

Mathematical Modelling: Can It Be Taught And Learnt?

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Abstract

Mathematical modelling (the process of translating between the real world and mathematics in both directions) is one of the topics in mathematics education that has been discussed and propagated most intensely during the last few decades. In classroom practice all over the world, however, modelling still has a far less prominent role than is desirable. The main reason for this gap between the goals of the educational debate and everyday school practice is that modelling is difficult both for students' and for teachers. In our paper, we will show examples of how students and teachers deal with demanding modelling tasks. We will refer both to results from our own projects DISUM and COM² as well as to empirical findings from various other research studies. First, we will present some examples of students' difficulties with modelling tasks and of students' specific modelling routes when solving such tasks (also dependent on their mathematical thinking styles), and try to explain these difficulties by the cognitive demands of these tasks. We will emphasise that mathematical modelling has to be learnt specifically by students, and that modelling can indeed be learned if teaching obeys certain quality criteria, in particular maintaining a permanent balance between teacher's guidance and students' independence. We will then show some examples of how teachers have successfully realised this subtle balance, and we will present interesting differences between individual teachers' handling of modelling tasks. In the final part of our paper, we will draw some consequences from the reported empirical findings and formulate corresponding implications for teaching mathematical modelling. Eventually, we will present some encouraging results from a recent intervention study in the context of the DISUM project where it is demonstrated that appropriate learning environments may indeed lead to a higher and more enduring progress concerning students' modelling competency.

Keywords: mathematical modelling, quality teaching, independent learning, mathematical thinking styles

1. What is mathematical modelling, and what is it for?

Here is an example of a modelling task:

Example 1: "Giant's shoes"

In a sports centre on the Philippines, Florentino Anonuevo Jr. polishes a pair of shoes. They are, according to the Guinness Book of Records, the world's biggest, with a width of 2.37 m and a length of 5.29 m.

Approximately how tall would a giant be for these shoes to fit? Explain your solution.



This task requires translations between reality and mathematics what, in short, can be called *mathematical modelling*. By reality, we mean according to Pollak (1979), the "rest of the world" outside mathematics including nature, society, everyday life and other scientific disciplines.

Here is how two students' from grade 9 (15 years old) in the German Hauptschule (the low-ability track in our tripartite system) solved this task:

“Well, to calculate, from these two figures, the height, the size of the man. If the width of the shoe is 2.37 m and the length 5.29 m, then ought, I believe, 2.37 m times 5.29 m. Then you have the height of the man, I believe.”

And here is the according solution of the students’:

$$2,37\text{ m} \cdot 5,29\text{ m} = 12,5373\text{ m}$$

Antwort: Der Mensch wäre 12,53 m groß.

In the following, we mean by a “modelling task” a task with a substantial modelling demand. The example shows that such tasks are usually *difficult* for students’. Why is modelling so difficult for students’? An important reason are certainly the cognitive demands of modelling tasks. Modelling is inseparably linked with other mathematical competencies (see Niss 2003) such as reading and communicating, designing and applying problem solving strategies, or working mathematically (reasoning, calculating, ...). Particularly helpful for cognitive analyses of modelling tasks is a model of the “modelling cycle” for solving these tasks. Here is the seven-step model (see Blum/Leiß 2007) that we use in both our projects:

