Hypothesis Testing Fundamentals

Loukia Meligkotsidou
Associate Professor of Statistics
Department of Mathematics
National and Kapodistrian University of Athens

Hypothesis Testing

- A hypothesis is a claim (assumption) about a population parameter:
  - population mean

    Example: The mean salary is μ = $1500

  - population proportion

    Example: The proportion of a candidate in a voting process is p = 0.60

The Null Hypothesis, H_0

- States the assumption (numerical) to be tested

Example: The average grade is 7

$$H_0 : \mu = 7$$

- Is always about a population parameter, not about a sample statistic

$$H_0 : \mu = 7$$

$$H_0 : \overline{X} = 7$$

The Null Hypothesis, H_0

- Begin with the assumption that the null hypothesis is true
  - Similar to the notion of innocent until proven guilty
- May or may not be rejected
- Decision will be made on the basis of a sample: either there is enough evidence to reject the null or not

The Alternative Hypothesis, H_1

- Is the opposite of the null hypothesis
  - e.g., The average grade is not equal to 7

    (H_1: $\mu \neq 7$)
- May or may not be supported by the data
- Is generally the hypothesis that the researcher is trying to support

In hypothesis testing, we use a statistic (function of the data, called the test statistic) and its sampling distribution under the null. Extreme (unlikely) values of the sample statistic show evidence against the null.

Level of Significance

- **Defines the unlikely values of the sample statistic if the null hypothesis is true**
  - Defines rejection region of the sampling distribution
- Is designated by α , (level of significance)
  - Typical values are .01, .05, or .10
- Is selected by the researcher at the beginning
- Provides the critical value(s) of the test
- Is the probability of rejecting the null, given it's true

Level of Significance and the Rejection Region

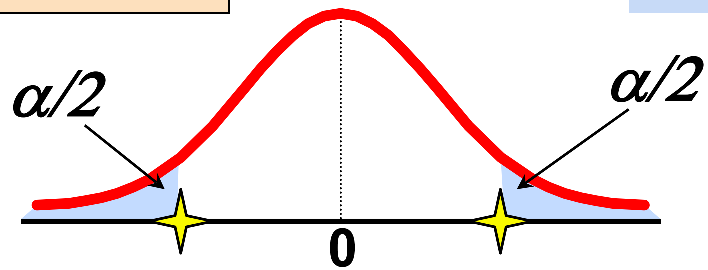
Level of significance = α

Represents critical value

Rejection region is shaded

$H_0: \mu = 7$

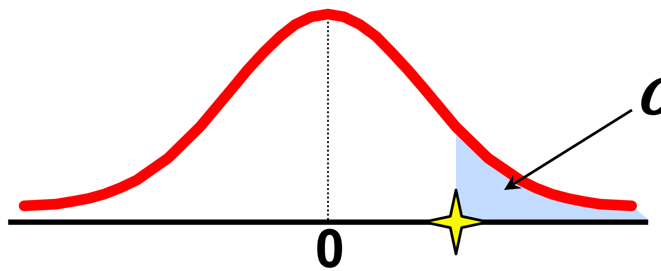
$H_1: \mu \neq 7$



Two-tail test

$H_0: \mu \leq 7$

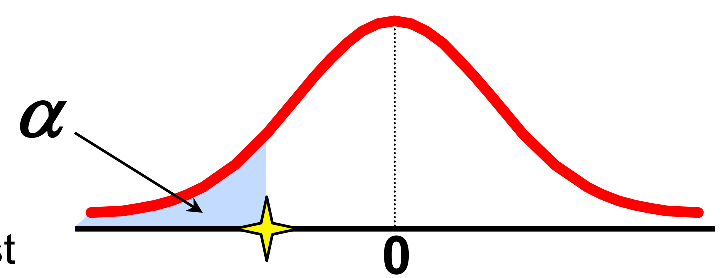
$H_1: \mu > 7$



Upper-tail test

$H_0: \mu \geq 7$

$H_1: \mu < 7$



Lower-tail test

Outcomes and Probabilities

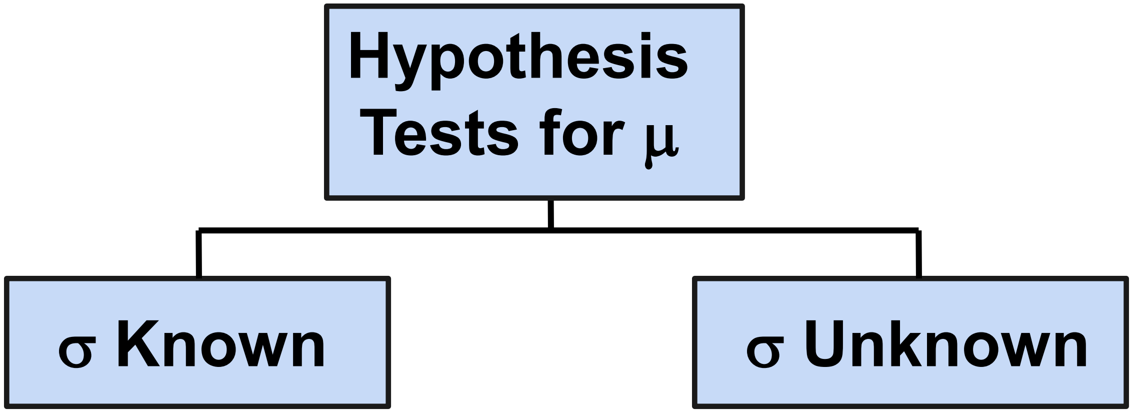
Key:
Outcome
(Probability)

