Chapter 2

Newton's Equation and Kepler's Law

We develop in this chapter the earliest important examples of differential equations, which in fact are connected with the origins of calculus. These equations were used by Newton to derive and unify the three laws of Kepler. These laws were found from the earlier astronomical observations of Tycho Brahe. Here we give a brief derivation of two of Kepler's laws, while at the same time setting forth some general ideas about differential equations.

The equations of Newton, our starting point, have retained importance throughout the history of modern physics and lie at the root of that part of physics called classical mechanics.

The first chapter of this book dealt with linear equations, but Newton's equations are nonlinear in general. In later chapters we shall pursue the subject of nonlinear differential equations somewhat systematically. The examples here provide us with concrete examples of historical and scientific importance. Furthermore, the case we consider most thoroughly here, that of a particle moving in a central force gravitational field, is simple enough so that the differential equations can be solved explicitly using exact, classical methods (just calculus!). This is due to the existence of certain invariant functions called *integrals* (sometimes called "first integrals"; we do not mean the integrals of elementary calculus). Physically, an integral is a conservation law; in the case of Newtonian mechanics the two integrals we find correspond to conservation of energy and angular momentum. Mathematically an integral reduces the number of dimensions.

We shall be working with a particle moving in a *field of force* F. Mathematically F is a *vector field* on the (configuration) space of the particle, which in our case we suppose to be Cartesian three space $\mathbf{R}^3$. Thus F is a map $F: \mathbf{R}^3 \to \mathbf{R}^3$ that assigns to a point x in $\mathbf{R}^3$ another point $F(x)$ in $\mathbf{R}^3$. From the mathematical point of view, $F(x)$ is thought of as a vector based at x. From the physical point of view, $F(x)$ is the force exerted on a particle located at x.

The example of a force field we shall be most concerned with is the gravitational field of the sun: $F(x)$ is the force on a particle located at x attracting it to the sun.

We shall go into details of this field in Section 6. Other important examples of force fields are derived from electrical forces, magnetic forces, and so on.

The connection between the physical concept of force field and the mathematical concept of differential equation is *Newton's second law*: $F = ma$. This law asserts that a particle in a force field moves in such a way that the force vector at the location of the particle, at any instant, equals the acceleration vector of the particle times the mass m. If $x(t)$ denotes the position vector of the particle at time t, where $x: \mathbf{R} \to \mathbf{R}^3$ is a sufficiently differentiable curve, then the acceleration vector is the second derivative of $x(t)$ with respect to time

$$a(t) = \ddot{x}(t).$$

(We follow tradition and use dots for time derivatives in this chapter.) Newton's second law states

$$F(x(t)) = m\ddot{x}(t).$$

Thus we obtain a second order differential equation:

$$\ddot{x} = \frac{1}{m} F(x).$$

In Newtonian physics it is assumed that m is a positive constant. Newton's law of gravitation is used to derive the exact form of the function $F(x)$. While these equations are the main goal of this chapter, we first discuss simple harmonic motion and then basic background material.

§1. Harmonic Oscillators

We consider a particle of mass m moving in one dimension, its position at time t given by a function $t \to x(t)$, $x: \mathbf{R} \to \mathbf{R}$. Suppose the force on the particle at a point $x \in \mathbf{R}$ is given by $-mp^2x$, where p is some real constant. Then according to the laws of physics (compare Section 3) the motion of the particle satisfies

$$\ddot{x} + p^2 x = 0. \tag{1}$$

This model is called the harmonic oscillator and (1) is the equation of the harmonic oscillator (in one dimension).

An example of the harmonic oscillator is the simple pendulum moving in a plane, when one makes an approximation of $\sin x$ by x (compare Chapter 9). Another example is the case where the force on the particle is caused by a spring.

It is easy to check that for any constants A, B, the function

$$x(t) = A \cos pt + B \sin pt \tag{2}$$

is a solution of (1), with initial conditions $x(0) = A$, $\dot{x}(0) = pB$. In fact, as is proved often in calculus courses, (2) is the only solution of (1) satisfying these initial conditions. Later we will show in a systematic way that these facts are true.

Using basic trigonometric identities, (2) may be rewritten in the form

$$x(t) = a \cos (pt + t_0), \tag{3}$$

where $a = (A^2 + B^2)^{1/2}$ is called the amplitude, and $\cos t_0 = A(A^2 + B^2)^{-1/2}$.

