
Δ.Π.Μ.Σ Μαθηματικά της Αγοράς και της Παραγωγής

Ειδικά Θέματα Επιχειρησιακής Έρευνας-
PERT/CPM Models ΜΑΘΗΜΑ 6^ο

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Τμήμα Μαθηματικών

Εθνικό και Καποδιστριακό Πανεπιστήμιο Αθηνών



Project Management

Project management can be defined as the coordination of activities with the potential use of many organizations, both internal and external to the business, in order to conduct a large scale project from beginning to end, e.g. a construction project or an R&D project.

Two OR/OM techniques used for project management:

- Program and Evaluation Review Technique (PERT) → it examines projects from the standpoint of uncertainty.
- Critical Path Method (CPM) → it examines projects from the standpoint of time associated with costs.

Both are merged to the so-called PERT/CPM technique – a network optimization based model. → planning and displaying the coordination of all the activities for a project.

Overview

- What is Project Management - Discuss PERT/CPM
- Motivating Case Study: The Reliable Construction Company
- Project Network Design
- Scheduling a project with PERT/CPM – The Critical Path notion
- Scheduling Individual Activities – Dealing with uncertainty
- Considering Time-Cost Trade-Offs – Crashed Activities and LP for making crashing decisions

The Reliable Construction Co. Problem

Reliable Construction Co. has just secured a contract to construct a new plant for a major manufacturer.

- The contract is for \$5.4 million to cover all costs and any profits.
- The plant must be finished in a year.
- A penalty of \$300,000 will be assessed if Reliable does not complete the project within 47 weeks.
- A bonus of \$150,000 will be paid to Reliable if the plant is completed within 40 weeks.



What we must know for designing:

- ✓ *Activities*
- ✓ *Immediate Predecessors*
- ✓ *Immediate Successors*
- ✓ *Estimated Duration*
- ✓ *Start and finish node (fictional)*
do not refer to an activity

Activity list for the Reliable Construction Co. project

Activity	Activity Description	Immediate Predecessors	Estimated Duration (Weeks)
A	Excavate	—	2
B	Lay the foundation	A	4
C	Put up the rough wall	B	10
D	Put up the roof	C	6
E	Install the exterior plumbing	C	4
F	Install the interior plumbing	E	5
G	Put up the exterior siding	D	7
H	Do the exterior painting	E, G	9
I	Do the electrical work	C	7
J	Put up the wallboard	F, I	8
K	Install the flooring	J	4
L	Do the interior painting	J	5
M	Install the exterior fixtures	H	2
N	Install the interior fixtures	K, L	6

Predecessors and Successors of a given activity

Activity

A distinct task that needs to be performed as part of the project.

Immediate Predecessor

These are activities that must be completed by no later than the start time of the given activity.

Immediate Successor

Given the immediate predecessor of an activity, this activity becomes the immediate successor of each of these immediate predecessors.

Comment

If an immediate successor has multiple immediate predecessors, then all must be finished before an activity can begin.

Questions – Goals of PM

A List of Questions – Objectives of Project Management

- (Q1) How can the project be displayed graphically?
- (Q2) How much time is required to finish the project if no delays occur?
- (Q3) When is earliest start and finish times for each activity if no delays occur?
- (Q4) When do the individual activities need to start and finish (at the latest) to meet this project completion time?
- (Q5) What activities are critical bottleneck activities where delays must be avoided to finish the project on time?
- (Q6) For non bottleneck activities, how much can an activity be delayed and yet still keep the project on time?
- (Q7) What is the probability of completing the project by the deadline?
- (Q8) What is the least amount of money needed to expedite the project to obtain the bonus?
- (Q9) How should costs be monitored to keep the project within budget?

(Q1) How can the project be displayed graphically?

Project Network

A project network is a network diagram that uses nodes and arcs to represent the progression of the activities in a project from start to finish.

What we need for drawing the network? Two alternative types of networks

- Activity information
 - Precedence relationship
 - Time information
- Activity – on – arc (AOA)
activity based on an arc, node is used to separate an activity from its immediate predecessors.
 - Activity – on – node (AON)
activity based on a node, arcs are used to show the precedence relationship between the activities.