Possible Hypothesis Test Outcomes		
	Actual Situation	
Decision	H_0 True	H_0 False
Do Not Reject H_0	**No error** **(1 - α)**	**Type II Error** **(β)**
Reject H_0	**Type I Error** **(α)**	**No Error** **(1 - β)**

Power of the Test

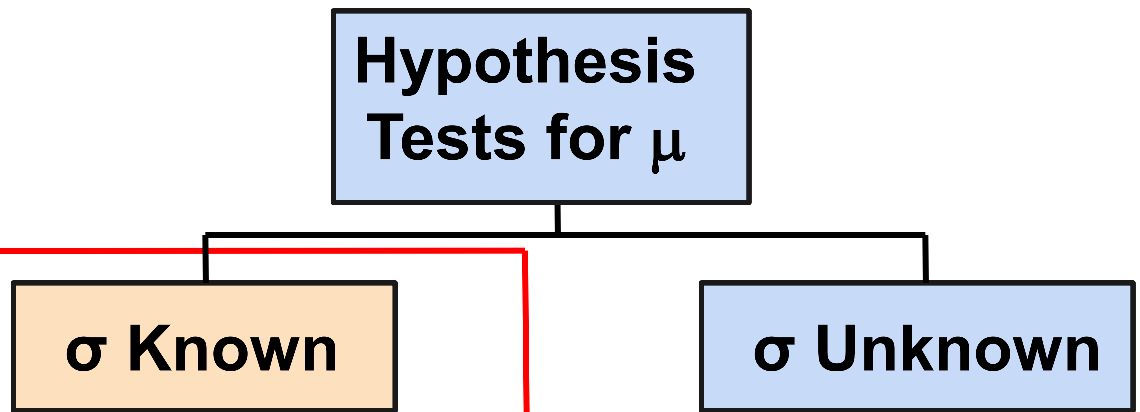
- The power of a test is the probability of rejecting a null hypothesis that is false
- i.e., Power = P(Reject H_0 | H_0 is false)
  = P(Reject H_0 | H_1 is true)
  = 1 – β =
  = 1 – P(not reject H_0 | H_0 is false)
  =1 – P(not reject H_0 | H_1 is true)
  - Power of the test increases as the sample size increases

Hypothesis Tests for the Mean



Test of Hypothesis for the Mean (σ Known)

- Convert sample result ($\bar{x}$) to a z value



Consider the test

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu > \mu_0$$

(Assume the population is normal)

The decision rule is:

$$\text{Reject } H_0 \text{ if } \; z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} > z_{1-\alpha}$$

Decision Rule

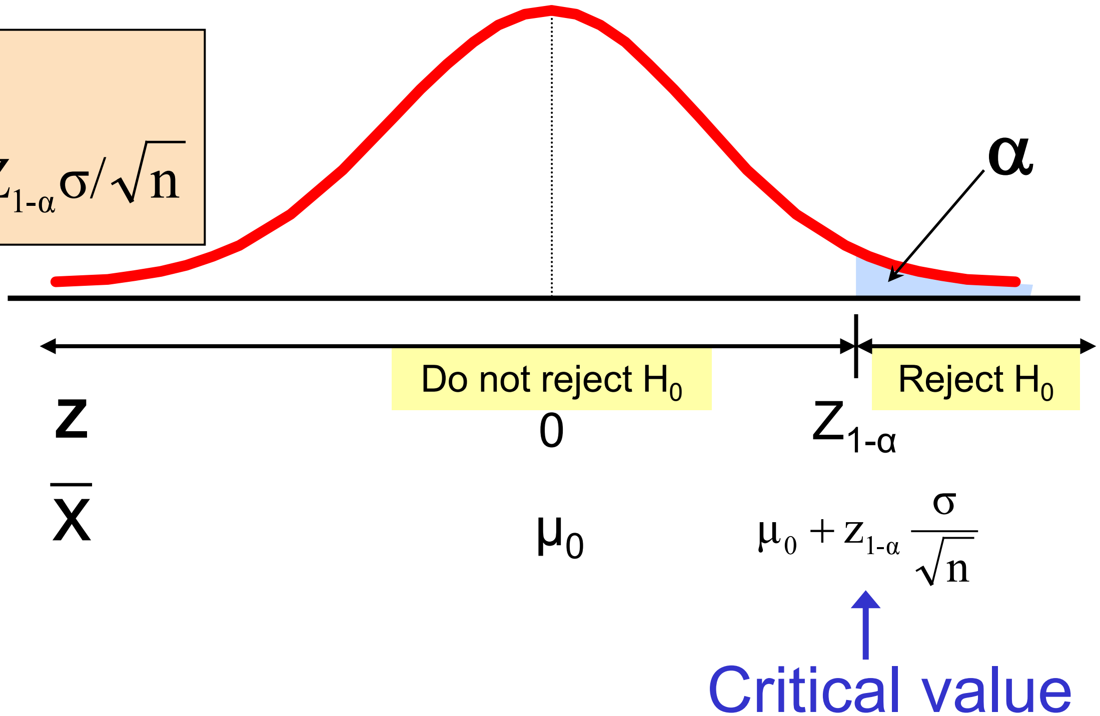
$$\text{Reject } H_0 \text{ if } z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} > z_{1-\alpha}$$

