The Annals of Probability
1994, Vol. 22, No. 3, 1576–1587

ON RUSSO'S APPROXIMATE ZERO-ONE LAW[1]

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Consider the product measure μ_p on $\{0,1\}^n$, when 0 (resp. 1) is given weight $1-p$ (resp. p). Consider a monotone subset A of $\{0,1\}^n$. We give a precise quantitative form to the following statement: if A does not depend much on any given coordinate, $d\mu_p(A)/dp$ is large. Thus, in that case, there is a threshold effect and $\mu_p(A)$ jumps from near 0 to near 1 in a small interval.

1. Introduction. A subset of $\{0,1\}^n$ is called monotone if

$$\forall\, x \in A,\ \forall\, y \in \{0,1\}^n,\ \forall\, i \le n, \qquad y(i) \ge x(i) \quad \Rightarrow \quad y \in A.$$

For $0 \le p \le 1$, consider on $\{0,1\}^n$ the measure μ_p, which is the product measure when 0 is given weight $1-p$ and 1 is given weight p. Thus $\mu(\{x\}) = (1-p)^{n-k}p^k$, where $k = \operatorname{card}\{i \le n;\, x_i = 1\}$. When A is monotone, $\mu_p(A)$ is an increasing function of p [see (3.9) below]. The question of understanding how $\mu_p(A)$ varies with p is of importance in percolation theory and in the theory of random graphs. For many sets of importance, there is a threshold effect, in the sense that $\mu_p(A)$ jumps from near 0 to near 1 in a small interval. It is therefore of interest to investigate general conditions under which this occurs. Several such conditions have been discovered, for example, by Margulis [3] (see also [5]). Intuitively, one would like to say that there is a threshold effect unless A is essentially determined by very few coordinates. A remarkable step has been made in this direction by Russo [4], who proved that there is a threshold effect as soon as A depends little on any given coordinate. (Since what we will mean by this is significantly simpler and weaker than Russo's definition, we will not recall his definitions.) For $x \in \{0,1\}^n$ and $i \le n$, we denote by $U_i(x)$ the point obtained from x by replacing x_i by $1 - x_i$ and leaving the other coordinates unchanged. We set

$$A_i = \{x \in \{0,1\}^n;\ x \in A, U_i(x) \notin A\}.$$

Our main result is an inequality that relates $\mu_p(A)$ and the measures of the sets A_i.

THEOREM 1.1. *There exists a universal constant K, such that, for any p and any monotone subset of $\{0,1\}^n$, we have*

$$\mu_p(A)\bigl(1-\mu_p(A)\bigr) \le K(1-p)\log\frac{2}{p(1-p)}\sum_{i\le n}\frac{\mu_p(A_i)}{\log[1/((1-p)\mu_p(A_i))]}. \tag{1.1}$$

Received September 1992.
[1]Work partially supported by the U.S.–Israel Binational Science Foundation.
AMS 1991 *subject classifications.* Primary 28A35, 60K35.
Key words and phrases. Approximate zero-one law, threshold effect, influence of variables.

COROLLARY 1.2. *Let* $\varepsilon = \sup_{i \le n} \mu_p(A_i)$. *Then*

$$\frac{d\mu_p(A)}{dp} \ge \frac{\log(1/\varepsilon)}{Kp(1-p)\log\left[2/\left(p(1-p)\right)\right]}\mu_p(A)\left(1-\mu_p(A)\right).$$

COROLLARY 1.3. *Let* $\varepsilon' = \sup_{0 \le p \le 1} \sup_{i \le n} \mu_p(A_i)$. *Then, if* $p_1 < p_2$, *we have*

$$\mu_{p_1}(A)\left(1-\mu_{p_2}(A)\right) \le (\varepsilon')^{(p_2-p_1)/K'},$$

where K' *is universal.*

COROLLARY 1.4. *We have*

$$\sup_{i \le n} \mu_p(A_i) \ge \frac{1}{K'(1-p)} U \log\frac{1}{U},$$

where K' *is universal and where* $U = \mu_p(A)(1-\mu_p(A))/n\log(2/p(1-p))$.