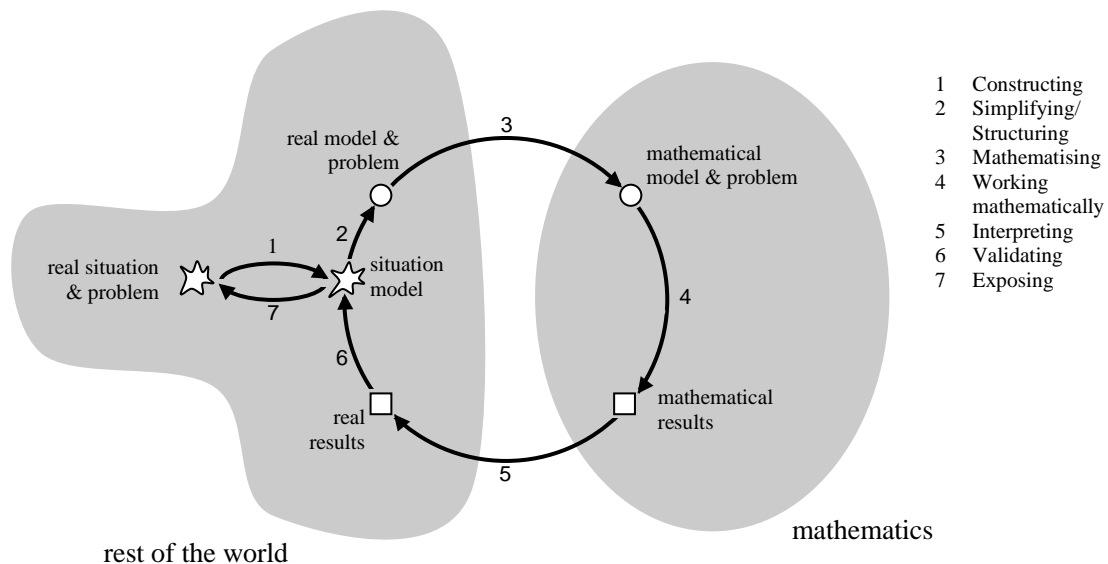


Figure 1 – Modelling cycle

We would like to illustrate this cycle by a second modelling task (Blum/Leiß 2006):

Example 2: “Filling up”

Mrs. Stone lives in Trier, 20 km away from the border of Luxemburg. To fill up her VW Golf she drives to Luxemburg where immediately behind the border there is a petrol station. There you have to pay 1.10 Euro for one litre of petrol whereas in Trier you have to pay 1.35 Euro.

Is it worthwhile for Mrs. Stone to drive to Luxemburg? Give reasons for your answer.



First, the problem situation has to be understood by the problem solver, that is a *situation model* has to be constructed. Then the situation has to be simplified, structured and made more precise, leading to a *real model* of the situation. In particular, the problem solver has to define here what “worthwhile” should mean. In the standard model, this means only “minimising the costs of filling up

and driving”. Mathematisation transforms the real model into a *mathematical model* which consists here of certain equations. Working mathematically (calculating, solving the equations, etc.) yields *mathematical results*, which are interpreted in the real world as *real results*, ending up in a recommendation for Mrs. Stone what to do. A validation of these results may show that it is necessary to go round the loop a second time, for instance in order to take into account more factors such as time or air pollution. Dependent on which factors have been taken, the recommendations for Mrs. Stone might be quite different.

There are a lot of models of the modelling process (compare the analyses in Borromeo Ferri 2006). The advantages of this particular model for research purposes are:

- Step 1 is separated this is a particularly individual construction process and the first cognitive barrier for students’ when solving modelling tasks (see, e.g., Kintsch/Greene 1985, DeCorte/Greer/Verschaffel 2000, Staub/Reusser 1995)
- All these steps are potential cognitive barriers for students’ as well as essential stages in actual modelling processes, though generally not in a linear order (Borromeo Ferri 2007, Leiß 2007, Matos/Carreira 1997); see our documentation of specific “modelling routes” in part ② of this paper.

On this basis, we can now concisely define “*modelling competency*” (see Blum et al. 2007) as the ability to construct models by carrying out those various steps appropriately as well as to analyse or compare given models.

Modelling and applications has been an increasingly important topic in mathematics education during the last two decades (see the Proceedings of the series of ICTMA Conferences and the corresponding sections in the series of ICMI Congresses, with survey papers such as Pollak 1979, Blum/Niss 1991, or Houston 2005; compare also the survey in Kaiser 2005 and in Kaiser/Blomhøj/Sriraman 2006). Recent interest in mathematical modelling has been stimulated by OECD’s PISA Study where students’ “Mathematical Literacy” (that is essentially the ability to deal with real world situations in a well-founded manner) is investigated. The present state-of-the-art is documented in the ICMI Study 14 Volume on “Modelling and Applications in Mathematics Education” (Blum et al. 2007).