In Section 6 we will consider equation (1) where a constant term is added (representing a constant disturbing force):

$$\ddot{x} + p^2 x = K. \tag{4}$$

Then, similarly to (1), every solution of (4) has the form

$$x(t) = a \cos (pt + t_0) + \frac{K}{p^2}. \tag{5}$$

The two-dimensional version of the harmonic oscillator concerns a map $x: \mathbf{R} \to \mathbf{R}^2$ and a force $F(x) = -mkx$ (where now, of course, $x = (x_1, x_2) \in \mathbf{R}^2$). Equation (1) now has the same form

$$\ddot{x} + k^2 x = 0 \tag{1'}$$

with solutions given by

$$\begin{aligned} x_1(t) &= A \cos kt + B \sin kt, \\ x_2(t) &= C \cos kt + D \sin kt. \end{aligned} \tag{2'}$$

See Problem 1.

Planar motion will be considered more generally and in more detail in later sections. But first we go over some mathematical preliminaries.

§2. Some Calculus Background

A path of a moving particle in $\mathbf{R}^n$ (usually $n \leq 3$) is given by a map $f: I \to \mathbf{R}^n$ where I might be the set $\mathbf{R}$ of all real numbers or an interval (a, b) of all real numbers strictly between a and b. The derivative of f (provided f is differentiable at each point of I) defines a map $f': I \to \mathbf{R}^n$. The map f is called C^1, or *continuously differentiable*, if f' is continuous (that is to say, the corresponding coordinate functions $f_i'(t)$ are continuous, $i = 1, \ldots, n$). If $f': I \to \mathbf{R}^n$ is itself C^1, then f is said to be C^2. Inductively, in this way, one defines a map $f: I \to \mathbf{R}^n$ to be C^r, where $r = 3, 4, 5$, and so on.

The *inner product*, or "dot product," of two vectors, x, y in $\mathbf{R}^n$ is denoted by $\langle x, y \rangle$ and defined by

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i.$$

Thus $\langle x, x\rangle = |x|^2$. If $x, y: I \to \mathbf{R}^n$ are C^1 functions, then a version of the Leibniz product rule for derivatives is

$$\langle x, y\rangle' = \langle x', y\rangle + \langle x, y'\rangle,$$

as can be easily checked using coordinate functions.

We will have occasion to consider functions $f: \mathbf{R}^n \to \mathbf{R}$ (which, for example, could be given by temperature or density). Such a map f is called C^1 if the map $\mathbf{R}^n \to \mathbf{R}$ given by each partial derivative $x \to \partial f/\partial x_i(x)$ is defined and continuous (in Chapter 5 we discuss continuity in more detail). In this case the gradient of f, called *grad* f, is the map $\mathbf{R}^n \to \mathbf{R}^n$ that sends x into $(\partial f/\partial x_1(x), \ldots, \partial f/\partial x_n(x))$. Grad f is an example of a vector field on $\mathbf{R}^n$. (In Chapter 1 we considered only linear vector fields, but grad f may be more general.)

Next, consider the composition of two C^1 maps as follows:

$$I \xrightarrow{f} \mathbf{R}^n \xrightarrow{g} \mathbf{R}.$$

The chain rule can be expressed in this context as

$$\frac{d}{dt} g(f(t)) = \langle \text{grad}\, g(f(t)), f'(t)\rangle;$$

using the definitions of gradient and inner product, the reader can prove that this is equivalent to

$$\sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(f(t)) \frac{df_i}{dt}(t).$$

§3. Conservative Force Fields

A vector field $F: \mathbf{R}^3 \to \mathbf{R}^3$ is called a force field if the vector $F(x)$ assigned to the point x is interpreted as a force acting on a particle placed at x.

Many force fields appearing in physics arise in the following way. There is a C^1 function

$$V: \mathbf{R}^3 \to \mathbf{R}$$

such that

$$F(x) = -\left(\frac{\partial V}{\partial x_1}(x), \frac{\partial V}{\partial x_2}(x), \frac{\partial V}{\partial x_3}(x)\right)$$
$$= -\text{grad}\, V(x).$$

(The negative sign is traditional.) Such a force field is called *conservative*. The function V is called the *potential energy* function. (More properly V should be called *a* potential energy since adding a constant to it does not change the force field $-\text{grad}\, V(x)$.) Problem 4 relates potential energy to *work*.