In the case $p = \frac{1}{2} = \mu_p(A)$, Corollary 1.4 is proved in [2] using harmonic analysis. Our approach will be an adaptation of these ideas to the present setting where one cannot use harmonic analysis. As it turns out, there is little specificity about sets in Theorem 1.1, so we will prove a more general result involving functions on $\{0,1\}^n$. For such a function f, we set $\Delta_i f(x) = (1-p)(f(x) - f(U_i(x)))$ if $x_i = 1$ and $\Delta_i f(x) = p(f(x) - f(U_i(x)))$ if $x_i = 0$.

THEOREM 1.5. *For some numerical constant K and each function f on* $\{0,1\}^n$, *such that* $\int f\, d\mu_p = 0$, *we have*

$$\|f\|_2^2 \le K \log\frac{2}{p(1-p)} \sum_{i \le n} \frac{\|\Delta_i f\|_2^2}{\log\left(e\|\Delta_i f\|_2/\|\Delta_i f\|_1\right)}. \tag{1.2}$$

In this statement, for $q = 1$ or $q = 2$, the norm $\|f\|_q$ is computed in $L_q(\mu_p)$. To deduce Theorem 1.1 from Theorem 1.5, we simply observe that if one takes $f = 1_A - \mu_p(A)$, then $\|f\|_2^2 = \mu_p(A)(1-\mu_p(A))$ and $\|\Delta_i f\|_q^q = p^{-1}\mu_p(A_i)(p(1-p)^q + (1-p)p^q)$.

Consider the function $\varphi(x) = x^2/\log(e+x)$ for $x \ge 1$. For a function f, we define

$$\|f\|_\varphi = \inf\left\{c > 0; \int \varphi\left(\frac{f}{c}\right) d\mu_p \le 1\right\}.$$

Then we have the following result.

THEOREM 1.6. *There is a universal constant K such that, for each function f on* $\{0,1\}^n$, *with* $\int f\, d\mu_p = 0$, *we have*

$$\|f\|_2^2 \le K\log\frac{2}{p(1-p)} \sum_{i \le n} \|\Delta_i f\|_\varphi^2.$$

It will be shown that Theorem 1.6 improves upon Theorem 1.5. The reason for which we give two separate statements is that Theorem 1.5 is both easier to prove and to understand and is sufficient to yield Theorem 1.1.

To conclude this section, we show that Theorem 1.1 is sharp for each p, by adapting the example of [2]. Consider first the case where $p \leq \frac{1}{2}$. Consider $k \geq 1$ and assume for simplicity that $r = p^{-k}$ is an integer. Take $n = kr$ and think of n as r blocks of length k. Consider the set A of sequences such that at least one block of coordinates consists of 1's only. Then $\mu_p(A) = (1 - p^k)^r$ is close to e^{-1}, so that the left-hand side of (1.1) is of order 1. Moreover, a simple computation shows that for each i we have $\mu_p(A_i) = p^k(1-p^k)^{r-1}$ is close to p^k/e and $n\mu_p(A_i)$ is of order k. Also, $\log(1/(1-p)\mu_p(A_i))$ is approximately $k\log(1/p)$, so that the right-hand side of (1.1) is of order 1.

In the case $p \geq \frac{1}{2}$, one can now take $r = (1-p)^{-k}$ and take for A the set of sequences such that no block consists of 0's only.