The Reliable Construct Co. Project Network

AON or AOA?

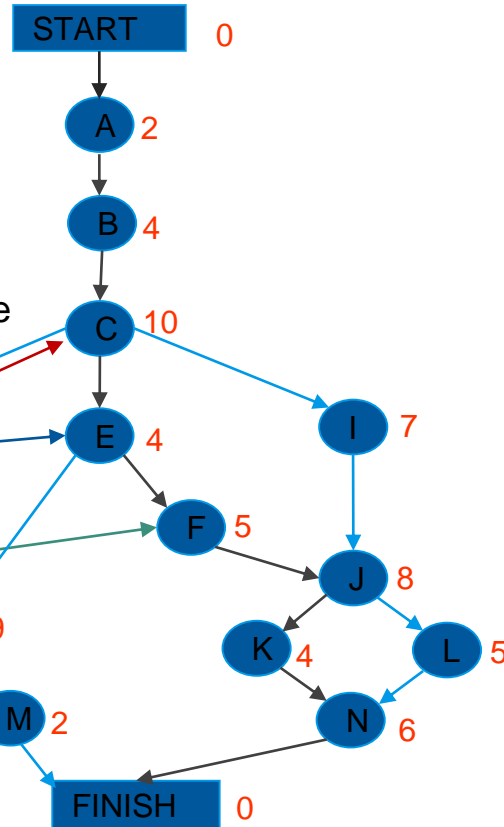
1. AON project networks are easier to construct than AOA.
2. AON are easier to understand than AOA for inexperienced users, including many managers.
3. AON are easier to revise than AOA when there are changes in the project.

Node E: Immediate predecessors
successors

A **path** through a project network is a route that follows a set of arcs from the start node to the finish node with **length** equal to the sum of the (estimated) of the activities of this path.

Start->A->B->C->E->F->J->K->N->Finish

with length = $2+4+10+4+5+8+4+6=43$ weeks.



- Activity Code
- A. Excavate
 - B. Foundation
 - C. Rough wall
 - D. Roof
 - E. Exterior plumbing
 - F. Interior plumbing
 - G. Exterior siding
 - H. Exterior painting
 - I. Electrical work
 - J. Wallboard
 - K. Flooring
 - L. Interior painting
 - M. Exterior fixtures
 - N. Interior fixtures

Is the best path to follow?

(Q2) How much time is required to finish the project?

A **path** through a project network is a route from START to FINISH.

The **length** of path is the sum of the task times (durations) of the nodes (activities) on the path.

The **critical path** is the longest path, and its length refers to the **project duration**

Path	Length
START →A →B →C →D →G →H →M→FINISH	$2 + 4 + 10 + 6 + 7 + 9 + 2 = 40$ weeks
START →A →B →C →E →H →M →FINISH	$2 + 4 + 10 + 4 + 9 + 2 = 31$ weeks
START →A →B →C →E →F →J →K →N →FINISH	$2 + 4 + 10 + 4 + 5 + 8 + 4 + 6 = 43$ weeks
START →A →B →C →E →F →J →L →N →FINISH	$2 + 4 + 10 + 4 + 5 + 8 + 5 + 6 = 44$ weeks
START →A →B →C →I →J →K →N →FINISH	$2 + 4 + 10 + 7 + 8 + 4 + 6 = 41$ weeks
START →A →B →C →I →J →L →N →FINISH	$2 + 4 + 10 + 7 + 8 + 5 + 6 = 42$ weeks

What do we get from the critical path?

- If no delays occur, we get the total time required to complete the project.
- The activities on the critical path are the critical bottlenecks, where any delays in their completion must be avoided to prevent delaying project completion.
- These activities must be kept on schedule.
- To reduce the duration of the project, these are the main activities to interfere in order to reduce their individual duration.

Is it always easy to derive the critical path?

If so, we need to determine specifically which activities should have their durations reduced and by how much (if we want to make it in 40 weeks) in the least expensive way.