The planar harmonic oscillation of Section 1 corresponds to the force field

$$F\colon \mathbf{R}^2 \to \mathbf{R}^2, \qquad F(x) = -mkx.$$

This field is conservative, with potential energy

$$V(x) = \tfrac{1}{2}mk\,|x|^2$$

as is easily verified.

For any moving particle $x(t)$ of mass m, the *kinetic energy* is defined to be

$$T = \tfrac{1}{2}m\,|\dot{x}(t)|^2.$$

Here $\dot{x}(t)$ is interpreted as the *velocity vector* at time t; its length $|\dot{x}(t)|$ is the *speed* at time t. If we consider the function $x\colon \mathbf{R} \to \mathbf{R}^3$ as describing a curve in $\mathbf{R}^3$, then $\dot{x}(t)$ is the *tangent vector* to the curve at $x(t)$.

For a particle moving in a conservative force field $F = -\operatorname{grad} V$, the potential energy at x is defined to be $V(x)$. Note that whereas the kinetic energy depends on the velocity, the potential energy is a function of position.

The *total energy* (or sometimes simply *energy*) is

$$E = T + V.$$

This has the following meaning. If $x(t)$ is the trajectory of a particle moving in the conservative force field, then E is a real-valued function of time:

$$E(t) = \tfrac{1}{2}\,|m\dot{x}(t)|^2 + V(x(t)).$$

Theorem (Conservation of Energy) *Let $x(t)$ be the trajectory of a particle moving in a conservative force field $F = -\operatorname{grad} V$. Then the total energy E is independent of time.*

Proof. It needs to be shown that $E(x(t))$ is constant in t or that

$$\frac{d}{dt}(T + V) = 0,$$

or equivalently,

$$\frac{d}{dt}\left(\frac{1}{2}m\,|\dot{x}(t)|^2 + V(x(t))\right) = 0.$$

It follows from calculus that

$$\frac{d}{dt}|\dot{x}|^2 = 2\langle \ddot{x}, \dot{x}\rangle$$

(a version of the Leibniz product formula); and also that

$$\frac{d}{dt}(V(\dot{x})) = \langle \operatorname{grad} V(x), \dot{x}\rangle$$

(the chain rule).

These facts reduce the proof to showing that

$$m\langle \ddot{x}, \dot{x}\rangle + \langle \text{grad } V, \dot{x}\rangle = 0$$

or $\langle m\ddot{x} + \text{grad } V, \dot{x}\rangle = 0$. But this is so since Newton's second law is $m\ddot{x} + \text{grad } V(x) = 0$ in this instance.

§4. Central Force Fields

A force field F is called *central* if $F(x)$ points in the direction of the line through x, for every x. In other words, the vector $F(x)$ is always a scalar multiple of x, the coefficient depending on x:

$$F(x) = \lambda(x)x.$$

We often tacitly exclude from consideration a particle at the origin; many central force fields are not defined (or are "infinite") at the origin.

Lemma *Let F be a conservative force field. Then the following statements are equivalent*:

(a) *F is central,*
(b) $F(x) = f(|x|)x$,
(c) $F(x) = -\text{grad } V(x)$ *and* $V(x) = g(|x|)$.

Proof. Suppose (c) is true. To prove (b) we find, from the chain rule:

$$\frac{\partial V}{\partial x_j} = g'(|x|)\frac{\partial}{\partial x_j}(x_1^2 + x_2^2 + x_3^2)^{1/2}$$

$$= \frac{g'(|x|)}{|x|}x_j;$$

this proves (b) with $f(|x|) = g'(|x|)/|x|$. It is clear that (b) implies (a). To show that (a) implies (c) we must prove that V is constant on each sphere.

$$S_\alpha = \{x \in \mathbf{R}^3 \mid |x| = \alpha\}, \qquad \alpha > 0.$$

Since any two points in S_α can be connected by a curve in S_α, it suffices to show that V is constant on any curve in S_α. Hence if $J \subset \mathbf{R}$ is an interval and $u: J \to S_\alpha$ is a C^1 map, we must show that the derivative of the composition $V \circ u$

$$J \xrightarrow{u} S_\alpha \subset \mathbf{R}^3 \xrightarrow{V} \mathbf{R}$$

is identically 0. This derivative is

$$\frac{d}{dt}V(u(t)) = \langle \text{grad } V(u(t)), u'(t)\rangle$$

as in Section 2. Now grad $V(x) = -F(x) = -\lambda(x)x$ since F is central:

$$\frac{d}{dt} V(u(t)) = -\lambda(u(t))\langle u(t), u'(t)\rangle$$

$$= \frac{-\lambda u(t))}{2} \frac{d}{dt} | u(t) |^2$$

$$= 0$$

because $| u(t) | \equiv \alpha$.