2. Preliminaries. For simplicity we shall write μ rather than μ_p. We consider the function r_i on $\{0,1\}^n$ given by

$$r_i(x) = \sqrt{\frac{1-p}{p}} \qquad \text{if } x_i = 1,$$

$$r_i(x) = -\sqrt{\frac{p}{1-p}} \qquad \text{if } x_i = 0.$$

Thus $\int r_i \, d\mu = 0$ and $\int r_i^2 \, d\mu = 1$. Consider a subset S of $\{1,\dots,n\}$. We write

$$r_S(x) = \prod_{i \in S} r_i(x)$$

(And $r_\emptyset = 1$). The functions r_S, for $S \subset \{1,\dots,n\}$, form an orthogonal basis of $L^2(\mu)$. This is a substitute for the Walsh system used in [2]. For a function $g = \Sigma a_S r_S$ on $\{0,1\}^n$, with $a_\emptyset = \int g \, d\mu = 0$, we define $M(g)$ by

$$M(g)^2 = \sum_{S \subset \{1,\dots,n\}} \frac{a_S^2}{|S|}.$$

The key to our results are suitable bounds for the quantity $M(g)$. To see how these relate to Theorem 1.5, consider a function f on $\{0,1\}^n$, with $\int f \, d\mu = 0$. Then $f = \sum_S b_S r_S$, with $b_\emptyset = 0$. The operator Δ_i has been designed so that $\Delta_i(r_S) = 0$ if $i \notin S$, and $\Delta_i(r_S) = r_S$ if $i \in S$. Thus

$$\Delta_i f = \sum_{i \in S} b_S r_S,$$

so that

$$M(\Delta_i f)^2 = \sum_{i \in S} \frac{b_S^2}{|S|}$$

and thus

$$\|f\|_2^2 = \sum_S b_S^2 = \sum_{i \le n} M(\Delta_i f)^2. \tag{2.1}$$

A crucial property of the functions r_S is as follows, where $|S|$ denotes the cardinality of S.

LEMMA 2.1. *Consider $q \ge 2$ and set $\theta = 1/\sqrt{p(1-p)}$. Then, for any k and for numbers $(a_S)_{|S|=k}$, we have*

$$\left\| \sum_{|S|=k} a_S r_S \right\|_q \le (q-1)^{k/2} \theta^k \left(\sum_{|S|=k} a_S^2 \right)^{1/2}. \tag{2.2}$$

COMMENT. In the case $p = \frac{1}{2}$, this is well known (see step 1 below) with even $\theta = 1$, and we will reduce to that case using symmetrization.

PROOF. *Step* 1. Consider $\{-1,1\}^n$, provided with the Haar (= uniform) measure λ. For $S \subset \{1,\dots,n\}$, set

$$w_S(\varepsilon) = \prod_{i \in S} \varepsilon_i,$$

so that w_S is an orthonormal basis of $L^2(\lambda)$. The key fact, proved in [1], is that, for $\delta = 1/\sqrt{q-1}$, the operator

$$T_\delta \colon \sum b_S w_S \to \sum b_S \delta^{|S|} w_S \tag{2.3}$$

is of norm 1 from $L_2(\lambda)$ to $L_q(\lambda)$. In particular, we get

$$\left\| \sum_{|S|=k} b_S w_S \right\|_{L_q(\lambda)} \le (q-1)^{k/2} \left| \sum_{|S|=k} b_S^2 \right|^{1/2}. \tag{2.4}$$

Step 2. Provide the product $G = \{0,1\}^n \times \{0,1\}^n$ with the measure $\mu' = \mu \otimes \mu$ and provide the product $H = G \times \{-1,1\}^n$ with $\nu = \mu' \otimes \lambda$. Given $S \subset \{1,\dots,n\}$, we consider for $x, y \in \{0,1\}^n$ and $\varepsilon \in \{-1,1\}^n$ the functions g_S and $g_{S,\varepsilon}$ on G given by

$$g_S(x,y) = \prod_{i \in S} \bigl(r_i(x) - r_i(y)\bigr),$$

$$g_{S,\varepsilon}(x,y) = \prod_{i \in S} \bigl(r_i(x) - r_i(y)\bigr)\varepsilon_i = g_S(x,y) w_S(\varepsilon)$$

and the function h_S on H given by $h_S(x,y,\varepsilon) = g_{S,\varepsilon}(x,y)$.