Ways to derive the critical path

- Examine all the paths and find the path with the maximum length.
- Calculate the slack for an activity.
 - If the slack is zero, it is on the critical path.
 - If the slack is positive, it is not on the critical path.

So, what is the slack of a given node?

We need some terminology first:

ES (Earliest start time) = The time at which an activity will begin if there are no delays.

EF (Earliest finish time) = The time at which an activity will finish if there are no delays.

LS (Latest start time) = The latest possible time that an activity can start without delaying the project.

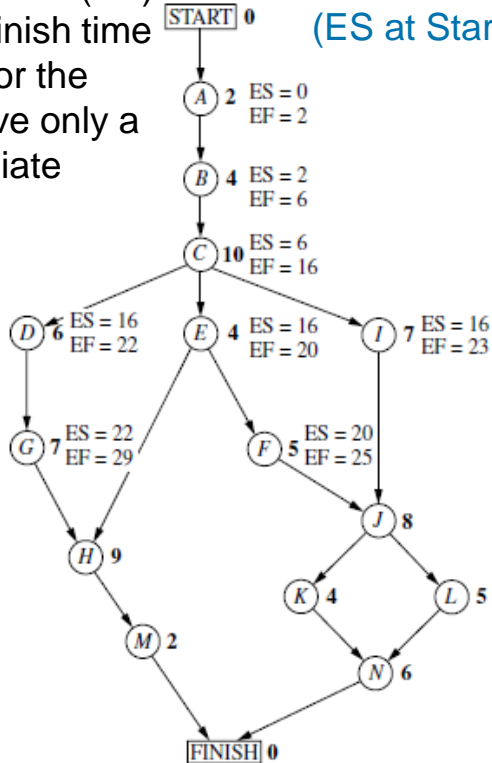
LF (Latest finish time) = The latest possible time that an activity can be completed without delaying the project.

To Find the Critical Path and Slacks

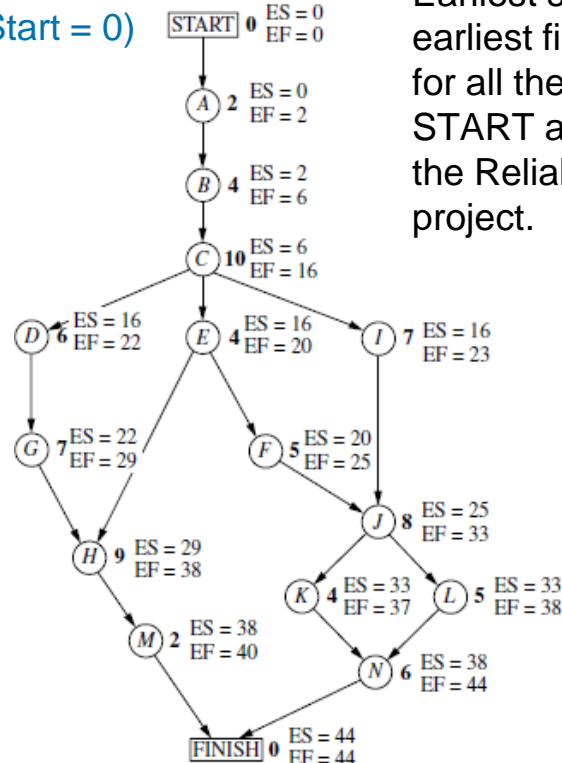
1. Work from top to bottom in the network (**Forward Pass**), calculating ES and EF for all activities, as follows:
 - ES for activity i = largest EF of the immediate predecessors
 - ES = 0 if no immediate predecessors
 - EF = ES + activity duration time
2. Work from bottom to top in the network (**Backward Pass**), calculating LS and LF for all activities, as follows:
 - LS = LF – activity duration time
 - LF for activity i = smallest LS of the immediate successors
 - LF at Finish = EF at Finish if no immediate successors
3. **Slack = LF - EF = LS - ES**
 - If slack is zero, the activity is on the critical path.

