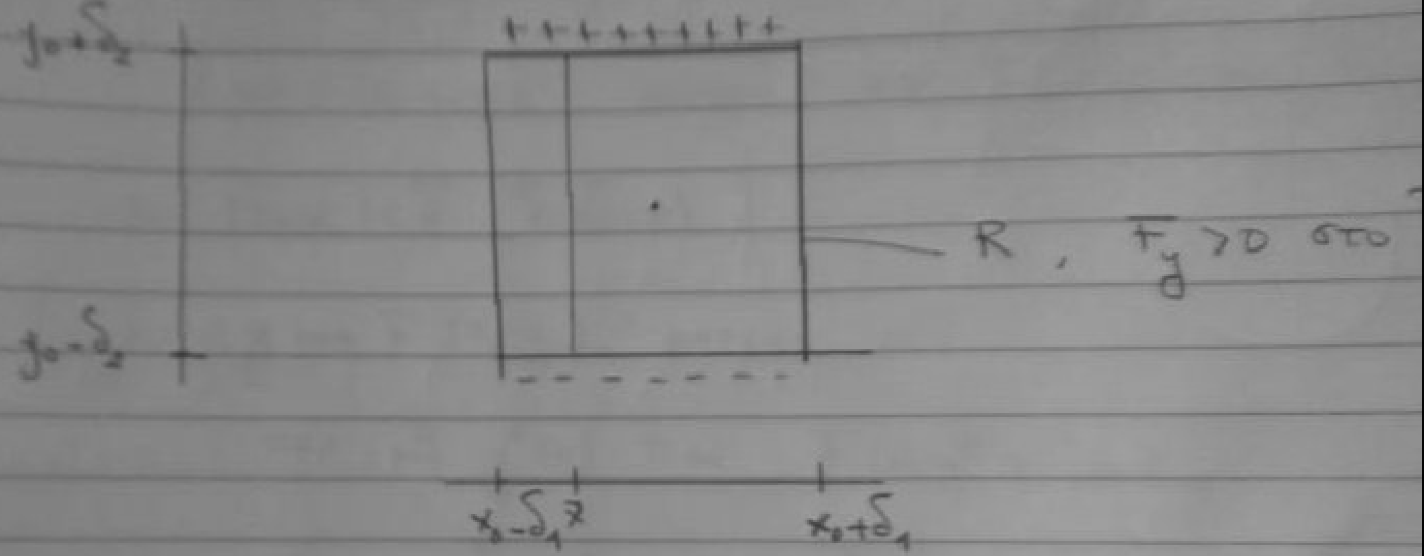
(ii) $f'(x)=-\frac{F_x(x,f(x))}{F_y(x,f(x))}$

Αν

x.p.y $\quad F_{y_0}(x_0,y_0)>0 \Rightarrow y \to F(x_0,y)$ ↑↑

$\therefore \quad F(x_0,y_0-\delta_2)<0 \quad , F(x_0,y_0+\delta_2)>0$

$\therefore \quad F(x,y_0-\delta_2)<0 \quad , F(x,y_0+\delta_2)>0 \quad , |x-x$



$\bar{x} \in (x_0-\delta_1, x_0+\delta_1)$ αυθαίρετο $\Rightarrow F(\bar{x},y_0-\delta_2)<0$, F

$\Rightarrow \exists\ \bar{y}\ , F(\bar{x},\bar{y})=0. \quad \exists! \quad (F_y(\bar{x},y)>0).$

Ορισμός f : $\quad f(\bar{x})=\bar{y}$

- f συνεχής
- f διαφορίσιμη
- $f \in C^1$

$\boxed{\text{Συνεχεια}}$

(μικρου)

Δοθεντος $\varepsilon > 0$ $\quad \exists\ \delta > 0$ τ.ω.

(1.) $F(x, y_0 + \varepsilon) > 0, \quad F(x, y_0 - \varepsilon) < 0 \quad , \quad |x - x_0| <$

(Διοτι: $F(x_0, y_0) = 0, \quad F_y(x_0, y_0) > 0 \Rightarrow F(x_0, y_0 + \varepsilon) > 0,$

$x \to F(x, y_0 \pm \varepsilon)$ συνεχης $\Rightarrow F(x, y_0 + \varepsilon) > 0,$

για $|x - x_0| < \delta \ , \ \delta = \delta(\varepsilon)$).

(2.) $y \to F(x, y)$ συνεχης

$$\Longrightarrow \exists\ y^*(x) \text{ τ.ω. } F(x, y^*(x)) = 0$$

$$y^*(x) \in (y_0 - \varepsilon, y_0 + \varepsilon)$$

Αναγκαστικα $y^*(x) = f(x)$

$\therefore$

$$|f(x) - f(x_0)| = |y^*(x) - y^*(x_0)| = |y^*(x) - y_0$$

για $|x - x_0| < \delta$. $\Longrightarrow$ ΣΥΝΕΧΕΙΑ ΑΠΕΔΕΙΧΘΗ.

Διαφορισιμότητα

Φιξάρουμε $x \in (x_0 - \delta_1, x_0 + \delta_1)$

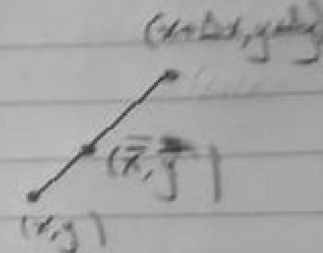
Θα δείξουμε ότι για $h \neq 0$ τ.ώ. $x+h \in (x_0-\delta_1, x_0+\delta_1)$ ισχύει

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = -\frac{F_x(x,f(x))}{F_y(x,f(x))}$$

Πράγματι

$$F(x,y)=0 = F(x+h, f(x+h)) = F(x+h, f(x) + (f(x+h)-f(x)))$$

$$\Delta y := f(x+h) - f(x) \quad , \quad \Delta x := h$$



$\therefore$

$$\begin{aligned} 0 &= F(x+\Delta x, f(x)+\Delta y) \\ &= F(x+\Delta x, f(x)+\Delta y) - F(x,y) \\ &\underset{\Theta M T}{=} F_x(\bar{x},\bar{y})\Delta x + F_y(\bar{x},\bar{y})\Delta y \end{aligned}$$

$(F \in C^1)$

$(\bar{x},\bar{y})$ στο ευθύγραμμο τμήμα

$\therefore$

$$\frac{\Delta y}{\Delta x} = -\frac{F_x(\bar{x},\bar{y})}{F_y(\bar{x},\bar{y})} \iff$$

$$\frac{f(x+h)-f(x)}{h} = -\frac{F_x(\bar{x},\bar{y})}{F_y(\bar{x},\bar{y})}$$

$$\lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\left(-\frac{F_x(\bar{x},\bar{y})}{F_y(\bar{x},\bar{y})}\right) \overset{C^1}{=} -\frac{F_x(x,y)}{F_y(x,y)} \, . \quad \square$$