It should be clear that, given numbers (a_S), for all ε we have

$$\left\| \sum a_S g_S \right\|_{L^q(\mu')} = \left\| \sum a_S g_{S,\varepsilon} \right\|_{L^q(\mu')}.$$

Thus, by Fubini's theorem, we have

$$\left\| \sum a_S g_S \right\|_{L^q(\mu')} = \left\| \sum a_S h_S \right\|_{L^q(\nu)}. \tag{2.5}$$

Now, using (2.4) for $b_S = a_S g_S(x,y)$, we have, for all x,y,

$$\int \left| \sum_{|S|=k} a_S g_S(x,y) w_S(\varepsilon) \right|^q d\lambda(\varepsilon) \le (q-1)^{kq/2} \left(\sum_{|S|=k} a_S^2 g_S^2(x,y) \right)^{q/2}. \tag{2.6}$$

We note that

$$|r_i(x) - r_i(y)| \le \theta,$$

so that $g_S^2(x,y) \le \theta^{2k}$. Using this bound in (2.6), integrating in x,y and taking the qth root yields

$$\left\| \sum_{|S|=k} a_S h_S \right\|_{L^q(\nu)} \le \theta^k (q-1)^{k/2} \left(\sum_{|S|=k} a_S^2 \right)^{1/2}. \tag{2.7}$$

Step 3. Since $\int r_i(y)\, d\mu(y) = 0$, using Fubini's theorem and integrating in y inside rather than outside the norm yields

$$\left\| \sum_{|S|=k} a_S r_S \right\|_{L^q(\mu)} \le \left\| \sum_{|S|=k} a_S g_S \right\|_{L^q(\mu')},$$

from which the result follows by combining with (2.5) and (2.7). □

We will use this result through duality.

PROPOSITION 2.2. *Consider a function g on $\{0,1\}^n$ and set $a_S = \int r_S g \, d\mu$. Then*

$$\sum_{|S|=k} a_S^2 \le (q-1)^k \theta^{2k} \|g\|_{q'}^2, \tag{2.8}$$

where q' is the conjugate exponent of q.

PROOF. We have, by Hölder's inequality and (2.2),

$$\begin{aligned}\sum_{|S|=k} a_S^2 &= \int \Big(\sum_{|S|=k} a_S r_S \Big) g \, d\mu \\ &\le \Big\| \sum_{|S|=k} a_S r_S \Big\|_q \|g\|_{q'} \\ &\le (q-1)^{k/2} \theta^k \Big(\sum_{|S|=k} a_S^2 \Big)^{1/2} \|g\|_{q'},\end{aligned}$$

from which (2.7) follows. □

Theorem 1.5 follows by combining (2.1) and the following.

PROPOSITION 2.3. *For some universal constant K, if $\int g \, d\mu = 0$, we have*

$$M(g)^2 \le K \log \frac{2}{p(1-p)} \frac{\|g\|_2^2}{\log\big(\|g\|_2/(e\|g\|_1)\big)}. \tag{2.9}$$

PROOF. We use again the notation $a_S = \int r_S g \, d\mu$. We use Proposition 2.2 with $q = 3, q' = \frac{3}{2}$, so that

$$\sum_{|S|=k} a_S^2 \le \left(2\theta^2\right)^k \|g\|_{3/2}^2.$$

Consider now an integer $m \ge 0$. Then

$$\sum_{|S| \le m} \frac{a_S^2}{|S|} \le \sum_{k \le m} \frac{\left(2\theta^2\right)^k}{k} \|g\|_{3/2}^2.$$

If we set $x_k = (2\theta^2)^k/k$, since $\theta^2 = 1/[p(1-p)] \ge 4$, we observe that $x_{k+1}/x_k \ge 2$. Thus $\sum_{k \le m} x_k \le 2x_m$. We have $M(g)^2 \le \|g\|_2^2$, and, for $m \ge 1$, we have

$$\begin{aligned}M(g)^2 &\le \sum_{1 \le |S| \le m} \frac{a_S^2}{|S|} + \sum_{|S| > m} \frac{a_S^2}{|S|} \\ &\le 2 \frac{\left(2\theta^2\right)^m}{m} \|g\|_{3/2}^2 + \frac{1}{m+1} \sum_{|S| > m} a_S^2 \\ &\le \frac{1}{m+1} \Big(4 \left(2\theta^2\right)^m \|g\|_{3/2}^2 + \|g\|_2^2 \Big).\end{aligned} \tag{2.10}$$