Calculating ES and EF – Forward Pass

Earliest start time (ES) and earliest finish time (EF) values for the initial that have only a single immediate predecessor.



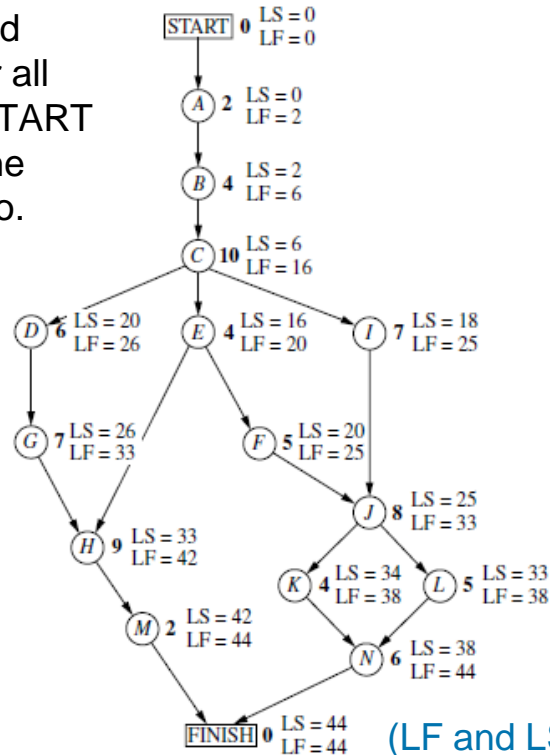
(ES at Start = 0, EF at Start = 0)



Earliest start time (ES) and earliest finish time (EF) values for all the activities (plus the START and FINISH nodes) of the Reliable Construction Co. project.

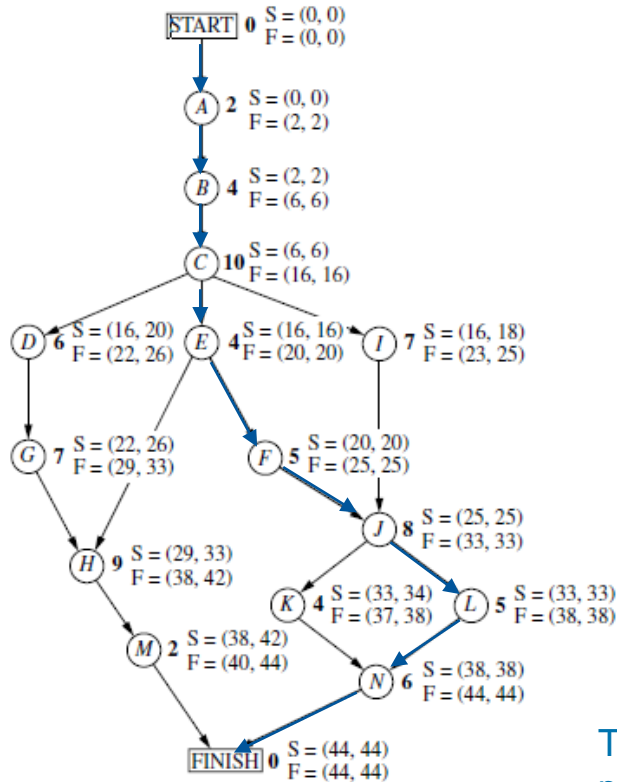
Calculating LS and LF – Backward Pass

Latest start time (LS) and latest finish time (LF) for all the activities (plus the START and FINISH nodes) of the Reliable Construction Co. project.



(LF and LS at Finish is EF and ES at Finish)

Calculating each slack and finding the critical path



The complete project network showing:
ES and LS (above the node) and
EF and LF (below the node) for
each activity of the Reliable
Construction Co. project.

$S = (ES, LS)$

$F = (EF, LF)$

Slack = $LS - ES = LF - EF$

The dark blue arrows show the critical path through the project network.