Why is modelling so *important* for students’? Mathematical models and modelling are everywhere around us, often in connection with powerful technological tools. Preparing students’ for responsible citizenship and for participation in societal developments requires them to build up modelling competency. More generally: mathematical modelling is meant to

- help students’ to better understand the world,
- support mathematics learning (motivation, concept formation, comprehension, retaining),
- contribute to develop various mathematical competencies and appropriate attitudes,
- contribute to an adequate picture of mathematics.

By modelling, mathematics becomes more meaningful for learners (this is, of course, not the only possibility for that). Underlying all these justifications of modelling are the main goals of mathematics teaching in secondary schools.

There is in fact a tendency in several countries to include more mathematical modelling in the *curriculum*. In Germany, for instance, mathematical modelling is one of six compulsory competencies in the new national “Educational Standards” for mathematics. However, in *everyday mathematics teaching* in most countries there is still only few modelling. Mostly “*word problems*” are treated where, after “undressing” the context, the essential aim is exercising mathematics. For competency development and for learning support also word problems are legitimate and helpful; it is only important to be honest about the true nature of reality-oriented tasks and problems.

Why do we find only so few modelling in everyday classrooms, why is there this *gap* between the educational debate (and even official curricula), on the one hand, and classroom practice, on the other hand? The main reason is that modelling is difficult also for teachers, for real world knowledge is needed, and teaching becomes more open and less predictable (see, e.g., Freudenthal 1973, Pollak 1979, DeLange 1987, Burkhardt 2004, Blum et al. 2007).

In the following, we will investigate more deeply how students’ and teachers deal with mathematical modelling. All the examples we use in this paper are taken from our own projects

DISUM and COM². **DISUM** means¹ “Didaktische Interventionsformen für einen Selbstständigkeitsorientierten aufgabengesteuerten Unterricht am Beispiel Mathematik” (“Didactical intervention modes for mathematics teaching oriented towards self-regulation and directed by tasks”; see Blum/Leiß 2008). **COM²** means² “Cognitive-psychological analysis of modelling processes in mathematics lessons” (see Borromeo Ferri 2006). Both projects analyse how students’ and teachers deal with cognitively demanding modelling tasks, with a focus on grades 8-10. Consequently, this age group (14-16-year-olds) will also be the focus of this paper.

2. How do students’ deal with modelling tasks?

The PISA-2006 results (OECD 2007) have revealed again that students’ all around the world have problems with modelling tasks. Analyses carried out by the PISA Mathematics Expert Group (whose member is the first author) have shown that the difficulty of modelling tasks can indeed be substantially explained by the inherent cognitive complexity of these tasks, that is by the demands on students’ competencies. Our own studies have shown that all potential cognitive barriers (according to the steps of the modelling cycle, see ①) can actually be observed empirically, specific for individual tasks and individual students’ (compare also Galbraith/Stillman 2006). Here are some selected *examples of students’ difficulties*:

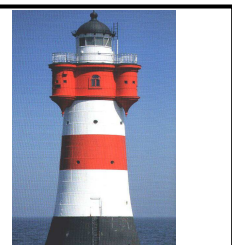
- Step 1 “constructing”: See the introductory example 1 “Giant’s shoes”! This is an instance of the well-known superficial solution strategy “Ignore the context, just extract all numbers from the text and calculate with these according to a familiar schema” which in everyday classrooms is very often rather successful for solving word problems (Baruk 1985, Verschaffel/Greer/DeCorte 2000).
- Step 2 “simplifying”: Here is an authentic solution of modelling example 2 “Filling up”: “*You cannot know if it is worthwhile since you don’t know what the Golf consumes. You also don’t know how much she wants to fill up.*” Obviously, the student has constructed an appropriate situation model, but he is not able to make assumptions.
- Step 6 “validating” seems to be particularly problematic. Mostly, students’ do not check at all whether their task solutions are reasonable and appropriate, the teacher seems to be exclusively responsible for the correctness of solutions.