In Section 5 we shall consider a special conservative central force field obtained from Newton's law of gravitation.

Consider now a central force field, not necessarily conservative.

Suppose at some time t_0, that $P \subset \mathbf{R}^3$ denotes the plane containing the particle, the velocity vector of the particle and the origin. The force vector $F(x)$ for any point x in P also lies in P. This makes it plausible that the particle stays in the plane P for all time. In fact, this is true: a particle moving in a central force field moves in a fixed plane.

The proof depends on the *cross product* (or vector product) $u \times v$ of vectors u, v in $\mathbf{R}^3$. We recall the definition

$$u \times v = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1) \in \mathbf{R}^3$$

and that $u \times v = -v \times u = | u | | v | N \sin\theta$, where N is a unit vector perpendicular to u and v, (U, v, N) oriented as the axes ("right-hand rule"), and θ is the angle between u and v.

Then the vector $u \times v = 0$ if and only if one vector is a scalar multiple of the other; if $u \times v \neq 0$, then $u \times v$ is orthogonal to the plane containing u and v. If u and v are functions of t in $\mathbf{R}$, then a version of the Leibniz product rule asserts (as one can check using Cartesian coordinates):

$$\frac{d}{dt}(u \times v) = \dot{u} \times v + u \times \dot{v}.$$

Now let $x(t)$ be the path of a particle moving under the influence of a central force field. We have

$$\frac{d}{dt}(x \times \dot{x}) = \dot{x} \times \dot{x} + x \times \ddot{x}$$

$$= x \times \ddot{x}$$

$$= 0$$

because $\ddot{x}$ is a scalar multiple of x. Therefore $x(t) \times \dot{x}(t)$ is a constant vector y. If $y \neq 0$, this means that x and $\dot{x}$ always lie in the plane orthogonal to y, as asserted. If $y = 0$, then $\dot{x}(t) = g(t)x(t)$ for some scalar function $g(t)$. This means that the velocity vector of the moving particle is always directed along the line through the origin and the particle, as is the force on the particle. This makes it plausible that the particle always moves along the same line through the origin. To prove this let $(x_1(t), x_2(t), x_3(t))$ be the coordinates of $x(t)$. Then we have three differential equations

$$\frac{dx_k}{dt} = g(t)x_k(t), \qquad k = 1, 2, 3.$$

By integration we find

$$x_k(t) = e^{h(t)}x_k(t_0), \qquad h(t) = \int_{t_0}^{t} g(s)\, ds.$$

Therefore $x(t)$ is always a scalar multiple of $x(t_0)$ and so $x(t)$ moves in a fixed line, and hence in a fixed plane, as asserted.

We restrict attention to a conservative central force field in a plane, which we take to be the Cartesian plane $\mathbf{R}^2$. Thus x now denotes a point of $\mathbf{R}^2$, the potential energy V is defined on $\mathbf{R}^2$ and

$$F(x) = -\operatorname{grad} V(x) = -\left(\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}\right).$$

Introduce polar coordinates (r, θ), with $r = |x|$.

Define the *angular momentum* of the particle to be

$$h = mr^2\dot{\theta},$$

where $\dot{\theta}$ is the time derivative of the angular coordinate of the particle.

Theorem (Conservation of Angular Momentum) *For a particle moving in a central force field*:

$$\frac{dh}{dt} = 0, \qquad \textit{where} \quad h = mr^2\dot{\theta}.$$

Proof. Let $i = i(t)$ be the unit vector in the direction $x(t)$ so $x = ri$. Let $j = j(t)$ be the unit vector with a 90° angle from i to j. A computation shows that $di/dt = \dot{\theta}j$, $dj/dt = -\dot{\theta}i$ and hence

$$\dot{x} = \dot{r}i + r\dot{\theta}j.$$

Differentiating again yields

$$\ddot{x} = (\ddot{r} - r\dot{\theta}^2)i + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})j.$$

If the force is central, however, it has zero component perpendicular to x. Therefore, since $\ddot{x} = m^{-1}F(x)$, the component of $\ddot{x}$ along j must be 0. Hence

$$\frac{d}{dt}(r^2\dot{\theta}) = 0,$$

proving the theorem.