Slack for Reliable Activities and Interpretation

Activity	Slack (LF - EF)	On Critical Path?
A	0	Yes
B	0	Yes
C	0	Yes
D	4	No
E	0	Yes
F	0	Yes
G	4	No
H	4	No
I	2	No
J	0	Yes
K	1	No
L	0	Yes
M	4	No
N	0	Yes

Activity M can be delayed up to 4 weeks beyond the earliest time schedule without delaying the completion of the project at 44 weeks. As long as *N* stays on schedule, the project still will finish at 44 weeks if any delays in starting and in performing *M* do not cumulate more than 4 weeks.



Dealing with (Q3) to (Q6)

A List of Questions – Objectives of Project Management

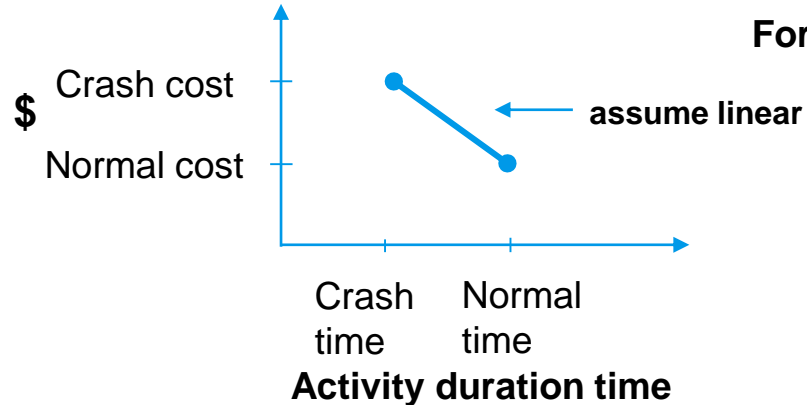
- (Q3) When is earliest start and finish times for each activity if no delays occur?
These times are the earliest start times (ES) and earliest finish times (EF). These times usually are used to establish the initial schedule for the project.
- (Q4) When do the individual activities need to start and finish (at the latest) to meet this project completion time?
These times are the latest start times (LS) and latest finish times (LF). These times provide a “last chance schedule” to complete the project in 44 weeks if no further delays occur.
- (Q5) What activities are critical bottleneck activities where delays must be avoided to finish the project on time?
These are the activities on the critical path shown by the blue arrows. The Project Manager needs to focus most of his attention on keeping these particular activities on schedule in striving to keep the overall project on schedule.
- (Q6) For non bottleneck activities, how much can an activity be delayed and yet still keep the project on time?
These tolerable delays are the positive slacks given in the middle column of the Table.

Time Cost Trade-Offs for Individual Activities

Recall that, Reliable had an incentive bonus of \$150,000 to finish the project in 40 weeks.

Is it worth while for Reliable to speed-up the project?

If one can expedite the project, use money/resources to reduce task times, what is the best way to allocate money?



$$\begin{aligned} & \text{crash cost per week saved} \\ &= \frac{\text{crash cost} - \text{normal cost}}{\text{time saved}} \\ &= \frac{\text{crash cost} - \text{normal cost}}{\text{normal time} - \text{crash time}} \\ &= -\text{slope} \end{aligned}$$

Crashing and CPM of time-cost trade-offs

What crashing means?

Crashing an activity refers to taking on extra expenditures in order to reduce the duration of an activity below its expected value.

Crashing a project refers to crashing a number of activities to reduce the duration of the project.

The CPM method of time-cost trade-offs

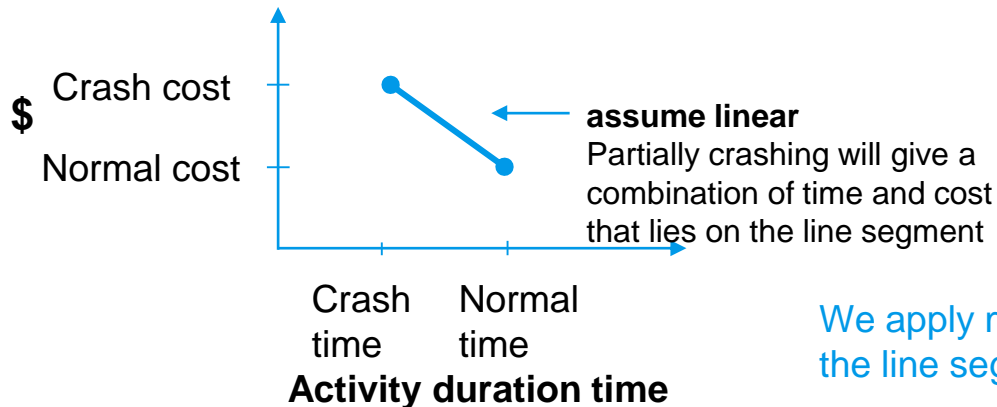
Is it worthwhile to **crash activities** to reduce the anticipated duration of the project to a desired value?