We now pick for m the largest integer such that $(2\theta^2)^m \|g\|_{3/2}^2 \le \|g\|_2^2$. Thus $m \ge 0$ and $(2\theta^2)^{m+1} \|g\|_{3/2}^2 \ge \|g\|_2^2$, so that

$$m + 1 \ge \frac{2 \log\big(\|g\|_2 / \|g\|_{3/2}\big)}{\log 2\theta^2}.$$

Then, since $m+1 \geq 1$, (2.10) yields

$$M(g)^2 \leq \frac{K \log 2\theta^2}{\log(e\|g\|_2/\|g\|_{3/2})} \|g\|_2^2.$$

Since $2\theta^2 = 2/[p(1-p)]$, to conclude the proof, it suffices to show the following.

LEMMA 2.4.

$$\frac{\|g\|_2}{\|g\|_1} \leq \left(\frac{\|g\|_2}{\|g\|_{3/2}}\right)^3.$$

Indeed, this is equivalent to

$$\|g\|_{3/2}^3 \leq \|g\|_1 \|g\|_2^2$$

or, equivalently,

$$\left(\int |g|^{3/2}\, d\mu\right)^2 \leq \int |g|\, d\mu \int g^2\, d\mu,$$

which holds by the Cauchy–Schwarz inequality. □

We now turn to the study of the Orlicz norm $\| \ \|_\varphi$. The following motivates Theorem 1.6.

LEMMA 2.5. *For any function f, we have*

$$\|f\|_\varphi^2 \leq \frac{K\|f\|_2^2}{\log(e\|f\|_2/\|f\|_1)}.$$

PROOF. We can assume $f \geq 0$. By homogeneity it suffices to prove that

$$\int \frac{f^2}{\log(e+f)}\, d\mu \geq 1 \quad \Rightarrow \quad \|f\|_2^2 \geq \frac{1}{K} \log \frac{e\|f\|_2}{\|f\|_1}.$$

Consider a number $a > 0$.
Case 1.

$$\int_{\{f \geq a\}} \frac{f^2}{\log(e+f)}\, d\mu \geq \frac{1}{2}.$$

Then

$$\|f\|_2^2 \geq \log(e+a) \int_{\{f \geq a\}} \frac{f^2}{\log(e+f)}\, d\mu \geq \frac{1}{2} \log(e+a). \tag{2.11}$$

Case 2.

$$\int_{\{f \geq a\}} \frac{f^2}{\log(e+f)}\, d\mu < \frac{1}{2}.$$

Then

$$\int_{\{f < a\}} \frac{f^2}{\log(e+f)}\, d\mu \geq \frac{1}{2},$$

so that, since $f^2/\log(e+f) \leq af$ when $f \leq a$, we have $\|f\|_1 \geq 1/2a$, and thus, setting $b = \log(e\|f\|_2/\|f\|_1)$,

$$b \leq \log\left(2ea\|f\|_2\right) \leq \log a + \log\left(2e\|f\|_2\right). \tag{2.12}$$

We now choose $a = (e\|f\|_2/\|f\|_1)^{1/2}$, so that $b = 2\log a$ and, in case 2, by (2.12),

$$\log\left(2e\|f\|_2\right) \geq \log a,$$

so that

$$\|f\|_2 \geq \frac{1}{2e}\left(\frac{e\|f\|_2}{\|f\|_1}\right)^{1/2} \geq \frac{1}{2e}\left(\log\frac{e\|f\|_2}{\|f\|_1}\right)^{1/2},$$

since $x \geq \log x$ for $x \geq 0$. In case 1, by (2.11), we have $\|f\|_2^2 \geq \frac{1}{4}b$.