We can now prove one of Kepler's laws. Let $A(t)$ denote the area swept out by the vector $x(t)$ in the time from t_0 to t. In polar coordinates $dA = \frac{1}{2}r^2\,d\theta$. We define the *areal velocity* to be

$$\dot{A} = \tfrac{1}{2}r^2\dot{\theta},$$

the rate at which the position vector sweeps out area. Kepler observed that the line segment joining a planet to the sun sweeps out equal areas in equal times, which we interpret to mean $\dot{A}$ = constant. We have proved more generally that this is true for any particle moving in a conservative central force field; this is a consequence of conservation of angular momentum.

§5. States

We recast the Newtonian formulation of the preceding sections in such a way that the differential equation becomes first order, the states of the system are made explicit, and energy becomes a function on the space of states.

A *state* of a physical system is information characterizing it at a given time. In particular, a state of the physical system of Section 1 is the position and velocity of the particle. The space of states is the Cartesian product $\mathbf{R}^3 \times \mathbf{R}^3$ of pairs (x, v), x, v in $\mathbf{R}^3$; x is the position, v the velocity that a particle might have at a given moment.

We may rewrite Newton's equation

$$m\ddot{x} = F(x) \tag{1}$$

as a first order equation in terms of x and v. (The *order* of a differential equation is the order of the highest derivative that occurs explicitly in the equation.) Consider the differential equation

$$\frac{dx}{dt} = v, \qquad m\frac{dv}{dt} = F(x). \tag{1'}$$

A solution to (1′) is a curve $t \to (x(t), v(t))$ in the state space $\mathbf{R}^3 \times \mathbf{R}^3$ such that

$$\dot{x}(t) = v(t) \quad \text{and} \quad \dot{v}(t) = m^{-1}F(x(t)) \quad \text{for all } t.$$

It can be seen then that the solutions of (1) and (1′) correspond in a natural fashion. Thus if $x(t)$ is a solution of (1), we obtain a solution of (1′) by setting $v(t) = \dot{x}(t)$. *The map* $\mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3 \times \mathbf{R}^3$ *that sends* (x, v) *into* $(v, m^{-1}F(x))$ *is a vector field on the space of states, and this vector field defines the differential equation,* (1′).

A solution $(x(t), v(t))$ to (1′) gives the passage of the state of the system in time.

Now we may interpret energy as a function on the state space, $\mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}$, defined by $E(x, v) = \frac{1}{2}m\,|v|^2 + V(x)$. The statement that "the energy is an integral" then means that the composite function

$$t \to (x(t), v(t)) \to E(x(t), v(t))$$

is constant, or that *on a solution curve in the state space, E is constant.*

We abbreviate $\mathbf{R}^3 \times \mathbf{R}^3$ by $\mathbf{S}$. An *integral* (for (1′)) on $\mathbf{S}$ is then any function that is constant on every solution curve of (1′). It was shown in Section 4 that in addition to energy, angular momentum is also an integral for (1′). In the nineteenth century, the idea of solving a differential equation was tied to the construction of a sufficient number of integrals. However, it is realized now that integrals do not exist for differential equations very generally; the problems of differential equations have been considerably freed from the need for integrals.

Finally, we observe that the force field may not be defined on all of $\mathbf{R}^3$, but only on some portion of it, for example, on an open subset $U \subset \mathbf{R}^3$. In this case the path $x(t)$ of the particle is assumed to lie in U. The force and velocity vectors, however, are still allowed to be arbitrary vectors in $\mathbf{R}^3$. The force field is then a vector field on U, denoted by $F: U \to \mathbf{R}^3$. The state space is the Cartesian product $U \times \mathbf{R}^3$, and (1′) is a first order equation on $U \times \mathbf{R}^3$.

§6. Elliptical Planetary Orbits

We now pass to consideration of Kepler's first law, that planets have elliptical orbits. For this, a central force is not sufficient. We need the precise form of V as given by the "inverse square law."

We shall show that in polar coordinates (r, θ), an orbit with nonzero angular momentum h is the set of points satisfying

$$r(1 + \epsilon \cos\theta) = l = \text{constant}; \qquad \epsilon = \text{constant},$$

which defines a conic, as can be seen by putting $r\cos\theta = x$, $r^2 = x^2 + y^2$.