Particularly interesting are students’ specific modelling routes during the process of solving modelling tasks. A “*modelling route*” (see Borromeo Ferri 2007) describes an individual modelling process in detail, referring to the various phases of the modelling cycle. The individual starts this process in a certain phase, according to his/her preferences, and then goes through different phases, focussing on certain phases or ignoring others. To be more precise, one ought to speak of *visible* modelling routes since one can only refer to verbal utterances or external representations for the reconstruction of the starting point and the course of a modelling route. We will illustrate the concept of modelling routes more concretely by means of the modelling task “Lighthouse”:

Example 3: “Lighthouse”

In the bay of Bremen, directly on the coast, a lighthouse called “Roter Sand” was built in 1884, measuring 30.7 m in height. Its beacon was meant to warn ships that they were approaching the coast.

How far, approximately, was a ship from the coast when it saw the lighthouse for the first time? Explain your solution.



A short analysis of the lighthouse task by means of the modelling cycle will give more insight into students’ thinking processes and make their statements (see below) more transparent. The first solution step is mentally imagining the situation, consisting of the lighthouse, the ship and the surface

¹ DISUM runs since 2002 and is funded by the German Research Foundation since 2005. It is directed by W. Blum (Mathematics Education), R. Messner (Pedagogy, both University of Kassel), and R. Pekrun (Pedagogical Psychology, University of München); the present project staff consists also of D. Leiß, S. Schukajlow, and J. Krämer (all Kassel).

² COM² runs since 2004 and is directed by R. Borromeo Ferri together with G. Kaiser (both University of Hamburg).

of the earth in between (already a non-trivial step for many students’). The resulting situation model has to be simplified: the earth as a sphere, the ship as a point, and free sight between lighthouse and ship. Mathematisation leads to a mathematical model of the real situation, with $H \approx 30,7$ m as the height of the lighthouse, $R \approx 6,37$ km as the radius of the earth and S as the unknown distance lighthouse-ship. Mathematical considerations show that there is a right-angled triangle, and the Pythagorean theorem gives $S^2 + R^2 = (R+H)^2$, hence $S = \sqrt{2RH + H^2} \approx \sqrt{2RH} \approx 19,81$ km. Interpreting this mathematical result leads to the answer “approximately 20 km” for the initial question. Now this real result has to be validated: Is it reasonable, are the assumptions appropriate (the ship is certainly not a point, etc.)? If need be, the cycle may start once again with new assumptions.

The following quotes made by two students’ during their videotaped problem solving processes can only give some exemplary illustrations of the various changes during these processes. The actual processes are too long and too complex for giving an account of all the utterances in detail.

Both students’, Max and Sebastian, worked together in one group; nevertheless it was possible to reconstruct their individual modelling routes which are presented in fig. 2:

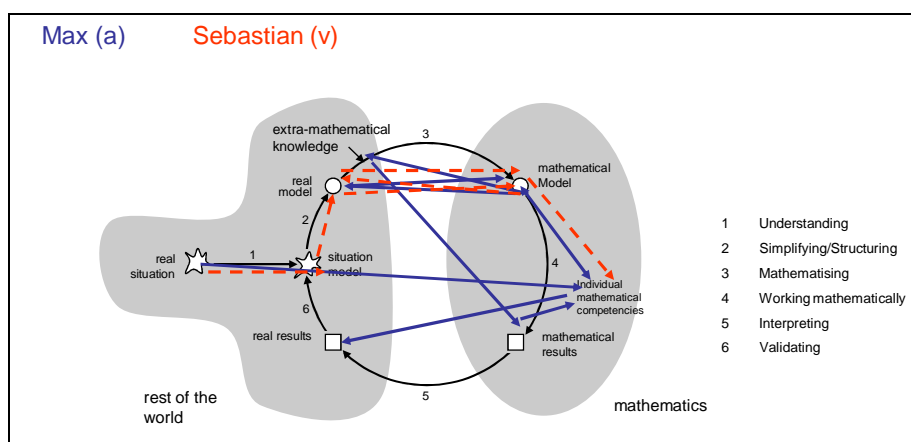


Figure 2 – Two Students’ individually modelling routes

Max’ modelling route (straight arrows):

Max read the “lighthouse task” and expressed the following thoughts shortly afterwards:

M: “Okay, what shall we do, I’d say we do Pythagoras!” (real situation => mathematical model)

Max changed immediately from the situation described in the task into the mathematical model, as he could see in his mind’s eye that he could apply Pythagoras. However, he did not make any progress with the mathematical model because he did not seem to have clarified the given situation sufficiently. He then changed to the real model in order to better imagine the situation described. Doing this, he started thinking aloud and intensively about the earth’s curvature, which shows that he was literally “picturing” the situation.

M: “Actually, it’s the earth’s curvature that makes the lighthouse disappear; if it was a smooth plane, it would be visible all the time!” (mathematical model => real model)

After Max had got a more precise mental picture, he changed quickly back to the mathematical model. He still remembered the Pythagorean Theorem and made a drawing.

M: “We have to mirror this on this cathetus, can you see the length, it’s the one up here.” (real model => mathematical model)

Max dwelled on the mathematical model for quite some time. He increasingly started wondering about what the earth’s curvature is and asked himself and the others for this extra-mathematical knowledge. Unlike the other group members, he held the opinion that the earth’s curvature would also have to be taken into account for the calculations.

M: “Yeah, see, we’ve got to include the earth’s curvature in our calculations.” (mathematical model => extra-mathematical knowledge)

Leaving the question of the earth’s curvature aside, Max returned to the mathematical model and remained in that phase for a long time. During that phase, he used his intra-mathematical skills

(Pythagoras' theorem) as well as extra-mathematical knowledge (the earth's diameter) to reach a conclusion.

M: *"It's twenty kilometres. I've got the lighthouse to the power of two minus the radius."* (extra-mathematical knowledge => mathematical results)

Max interpreted the result only to some extent and did not validate it with regard to the real situation; he assumed it to be "mathematically" correct.

M: *"I've got twenty kilometres, as the crow flies."* (mathematical results => real results)

Sebastian's modelling route (broken arrows):

Sebastian started immediately with a sketch and at first described the real situation given very vividly. That way, he got the situation described in the task clear in his mind and created a situation model.

S: *"Here's the ship, somewhat like this and this is the earth's curvature."* (real situation => situation model)

Starting with his mental picture, he kept simplifying the situation further and created a real model.

S: *"We're gonna do a triangle here."* (situation model => real model)

In his further statements, an increasing mathematisation became apparent, and he changed to the mathematical model.

S: *"We need an angle on this side in order to calculate the distance. (...) Cos I need this (points at Mark's drawing), then I could hundred and eighty minus ninety minus..."* (real model => mathematical model)

Sebastian did not stick to the mathematical model for long, as he had to keep "picturing" the situation. When the group started discussing the question of whether the earth's curvature should be included in the calculations, he remained rather neutral.

S: *"The only thing which otherwise prevents us from getting a clear view is mostly our eyes, if the plane was level, and probably particles in the air."* (mathematical model => real model)

From the real model Sebastian returned to the mathematical model and continued to work more mathematically. As it did not occur to him to work with Pythagoras, but with Sinus instead, he only focussed on applying this individual mathematical competence.