The method assumes that there is a **trade-off between time and cost that has an inverse relationship**, and it determines how much to crash each activity in order to reduced the anticipated project duration to a desired value.

Normal and Crash Point

Normal point is the time and cost of an activity when it is performed in a normal way.


Crash point show the time and cost when the activity is fully crashed, i.e., it is fully expedited with no cost spared to reduce its duration as much as possible.



As an approximation, CPM assumes that these times and costs can be reliably predicted without significant uncertainty.

We apply marginal cost analysis to derive the slope of the line segment

Calculating the slope

Activity	Time (weeks)		Cost		Maximum Reduction in Time (weeks)	Crash Cost per Week Saved	
	Normal	Crash	Normal	Crash			
A	2	1	\$180,000	\$280,000	1	\$100,000	
B	4	2	320,000	420,000	2	50,000	crash cost per week saved
C	10	7	620,000	860,000	3	80,000	= $\frac{\text{crash cost} - \text{normal cost}}{\text{time saved}}$
D	6	4	260,000	340,000	2	40,000	= $\frac{\text{crash cost} - \text{normal cost}}{\text{normal time} - \text{crash time}}$
E	4	3	410,000	570,000	1	160,000	= -slope
F	5	3	180,000	260,000	2	40,000	
G	7	4	900,000	1,020,000	3	40,000	
H	9	6	200,000	380,000	3	60,000	
I	7	5	210,000	270,000	2	30,000	
J	8	6	430,000	490,000	2	30,000	 For activity J:
K	4	3	160,000	200,000	1	40,000	
L	5	3	250,000	350,000	2	50,000	$\frac{430 - 490}{8 - 6} = -30 K/week$
M	2	1	100,000	200,000	1	100,000	
N	6	3	330,000	510,000	3	60,000	

Marginal Cost Analysis

It is a method of using the marginal cost of crashing individual activities on the current critical path to determine the least expensive way of reducing the project duration to an acceptable level.

This method requires you to calculate the cost per desired time unit and compare each cost with the other costs.

Once the marginal cost for crashing each activity has been conducted, you next want to choose the crashing that has the smallest marginal cost.

Next, calculate the effect that the crash has on each path.

Note: Crashing can potentially cause another path to become a critical path.

(Q8) What is the least amount of money needed to expedite the project to obtain the bonus?

For small networks, may reduce the project 1 week at a time, and observe the changes at all possible paths.

Activity to Crash	Crash Cost	Length of Path					
		ABCDGHM	ABCEHM	ABCEFJKN	ABCEFJLN	ABCIJKN	ABCIJLN
		40	31	43	44	41	42
J	\$ 30,000	40	31	42	43	40	41
J	\$ 30,000	40	31	41	42	39	40
F	\$ 40,000	40	31	40	41	39	40
F	\$ 40,000	40	31	39	40	39	40

(Q8) What is the least amount of money needed to expedite the project to obtain the bonus?

- We start with the least expensive crashed activity in the critical path. This is J, and we reduce the overall time by 1 week at all paths that include J. We find the next critical path. It may not be the same. Here, the critical path is still ABCEFJLN. (Row 2)
- We repeat the process. Again, the least expensive crashed activity in ABCEFJLN is J (we crash it until it is fully crashed by 2 weeks). We reduce the overall time by 1 week and the next CP is ABCIJKN, once again.
- In the next iteration, the least expensive crashed activity in ABCIJKN is J, but we cannot crash it anymore. We move to F, which can be fully crashed by 3 weeks. Here, we do it two times.