Astronomical observations have shown the orbits of planets to be (approximately) ellipses.

Newton's law of gravitation states that a body of mass m_1 exerts a force on a body of mass m_2. The magnitude of the force is gm_1m_2/r^2, where r is the distance between their centers of gravity and g is a constant. The direction of the force on m_2 is from m_2 to m_1.

Thus if m_1 lies at the origin of $\mathbf{R}^3$ and m_2 lies at $x \in \mathbf{R}^3$, the force on m_2 is

$$-gm_1m_2 \frac{x}{|x|^3}.$$

The force on m_1 is the negative of this.

We must now face the fact that *both* bodies will move. However, if m_1 is much greater than m_2, its motion will be much less since acceleration is inversely proportional to mass. We therefore make the simplifying assumption that one of the bodies does not move; in the case of planetary motion, of course it is the sun that is assumed at rest. (One might also proceed by taking the center of mass at the origin, without making this simplifying assumption.)

We place the sun at the origin of $\mathbf{R}^3$ and consider the force field corresponding to a planet of given mass m. This field is then

$$F(x) = -C\frac{x}{|x|^3},$$

where C is a constant. We then change the units in which force is measured to obtain the simpler formula

$$F(x) = -\frac{x}{|x|^3}.$$

It is clear this force field is central. Moreover, it is conservative, since

$$\frac{x}{|x|^3} = \text{grad } V,$$

where

$$V = \frac{-1}{|x|}.$$

Observe that $F(x)$ is not defined at 0.

As in the previous section we restrict attention to particles moving in the plane $\mathbf{R}^2$; or, more properly, in $\mathbf{R}^2 - 0$. The force field is the Newtonian gravitational field in $\mathbf{R}^2$, $F(x) = -x/|x|^3$.

Consider a particular solution curve of our differential equation $\ddot{x} = m^{-1}F(x)$. The angular momentum h and energy E are regarded as constants in time since they are the same at all points of the curve. The case $h = 0$ is not so interesting; it corresponds to motion along a straight line toward or away from the sun. Hence we assume $h \neq 0$.

Introduce polar coordinates (r, θ); along the solution curve they become functions of time $(r(t), \theta(t))$. Since $r^2\dot{\theta}$ is constant and not 0, *the sign of* $\dot{\theta}$ *is constant* along the curve. Thus θ is always increasing or always decreasing with time. Therefore r *is a function of* θ *along the curve.*

Let $u(t) = 1/r(t)$; then u is also a function of $\theta(t)$. Note that

$$u = -V.$$

We have a convenient formula for kinetic energy T.

Lemma

$$T = \frac{1}{2}\frac{h^2}{m}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right].$$

Proof. From the formula for $\dot{x}$ in Section 4 and the definition of T we have

$$T = \tfrac{1}{2}m[\dot{r}^2 + (r\dot{\theta})^2].$$

Also,

$$\dot{r} = \frac{-1}{u^2}\frac{du}{d\theta}\dot{\theta} = -\frac{h}{m}\frac{du}{d\theta}$$

by the chain rule and the definitions of u and h; and also

$$r\dot{\theta} = \frac{h}{mr} = \frac{hu}{m}.$$

Substitution in the formula for T proves the lemma.

Now we find a differential equation relating u and θ along the solution curve. Observe that $T = E - V = E + u$. From the lemma we get

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2m}{h^2}(E + u). \tag{1}$$

Differentiate both sides by θ, divide by $2\,du/d\theta$, and use $dE/d\theta = 0$ (conservation of energy). We obtain another equation

$$\frac{d^2u}{d\theta^2} + u = \frac{m}{h^2}, \tag{2}$$

where m/h^2 is a constant.

We re-examine the meaning of just what we are doing and of (2). A particular orbit of the planar central force problem is considered, the force being gravitational. Along this orbit, the distance r from the origin (the source of the force) is a function of θ, as is $1/r = u$. We have shown that this function $u = u(\theta)$ satisfies (2), where h is the constant angular momentum and m is the mass.

The solution of (2) (as was seen in Section 1) is

$$u = \frac{m}{h^2} + C\cos(\theta + \theta_0), \tag{3}$$

where C and θ_0 are arbitrary constants.