$H_0: \mu = \mu_0$

$H_1: \mu > \mu_0$

Alternate rule:

$$\text{Reject } H_0 \text{ if } \bar{X} > \mu_0 + Z_{1-\alpha}\sigma/\sqrt{n}$$



p-Value Approach to Testing

- p-value: Probability of obtaining a test statistic more extreme ($\leq$ or $\geq$) than the observed sample value given H_0 is true
  - Also called observed level of significance
  - The rule: Reject the null if p-value<α

p-Value Approach to Testing

- Convert sample result (e.g., $\bar{x}$) to test statistic (e.g., z statistic)
- Obtain the p-value
  - For an upper tail test:

$$\text{p-value} = P\left(Z > \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}, \text{ given that } H_0 \text{ is true}\right)$$

$$= P\left(Z > \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} \;\middle|\; \mu = \mu_0\right)$$

- Decision rule: compare the p-value to α
  - If p-value $< \alpha$, reject H_0
  - If p-value $\geq \alpha$, do not reject H_0

Example: Upper-Tail Z Test for Mean (σ Known)

A phone industry manager thinks that customer monthly cell phone bill is grater than \$52 per month. The company wishes to test this claim. (Assume σ = 10 is known)

Form hypothesis test:

H_0: $\mu \leq 52$ the average is not over \$52 per month

H_1: $\mu > 52$ the average **is** greater than \$52 per month (i.e., sufficient evidence exists to support the manager's claim)

Example: Sample Results

Obtain sample and compute the test statistic

Suppose a sample is taken with the following results: $n = 64$, $\bar{x} = 53.1$

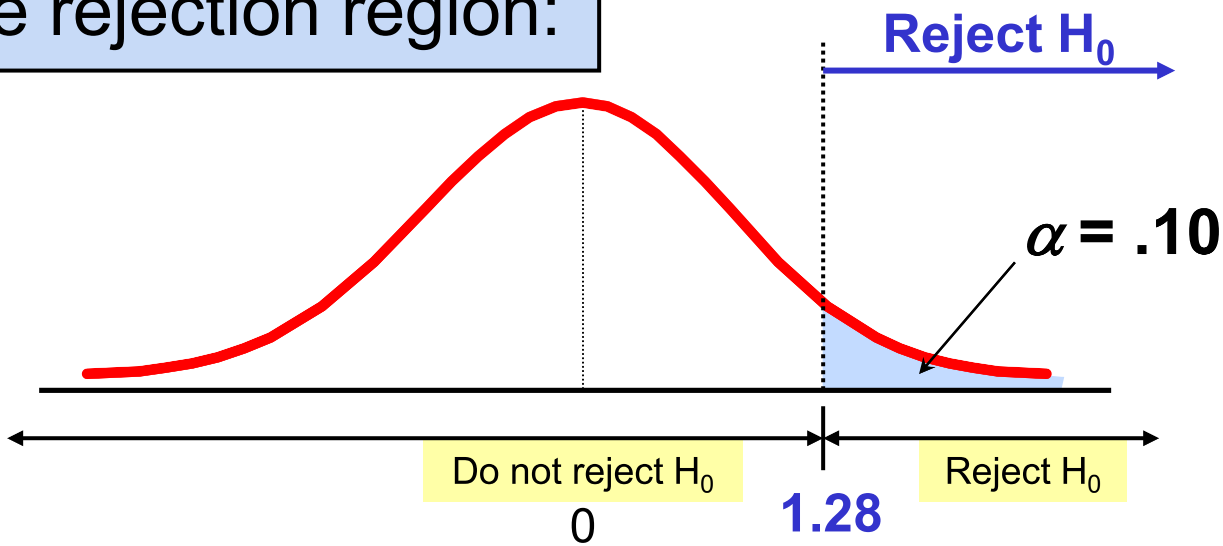
- Using the sample results,

$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{53.1 - 52}{\frac{10}{\sqrt{64}}} = 0.88$$

Example: Find Rejection Region

- Suppose that α = .10 is chosen for this test

Find the rejection region:

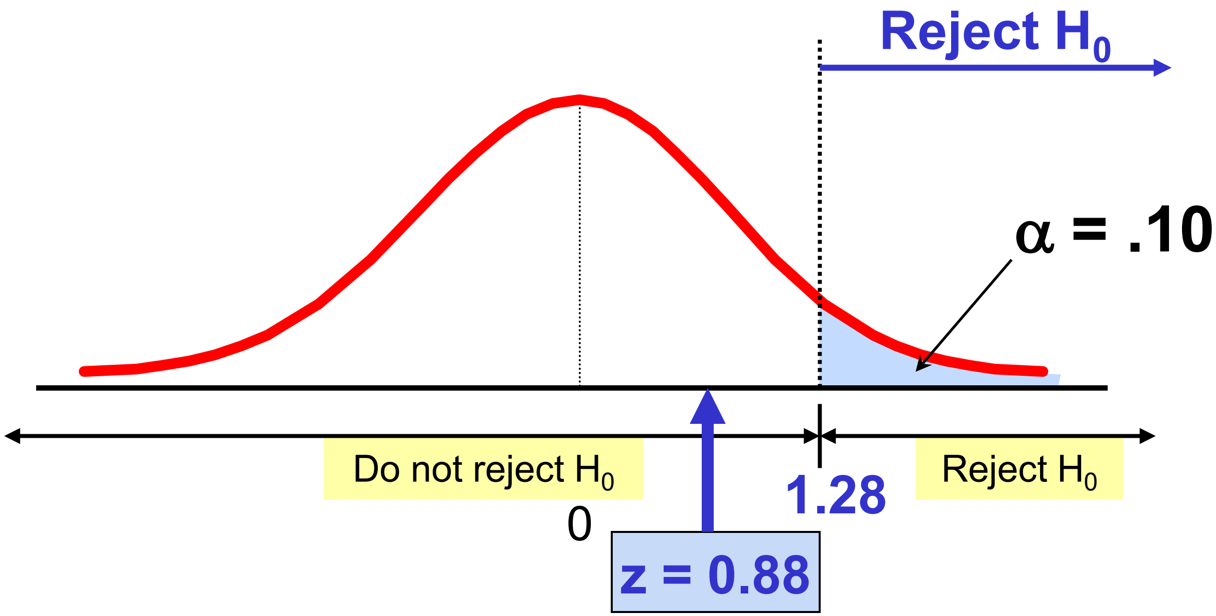


$$Z_{1-\alpha} = Z_{1-0.10} = Z_{0.90} = 1.28$$

$$\text{Reject } H_0 \text{ if } \quad z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} > 1.28$$

Example: Decision

Reach a decision and interpret the result:



Do not reject H_0 since z = 0.88 < 1.28

i.e.: there is not sufficient evidence that the mean bill is over $52

One-Tail Tests

- In many cases, the alternative hypothesis focuses on one particular direction

H_0: $\mu \le 3$

H_1: $\mu > 3$

→ This is an upper-tail test since the alternative hypothesis is focused on the upper tail above the mean of 3