The proof is complete. □

The following improves upon Proposition 2.3.

PROPOSITION 2.6. *For some universal constant K, we have*

$$M(g)^2 \leq K\|g\|_\varphi^2 \log\frac{2}{p(1-p)}. \tag{2.13}$$

PROOF. By homogeneity, we can assume $\|g\|_\varphi = \frac{1}{2}$, so that

$$\int \frac{g^2}{\log(e+g)}\, d\mu \leq 1. \tag{2.14}$$

We can assume $g \geq 0$. We set $g_0 = g\mathbf{1}_{\{g<2\}}$, and, for $m \geq 1$, we set

$$g_m = g\mathbf{1}_{\{2^{2^{m-1}} \leq g < 2^{2^m}\}}.$$

To simplify the computations, we will denote by K a universal constant, not necessarily the same at each occurrence. From (2.14) it follows that

$$\sum_{m \geq 0} \frac{\|g_m\|^2}{2^m} \leq K. \tag{2.15}$$

For a function $h = \sum h_S r_S (h_\emptyset = 0)$, we define

$$M_l(h)^2 = \sum_{\substack{2^l \le k < 2^{l+1} \\ |S| = k}} \frac{h_S^2}{|S|}.$$

We observe that

$$M_l(h)^2 \le 2^{-l} \sum h_S^2 = 2^{-l} \|h\|_2^2 \tag{2.16}$$

and

$$M(g)^2 = \sum_{l \ge 0} M_l(g)^2.$$

Clearly, M_l is a seminorm. Thus

$$M_l(g) \le \sum_{m \ge 0} M_l(g_m). \tag{2.17}$$

As in the proof of Proposition 2.3, it follows from Proposition 2.2 that

$$M_l(g_m)^2 \le \left(2\theta^2\right)^{2^{l+1}} \|g_m\|_{3/2}^2. \tag{2.18}$$

Since $g_m \ge 2^{2^m - 1}$ when $g_m \ne 0$, we have, using (2.15),

$$\int g_m^{3/2} d\mu \le \left(2^{2^{m-1}}\right)^{-1/2} \int g_m^2 \, d\mu$$
$$\le K 2^{m - 2^{m-2}} \le K 2^{-2^{m-3}}.$$

Thus $\|g_m\|_{3/2}^2 \le K 2^{-2^{m-3}}$, so that, by (2.17),

$$M_l(g_m)^2 \le K\left(2\theta^2\right)^{2^{l+1}} 2^{-2^{m-3}}.$$

Denote by $m(l)$ the smallest integer m such that $2^{l+1} \log(2\theta^2) \le 2^{m-4}$. Thus $M_l(g_m)^2 \le K 2^{-2^{m-4}}$ for $m \ge m(l)$, so that $M_l(g_m) \le K 2^{-2^{m-5}}$. From (2.16) and (2.18) we have

$$\begin{aligned} M_l(g) &\le \sum_{m < m(l)} M_l(g_m) + \sum_{m \ge m(l)} K 2^{-2^{m-5}} \\ &\le \sum_{m < m(l)} M_l(g_m) + K 2^{-2^{m(l)-5}} \\ &\le 2^{-l/2} \sum_{m < m(l)} \|g_m\|_2 + K 2^{-2^{m(l)-5}}. \end{aligned} \tag{2.19}$$