To obtain a solution to (1), use (3) to compute $du/d\theta$ and $d^2u/d\theta^2$, substitute the resulting expression into (1) and solve for C. The result is

$$C = \pm\frac{1}{h^2}(2mh^2E + m^2)^{1/2}.$$

Putting this into (3) we get

$$u = \frac{m}{h^2}\left[1 \pm \left(1 + 2\frac{Eh^2}{m}\right)^{1/2} \cos\,(\theta + q)\right],$$

where q is an arbitrary constant. There is no need to consider both signs in front of the radical since $\cos(\theta + q + \pi) = -\cos(\theta + q)$. Moreover, by changing the variable θ to $\theta - q$ we can put any particular solution in the form

$$u = \frac{m}{h^2}\left[1 + \left(1 + 2\frac{Eh^2}{m}\right)^{1/2} \cos\theta\right]. \tag{4}$$

We recall from analytic geometry that the equation of a conic in polar coordinates is

$$u = \frac{1}{l}\,(1 + \epsilon\cos\theta), \qquad u = \frac{1}{r}. \tag{5}$$

Here l is the *latus rectum* and $\epsilon \geq 0$ is the *eccentricity*. The origin is a focus and the three cases $\epsilon > 1$, $\epsilon = 1$, $\epsilon < 1$ correspond respectively to a hyperbola, parabola, and ellipse. The case $\epsilon = 0$ is a circle.

Since (4) is in the form (5) we have shown that *the orbit of a particle moving under the influence of a Newtonian gravitational force is a conic of eccentricity*

$$\epsilon = \left(1 + \frac{2Eh^2}{m}\right)^{1/2}$$

Clearly, $\epsilon \geq 1$ if and only if $E \geq 0$. Therefore the orbit is a hyperbola, parabola, or ellipse according to whether $E > 0$, $E = 0$, or $E < 0$.

The quantity $u = 1/r$ is always positive. From (4) it follows that

$$\left(1 + \frac{2Eh^2}{m}\right)^{1/2} \cos\theta > -1.$$

But if $\theta = \pm\pi$ radians, $\cos\theta = -1$ and hence

$$\left(1 + \frac{2Eh^2}{m}\right)^{1/2} < 1.$$

This is equivalent to $E < 0$. For some of the planets, including the earth, complete revolutions have been observed; for these planets $\cos\theta = -1$ at least once a year. Therefore their orbits are ellipses. In fact from a few observations of any planet it can be shown that the orbit is in fact an ellipse.

PROBLEMS

1. A particle of mass m moves in the plane $\mathbf{R}^2$ under the influence of an elastic band tying it to the origin. The length of the band is negligible. Hooke's law states that the force on the particle is always directed toward the origin and is proportional to the distance from the origin. Write the force field and verify that it is conservative and central. Write the equation $F = ma$ for this case and solve it. (Compare Section 1.) Verify that for "most" initial conditions the particle moves in an ellipse.

2. Which of the following force fields on $\mathbf{R}^2$ are conservative?
   (a) $F(x, y) = (-x^2, -2y^2)$
   (b) $F(x, y) = (x^2 - y^2, 2xy)$
   (c) $F(x, y) = (x, 0)$

3. Consider the case of a particle in a gravitational field moving directly away from the origin at time $t = 0$. Discuss its motion. Under what initial conditions does it eventually reverse direction?

4. Let $F(x)$ be a force field on $\mathbf{R}^3$. Let x_0, x_1 be points in $\mathbf{R}^3$ and let $y(s)$ be a path in $\mathbf{R}^3$, $s_0 \leq s \leq s_1$, parametrized by arc length s, from x_0 to x_1. The *work* done in moving a particle along this path is defined to be the integral

$$\int_{s_0}^{s_1} \langle F(y(s)), y'(s) \rangle \, ds,$$

   where $y'(s)$ is the (unit) tangent vector to the path. Prove that the force field is conservative if and only if the work is independent of the path. In fact if $F = -\text{grad } V$, then the work done is $V(x_1) - V(x_0)$.

5. How can we determine whether the orbit of (a) Earth and (b) Pluto is an ellipse, parabola, or hyperbola?

6. Fill in the details of the proof of the theorem in Section 4.

7. Prove the angular momentum h, energy E, and mass m of a planet are related by the inequality

$$E \geq -\frac{m}{2h^2}.$$

Notes

Lang's *Second Course in Calculus* [12] is a good background reference for the mathematics in this chapter, especially his Chapters 3 and 4. The physics material is covered extensively in a fairly elementary (and perhaps old-fashioned) way in