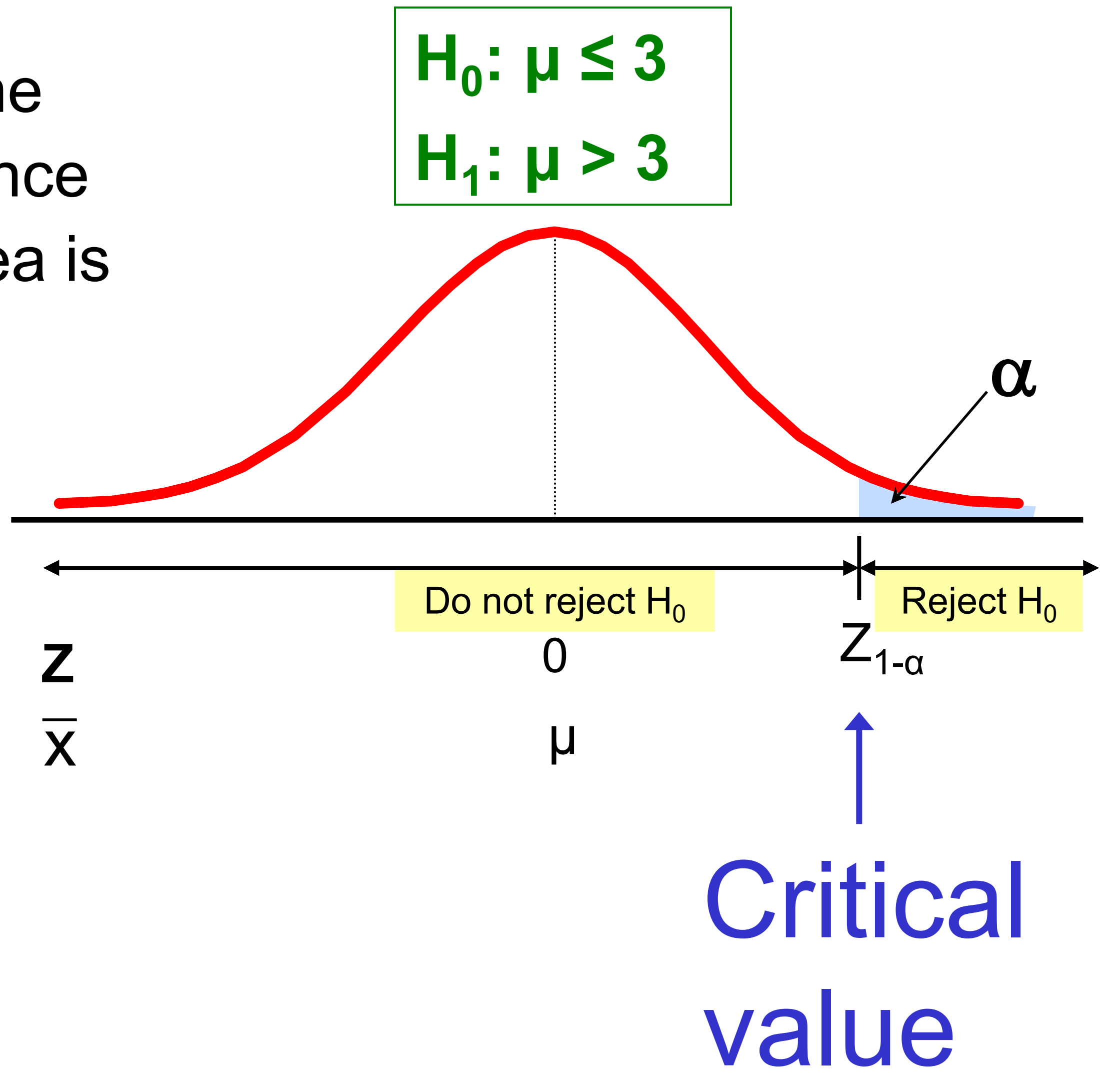
H_0: $\mu \ge 3$

H_1: $\mu < 3$

→ This is a lower-tail test since the alternative hypothesis is focused on the lower tail below the mean of 3