Activity to Crash	Crash Cost	Length of Path					
		ABCDGHM	ABCEHM	ABCEFJKN	ABCEFJLN	ABCIJKN	ABCIJLN
		40	31	43	44	41	42
J	\$ 30,000	40	31	42	43	40	41
J	\$ 30,000	40	31	41	42	39	40
F	\$ 40,000	40	31	40	41	39	40
F	\$ 40,000	40	31	39	40	39	40

(Q8) What is the least amount of money needed to expedite the project to obtain the bonus?

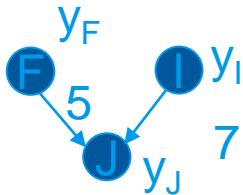
Activity to Crash	Crash Cost	Length of Path					
		ABCDGHM	ABCEHM	ABCEFJKN	ABCEFJLN	ABCIJKN	ABCIJLN
		40	31	43	44	41	42
J	\$ 30,000	40	31	42	43	40	41
J	\$ 30,000	40	31	41	42	39	40
F	\$ 40,000	40	31	40	41	39	40
F	\$ 40,000	40	31	39	40	39	40

Final Remarks

- We repeat the process until the total duration of the project is within 40 weeks.
- The overall crashing cost is 140K, which is slightly under the bonus of 150K.
- To achieve this we have to fully crash activity J by two weeks and activity F by two weeks (partially crashed). In this case, we find more critical paths, i.e., **ABCDGHM** and **ABCIJLN**, except from **ABCEFJLN**

Linear Programming Formulation to Make Crashing Decisions

- Let
- Z = total cost of crashing on any activity
 - x_j = reduction in the duration of activity j due to crashing $j = A, B, C, \dots, N$
 - $x_j \leq$ maximum reduction time = normal time – crash time
 - y_{FINISH} = project duration, time at which FINISH node is reached
 - y_j = start time of activity j
 - $y_j \geq y_i + \text{normal time}_i - x_i$, where i is an immediate predecessor of j



$$\begin{aligned}y_J &\geq y_F + 5 - x_F \\y_J &\geq y_I + 7 - x_I\end{aligned}$$

The Linear Programming Formulation

$$\begin{aligned} \text{Minimize crash costs} = Z &= \sum_{j=A}^N \left(\frac{\text{crash cost}_j}{\text{per week saved}} \right) (x_j) \\ &= 100,000x_A + 50,000x_B + \dots + 60,000x_N \end{aligned}$$

subject to

$$y_{FINISH} \leq 40 \quad \left. \vphantom{y_{FINISH} \leq 40} \right\} \text{(total project duration)}$$

$$x_A \leq 1$$

$$x_B \leq 2$$

$$\vdots$$

$$x_N \leq 3 \quad \left. \vphantom{x_N \leq 3} \right\} \text{(max. time reduction)}$$

$$y_B \geq 0 + 2 - x_A$$

$$y_C \geq y_B + 4 - x_B$$

$$y_D \geq y_C + 10 - x_C$$

$$y_E \geq y_C + 10 - x_C$$

$$y_I \geq y_C + 10 - x_C$$

$$y_G \geq y_D + 6 - x_D$$

$$y_F \geq y_E + 4 - x_E$$

$$y_H \geq y_G + 7 - x_G$$

$$y_H \geq y_E + 4 - x_E$$

$$\vdots$$

$$y_{Finish} \geq y_M + 2 - x_M$$

$$y_{Finish} \geq y_N + 6 - x_N$$

$$x_j \geq 0, y_j \geq 0, y_{FINISH} \geq 0 \quad \left. \vphantom{x_j \geq 0} \right\} \text{(non-negativity constrains)}$$

PERT/CPM Models – Ενδεικτική Βιβλιογραφία

- **Hillier and Lieberman, “Introduction to Operations Research”, 7th edition**
Chapter 10 , Project Management with PERT/CPM, p.p. 468-532.