To simplify the notation, we observe that $m(l) = l + s$, where s is the smallest such that $2^{s-5} \geq \log(2\theta^2)$ [so that $2^s \leq K \log(2\theta^2)$]. Also, we observe that by convexity of the function x^2, we have

$$\Big(\sum \alpha_i x_i\Big)^2 \leq \sum \alpha_i x_i^2,$$

whenever $\Sigma \alpha_i = 1, \alpha_i \geq 0$. Thus

$$\Big(\sum x_i\Big)^2 \leq \sum \frac{x_i^2}{\alpha_i}$$

and thus

$$\left(\sum_{i \geq 0} x_i\right)^2 \leq K \sum_{i \geq 0} 2^{i/2} x_i^2.$$

Using this, as well as the inequality $(A+B)^2 \leq 2A^2 + 2B^2$, we deduce from (2.19) that

$$M_l(g)^2 \leq K 2^{-l} \left(\sum_{m < l+s} \|g_m\|_2^2 2^{(l+s-m)/2} \right) + K 2^{-2^{l+s-4}}.$$

Thus, since $2^s \geq 1$,

$$\begin{aligned}
\sum_{l \geq 0} M_l(g)^2 &\leq K 2^{s/2} \sum_{\substack{m < l+s \\ l \geq 0}} \|g_m\|_2^2 2^{-(l+m)/2} + K \\
&\leq K + K 2^{s/2} \sum_m 2^{-m/2} \|g_m\|_2^2 \left(\sum_{l > m-s} 2^{-l/2} \right) \\
&\leq K + K 2^s \sum_m 2^{-m} \|g_m\|_2^2.
\end{aligned}$$

This completes the proof by (2.14). □

REMARK. In the case $p = \frac{1}{2}$ the inequality (2.13) can be obtained by duality from an inequality of L. Gross, that itself follows from (2.3). In this approach it is essential that one have the correct value for δ in (2.3). Unfortunately, this value is no longer valid in the natural adaptation of (2.3) to the case $p \neq \frac{1}{2}$. For this reason, this creates complications in using this approach when $p \neq \frac{1}{2}$, and we have chosen a simpler route.

3. End of proofs.

PROOF OF COROLLARY 1.2. This is an immediate consequence of Theorem 1.1 and of the formula

$$\frac{d\mu_p(A)}{dp} = \frac{1}{p}\sum_{i\le n}\mu_p(A_i). \tag{3.1}$$

This formula is often called Russo's formula, though it was proved earlier by Margulis [3]. □

PROOF OF COROLLARY 1.3. We observe that Corollary 1.2 means that if we set $g(x)=\log(x/(1-x))$, we have

$$\frac{d}{dp}\Big(g\big(\mu_p(A)\big)\Big) \ge \frac{\log(1/\varepsilon)}{Kp(1-p)\log\big[1/\big(p(1-p)\big)\big]} \ge \frac{1}{K'}\log\frac{1}{\varepsilon},$$

where K' is universal, since $p(1-p)\le\frac{1}{4}$. Thus, setting $x_1=\mu_{p_1}(A), x_2=\mu_{p_2}(A)$, we have

$$g(x_2)-g(x_1)\ge\frac{p_2-p_1}{K'}\log\frac{1}{\varepsilon}.$$

Thus

$$\begin{aligned} x_1(1-x_2) &\le \frac{x_1}{1-x_1}\frac{1-x_2}{x_2} \le \exp\big(g(x_1)-g(x_2)\big) \\ &\le \exp\left(-\frac{(p_2-p_1)}{K'}\log\frac{1}{\varepsilon}\right). \end{aligned}$$

□

PROOF OF COROLLARY 1.4. We simply observe that if we set $\varepsilon=\sup_{i\le n}\mu_p(A_i)$, by Theorem 1.1 and since the function $x/\log(1/x)$ increases for $x<1$, we have

$$\mu_p(A)\big(1-\mu_p(A)\big)\le K\left(\log\frac{2}{p(1-p)}\right)n\frac{(1-p)\varepsilon}{\log\big[1/((1-p)\varepsilon)\big]}.$$

Also, we observe that, if $x\le\frac{1}{2}$,

$$\frac{y}{\log(1/y)}\ge x \quad\Rightarrow\quad y\ge\frac{x}{K}\log\frac{1}{x},$$

and this implies the result. □

Acknowledgment. I am indebted to N. Alon for pointing out the paper by Kahn, Kalai and Linial [2].

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