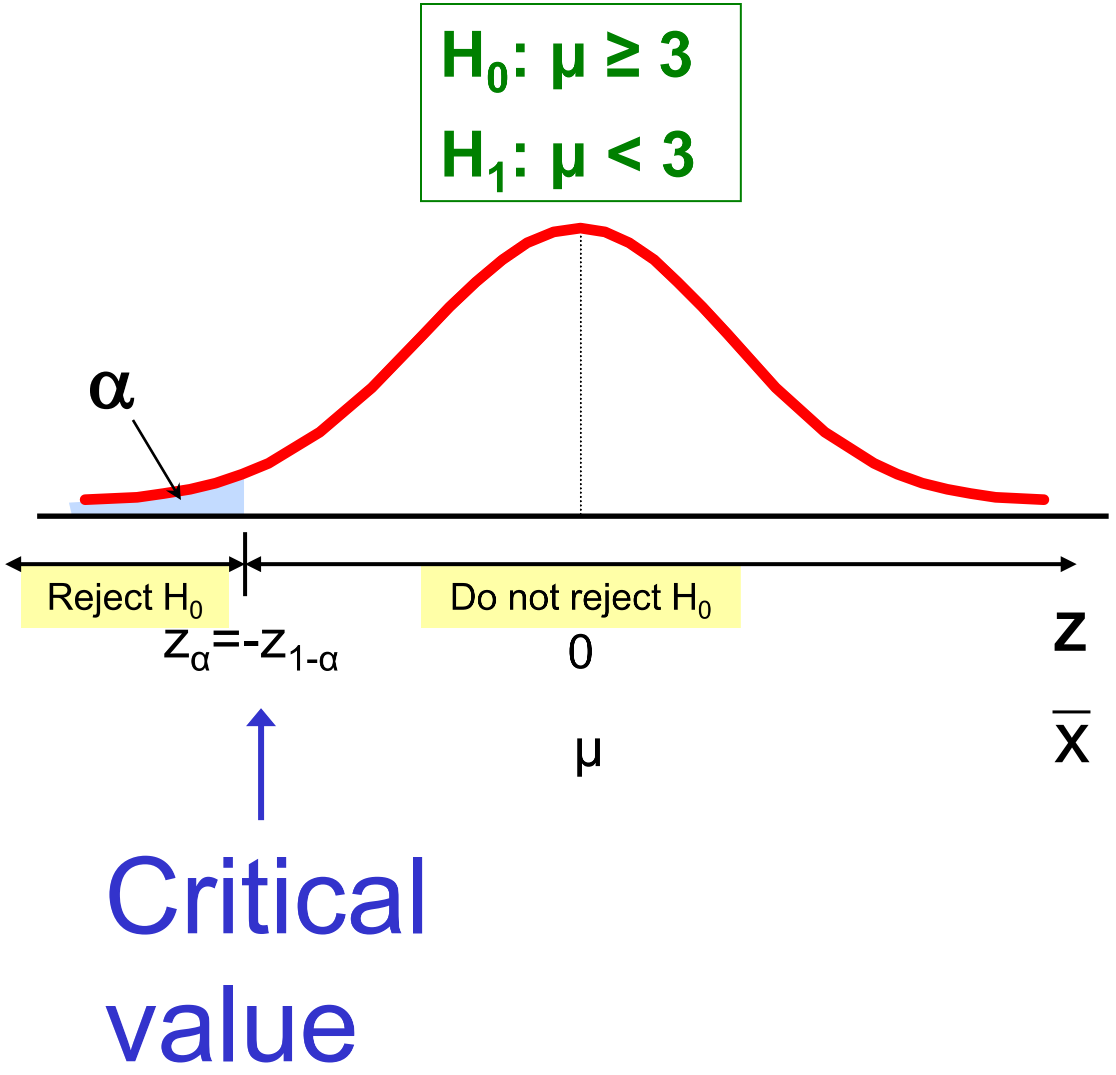
Upper-Tail Tests

- There is only one critical value, since the rejection area is in only one tail



Lower-Tail Tests

- There is only one critical value, since the rejection area is in only one tail

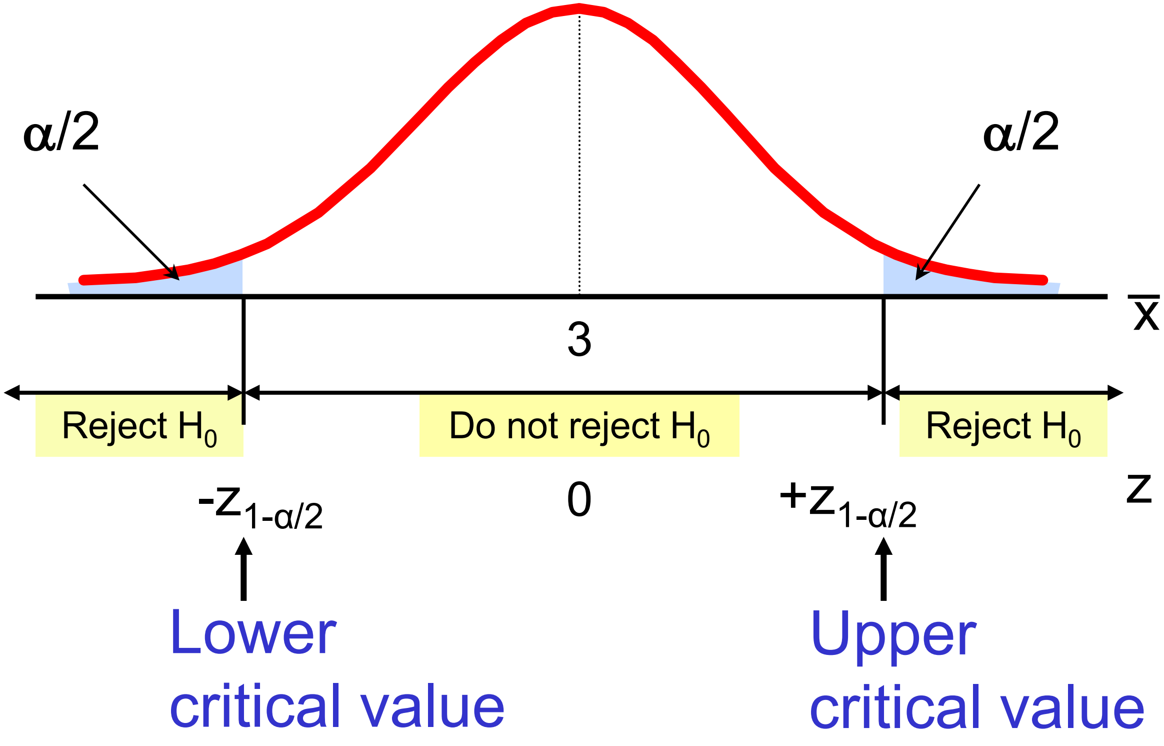


Two-Tail Tests

- In some settings, the alternative hypothesis does not specify a unique direction

H_0: $\mu = 3$
H_1: $\mu \neq 3$

- There are two critical values, defining the two regions of rejection



Hypothesis Testing Example

Test the claim that the true mean # of TV sets in US homes is equal to 3.

(Assume $\sigma = 0.8$)

- State the appropriate null and alternative hypotheses
  - H_0: $\mu = 3$, H_1: $\mu \neq 3$ (This is a two tailed test)
- Specify the desired level of significance
  - Suppose that $\alpha = .05$ is chosen for this test
- Choose a sample size
  - Suppose a sample of size $n = 100$ is selected

Hypothesis Testing Example

(continued)

- Determine the appropriate technique
  - σ is known so this is a z test
- Set up the critical values
  - For $\alpha = .05$ the critical z values are ±1.96
- Collect the data and compute the test statistic
  - Suppose the sample results are

    $n = 100$, $\bar{x} = 2.84$ ($\sigma = 0.8$ is assumed known)

So the test statistic is:

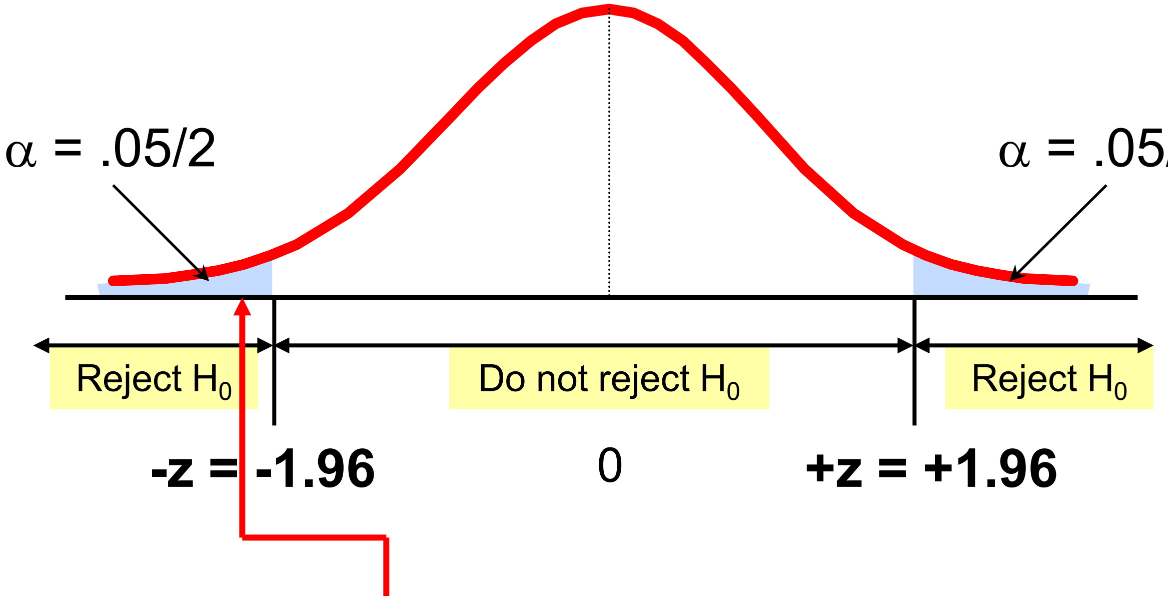
$$z = \frac{\bar{x} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{2.84 - 3}{\frac{0.8}{\sqrt{100}}} = \frac{-.16}{.08} = -2.0$$

Hypothesis Testing Example

(continued)

- Is the test statistic in the rejection region?

Reject H_0 if $z < -1.96$ or $z > 1.96$; otherwise do not reject H_0

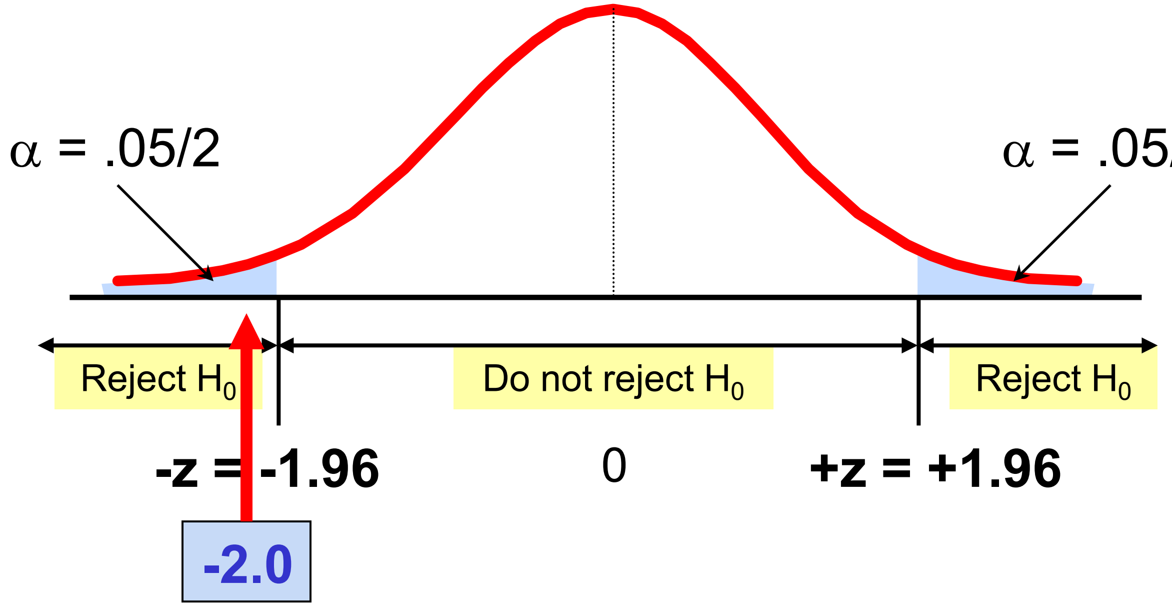


Here, $z = -2.0 < -1.96$, so the test statistic is in the rejection region

Hypothesis Testing Example

(continued)

- Reach a decision and interpret the result



Since $z = -2.0 < -1.96$, we reject the null hypothesis and conclude that there is sufficient evidence that the mean number of TVs in US homes is not equal to 3

Example: p-Value

- Example: How likely is it to see a sample mean of 2.84 (or something further from the mean, in either direction) if the true mean is μ = 3.0?

