Παράδειγμα (Μηχανή με 2 καταστάσεις

ή

$M/M/1/1$ ουρά)

λ μ

Μία μηχανή μπορεί να λειτουργεί ή να είναι χαλασμένη. Αν λειτουργεί, θα χαλάσει μετά από χρόνο $Exp(\mu)$. Ο χρόνος επιδιόρθωσης είναι $Exp(\lambda)$. Οι χρόνοι λειτουργίας & επιδιόρθωσης είναι ανεξάρτητοι.

$$X(t) = \begin{cases} 0, & \text{αν δεν λειτουργεί τη στιγμή } t \\ 1, & \text{αν λειτουργεί } \text{» » »} \end{cases}$$

(i) Να βρεθεί ο Q

(ii) Να βρεθεί ο $P(t)$.

Λύση

(i) $S=\{0,1\}$

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \end{tikzcd}$$

$$Q = \begin{array}{c c} & \begin{array}{cc} 0 & 1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix} \end{array}$$

(ii) $P'(t) = P(t)Q \Rightarrow$

$$\begin{bmatrix} p_{00}'(t) & p_{01}'(t) \\ p_{10}'(t) & p_{11}'(t) \end{bmatrix} = \begin{bmatrix} p_{00}(t) & p_{01}(t) \\ p_{10}(t) & p_{11}(t) \end{bmatrix} \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix}$$

$$p_{00}'(t) = -\lambda p_{00}(t) + \mu p_{01}(t) \quad (1)$$

$$p_{01}'(t) = \lambda p_{00}(t) - \mu p_{01}(t) \quad (2)$$

$$p_{10}'(t) = -\lambda p_{10}(t) + \mu p_{11}(t) \quad (3)$$

$$p_{11}'(t) = \lambda p_{10}(t) - \mu p_{11}(t) \quad (4)$$

$$p_{00}(0) = 1 \quad (7)$$

$$p_{01}(0) = 0 \quad (8)$$

$$p_{10}(0) = 0 \quad (9)$$

$$p_{11}(0) = 1 \quad (10)$$

$$p_{10}(t) + p_{11}(t) = 1 \quad (5)$$

$$p_{00}(t) + p_{01}(t) = 1 \quad (6)$$

$$(1) \overset{(6)}{\Longrightarrow} p_{00}'(t) = -\lambda p_{00}(t) + \mu\left(1 - p_{00}(t)\right) \Rightarrow$$

$$p_{00}'(t) = -(\lambda+\mu) p_{00}(t) + \mu \Rightarrow$$

$$p_{00}'(t) + (\lambda+\mu) p_{00}(t) = \mu \Rightarrow$$

$$e^{(\lambda+\mu)t} p_{00}'(t) + (\lambda+\mu) e^{(\lambda+\mu)t} p_{00}(t) = \mu e^{(\lambda+\mu)t} \Rightarrow$$

$$\left[e^{(\lambda+\mu)t} p_{00}(t)\right]' = \mu e^{(\lambda+\mu)t} \Rightarrow$$

$$e^{(\lambda+\mu)t} p_{00}(t) = \frac{\mu}{\lambda+\mu} e^{(\lambda+\mu)t} + c \Rightarrow$$

$$p_{00}(t) = \frac{\mu}{\lambda+\mu} + c\, e^{-(\lambda+\mu)t} \quad (11)$$

$$(7): \quad p_{00}(0) = 1 \overset{(11)}{\Longrightarrow} \frac{\mu}{\lambda+\mu} + c = 1 \Rightarrow c = 1 - \frac{\mu}{\lambda+\mu} \Rightarrow$$

$$c = \frac{\lambda}{\lambda+\mu}$$

Άρα $$\boxed{p_{00}(t) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}}$$

(6) $\Rightarrow$ $p_{01}(t) = 1 - p_{00}(t) \Rightarrow$

$$p_{01}(t) = 1 - \frac{\mu}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} \Rightarrow$$

$$\boxed{p_{01}(t) = \frac{\lambda}{\lambda+\mu} - \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t}}$$

Από τις (3), (5) και (9) προκύπτει ότι

$$\boxed{p_{10}(t) = \frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}}$$

και $$\boxed{p_{11}(t) = \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t}}$$

Οπότε

$$P(t) = \begin{bmatrix} \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu} & \frac{\lambda}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)t}\right) \\ \frac{\mu}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)t}\right) & \frac{\lambda}{\lambda+\mu} + \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t} \end{bmatrix}$$