S: *"And if we knew one angle now, then we could, we could use Sinus."* (real model => mathematical model)

Sebastian often switched between the real and the mathematical model because he had to transport himself into the real situation and needed always to picture the situation visually in order to keep working on the task. In contrast to Max, who solved the problem, Sebastian did not reach a conclusion and was stuck in the mathematical model.

The modelling routes of the two students' are rather different. One reason for that is the fact that students' problem solving behaviour substantially depends on their *mathematical thinking styles* (Borromeo Ferri 2004). According to their responses in questionnaires and interviews, Max is an "analytic" thinker and Sebastian a "visual" thinker. The term "*mathematical thinking style*" denotes the way in which an individual prefers to present, to understand and to think through mathematical facts and connections, using certain internal imaginations and/or externalised representations. Accordingly, a mathematical thinking style is constituted by two components: 1) internal imaginations and externalized representations, 2) the "holistic" respectively "dissecting" way of proceeding when solving mathematical problems. Mathematical thinking styles should not be seen as mathematical abilities but as preferences how mathematical abilities are used. Empirically, three mathematical thinking styles of students' attending grades 9/10 could be reconstructed:

- *"Visual" (pictorial-holistic) thinking style:* Visual thinkers show preferences for distinctive internal pictorial imaginations and externalized pictorial representations and for the understanding of mathematical facts and connections through existing illustrative representations, as well as preferences for a more holistic view on given problem situations. In modelling tasks, they tend to focus more on the real world part of the process.
- *"Analytical" (symbolic-dissecting) thinking style:* Analytic thinkers show preferences for internal formal imaginations and for externalized formal representations; they are able to comprehend and to express mathematical facts preferably through symbolic or verbal

representations, and they show preferences for a more step-by-step procedure when solving given problems. In modelling tasks, they tend to focus more on the mathematical part of the process.

- “*Integrated*” thinking style: These persons are able to combine visual and analytic ways of thinking to the same extent.

In the following, we will mention some more *empirical findings* concerning students’ dealing with modelling tasks.

- In most cases, there is no conscious use of *problem solving strategies* by students’. This explains a lot of the observed difficulties since we know from several studies that strategies (meta-cognitive activities) are helpful also for modelling (Tanner/Jones 1993, Matos/Carreira 1997, Schoenfeld 1994, Kramarski/Mevarech/Arami 2002, Burkhardt/Pollak 2006, Galbraith/Stillman 2006; for an overview see Greer/Verschaffel in Blum et al. 2007).
- We know from several studies in the context of Situated Cognition that learning is always dependent on the specific learning *context* and hence a simple transfer from one situation to others cannot be expected (Brown/Collins/Duguid 1989, DeCorte/Greer/Verschaffel 1996, Niss 1999). This holds for the learning of mathematical modelling in particular, modelling *has to be learnt* specifically. This is a “bad” message; the “good” counterpart is the following message:
- Several studies have shown that mathematical modelling *can be learnt* (Galbraith/Clatworthy 1990, Abrantes 1993, Kaiser 1987, Maaß 2007; see also④). The decisive variable for successful teaching seems to be “quality teaching”. This will be addressed in the next chapter.

3. How do teachers treat modelling in the classroom?

Perhaps the most important finding is the following: *Teachers are indispensable*, there is a fundamental distinction between students’ working independently with teacher’s support and students’ working alone. This may sound rather trivial but it is not at all trivial; here is a picture from a German best-seller on general pedagogy:

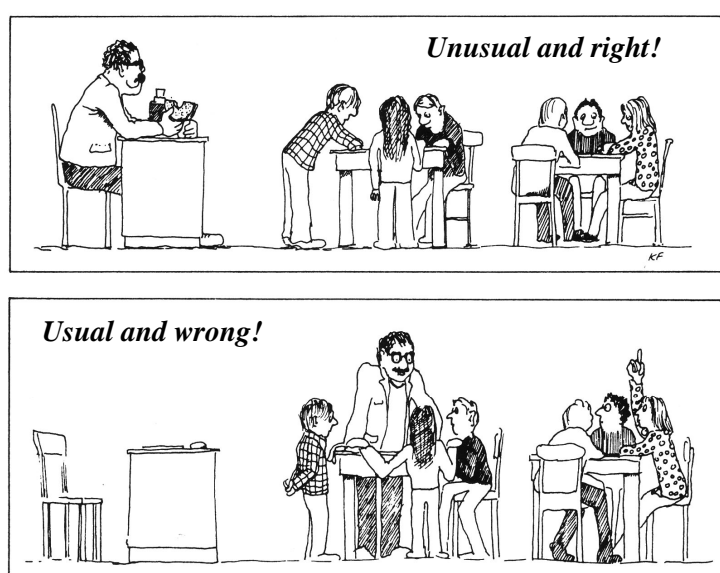


Figure 3 – A wrong view on students’ learning

According to the empirical findings, it should be just *the other way round!*

There is dense empirical evidence that teaching effects can only (to be more precise: at most) be expected on the basis of “*quality mathematics teaching*”. What could that mean? Here is the working definition we use in our projects (compare, e.g., Blum/Leiß 2008):

- *A demanding orchestration of teaching the mathematical subject matter* (by giving students' vast opportunities to acquire mathematical competencies and establishing connections within and outside mathematics)
- *Permanent cognitive activation of the learners* (by stimulating cognitive and meta-cognitive activities and fostering students' independence)
- *An effective and learner-oriented classroom management* (by varying methods flexibly, using time effectively, separating learning and assessment etc.)

For quality teaching, it is crucial that a permanent *balance* between (minimal) teacher's guidance and (maximal) students' independence is maintained (according to Maria Montessori's famous maxim: "Help me to do it by myself"). In particular, when students' are dealing with modelling tasks, this balance is best achieved by adaptive, independence-preserving teacher interventions. In this context, often *strategic* interventions are most adequate, that means interventions which give hints to students' on a meta-level ("Imagine the situation!", "What do you aim at?", "How far have you got?", "What is still missing?", "Does this result fit to the real situation?", etc.). In everyday mathematics teaching, those quality criteria are often violated. In particular, teacher's interventions are mostly not independence-preserving. Here, we will report on an example of a successful strategic intervention. The task students' (from a Realschule class 9, medium-ability-track) were dealing with was the following:

Example 4: "Fire-brigade"

Feuerwehr

Die Münchner Feuerwehr hat sich im Jahr 2004 ein neues Drehleiter-Fahrzeug angeschafft. Mit diesem kann man über einem am Ende der Leiter angebrachten Korb Personen aus großen Höhen retten. Dabei muss das Feuerwehrauto laut einer Vorschrift 12 m Mindestabstand vom brennenden Haus einhalten.



Die technischen Daten des Fahrzeugs sind:

| | |
|--------------------|---|
| Fahrzeugtyp: | Daimler Chrysler AG Eonic 18/28 LL - Diesel |
| Baujahr: | 2004 |
| Leistung: | 205kw (279 PS) |
| Hubraum: | 6374 cm ³ |
| Maße des Fahrzeug: | Länge 10m Breite 2,5m Höhe 3,19m |
| Maße der Leiter: | 30m Länge |
| Leergewicht: | 15540kg |
| Gesamtgewicht: | 18000 kg |

Aus welcher maximalen Höhe kann die Münchner Feuerwehr mit diesem Fahrzeug Personen retten?

From which maximal height can the Munich fire-brigade rescue persons with this engine?

A student thought that he was done but the teacher recognised that he had forgotten to include the engine's height into his calculations. Then, the following dialogue arose:

T.: "You have disregarded a little thing!"

S.: "... this calculation of a?"

T.: "No, you have calculated everything correctly, by the way. You only have to read once more precisely!"

The student found quickly and independently his mistake, and not only did he correct it, he also explained to the members of his table group what to do and why they ought, by all means, not to forget to include the engine's height ("*Look, it is said here in the text, 3.19 m!*").