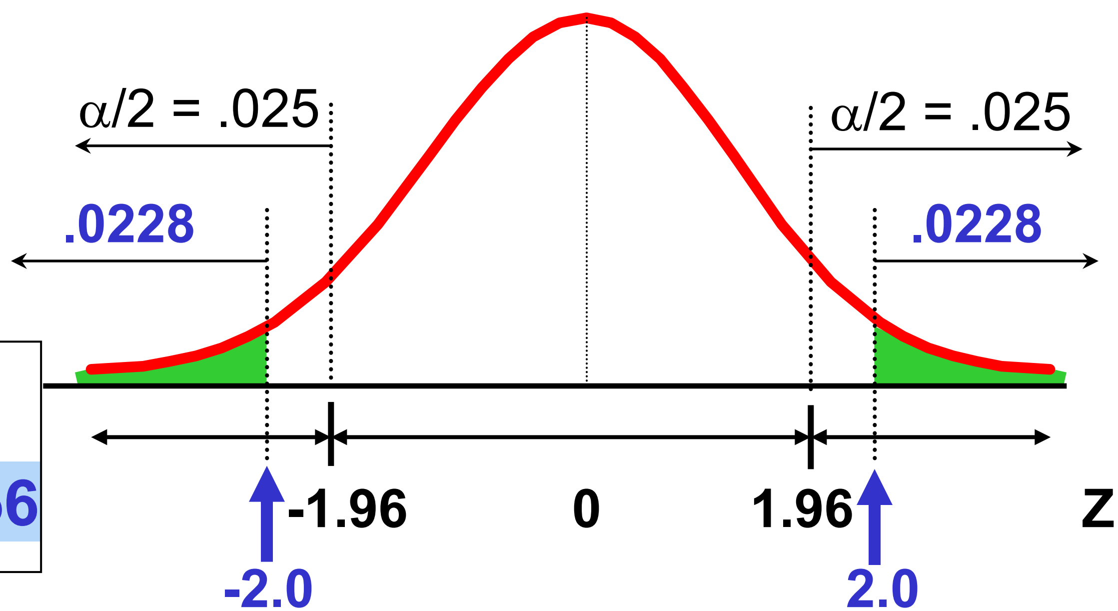
$\overline{x}$ = 2.84 is translated to a z score of z = -2.0

$$P(z < -2.0) = .0228$$

$$P(z > 2.0) = .0228$$

p-value

= .0228 + .0228 = .0456



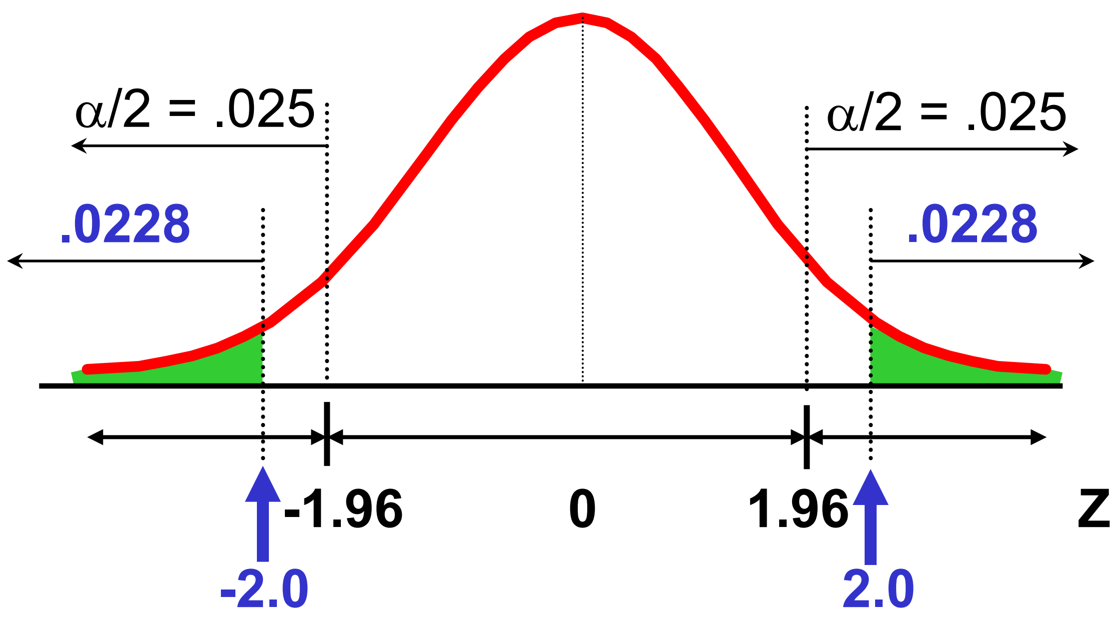
Example: p-Value

(continued)

- Compare the p-value with α
  - If p-value $<$ α, reject H_0
  - If p-value $\geq$ α, do not reject H_0

Here: p-value = .0456
α = .05

Since .0456 < .05, we reject the null hypothesis



t Test of Hypothesis for the Mean (σ Unknown)

- Convert sample result ($\bar{x}$) to a t test statistic

Hypothesis Tests for μ

σ Known | **σ Unknown**

Consider the test

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu > \mu_0$$

(Assume the population is normal)

The decision rule is:

$$\text{Reject } H_0 \text{ if } t = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} > t_{n-1,\alpha}$$

t Test of Hypothesis for the Mean (σ Unknown)

(continued)

- For a two-tailed test:

Consider the test

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0$$

(Assume the population is normal, and the population variance is unknown)

The decision rule is:

Reject H_0 if $t = \dfrac{\bar{x} - \mu_0}{\dfrac{s}{\sqrt{n}}} < -t_{n-1,\, \alpha/2}$ or if $t = \dfrac{\bar{x} - \mu_0}{\dfrac{s}{\sqrt{n}}} > t_{n-1,\, \alpha/2}$