We have reported on another successful strategic intervention (in the context of the "Lighthouse" task) in Borromeo Ferri/ Blum (2008). However, according to our observations, mathematics teacher's spontaneous interventions in modelling contexts were mostly not independence-preserving, they were mostly content-related or organisational, and next to never strategic. Mostly, only a narrow spectrum of interventions was available even for experienced teacher's (constituting the teacher's specific "intervention styles").

A common feature of many of our observations was that the teacher's own *favourite solution* of a given task was often imposed on the students' through his interventions, mostly without even noticing it, also due to an insufficient knowledge of the richness of the "task space" on the teacher's side. However, we know that it is important to encourage *various individual solutions*, also to match different thinking styles of students', and particularly as a basis for retrospective reflections after the students' presentations. To this end, it is necessary for teacher's to have an intimate knowledge of the cognitive demands of given tasks. In another project (COACTIV; see Krauss et al. 2008) we have found that the teacher's knowledge of task spaces is one significant predictor of his students' achievement gains.

An interesting question in this context is: How do the *mathematical thinking styles of teachers'* influence their way of dealing with modelling tasks? In the COM² project, three grade 10 classes of the Gymnasium (the German Grammar Schools, high-ability track) were chosen for an analysis of the teacher's behaviour when treating modelling tasks. The sample was comprised of 65 pupils and 3 teachers' (one male, two female). Focused interviews were conducted with each teacher to reconstruct his/her mathematical thinking style. Biographical questions were also included and questions were asked, among other things, about his/her current view of mathematics or about reasons why his/her view of mathematics might have changed in the course of his/her teaching life. After the lessons there was a stimulated recall with each of the teachers' where they were shown videotaped sequences of their acting in the classroom.

We will show here reactions of Mr. P (an analytic thinker) and Mrs. R (a visual thinker) after the students' presentation of their solutions of the lighthouse task. What can be seen here (in the validation phase) is typical also for other phases of the modelling process.

Reaction of Mr P.: "*That was really good. [...] But what I am missing as a maths teacher is that you can use more terms, more abstract terms and that you write down a formula and not only numbers. This way corresponds more to the way that thinking physicians and mathematicians prefer, when you use and transform terms and get a formula afterwards [...]*" [Mr. P. then developed with the pupils a formula after this statement.]

Reaction of Mrs. R.: "*So we have different solutions. But what I recognized and what I missed in our discussion till now is the fact that you are not thinking of what is happening in the reality! When you want to illustrate yourself the lighthouse and the distance to a ship, then think for example of the Dom [name of a famous fair in Hamburg]. I can see the Dom from my balcony. Or, whatever, think of taking off with a plane in the evening and so on. Two kilometres. Is that much? Is that less?*"

So, on the one hand, Mr P. as an analytical thinker obviously focussed less on interpretation and validation. For him, the subsequent formalisation of the task solutions in the form of abstract equations was important. Accordingly, the real situation became less important.

On the other hand, Mrs. R. as a visual thinker interpreted and, above all, validated the modelling processes with the learners. This became evident in her very vivid, reality-based descriptions she provided for the learners.

4. How can modelling be appropriately taught?

4.1. Some implications for teaching

There is, of course, no general “king’s route” for teaching modelling. However, some *implications* from the empirical findings are plausible (though not at all trivial!) for teaching modelling in an effective way.

Implication 1:

⇒ The *criteria for quality teaching* (see ③) have to be considered also for teaching modelling. The substance for quality teaching is constituted by appropriate modelling *tasks*. When treating modelling tasks, a permanent *balance* between maximal independence of students’ and minimal guidance by the teacher ought to be realised.

Implication 2:

⇒ It is important to support students’ *individual* modelling routes and to encourage *multiple solutions*. To this end, teachers have to be familiar with the *task spaces* and to be aware of their own potential preferences for special solutions.

Implication 3:

⇒ Teachers have to know a broad spectrum of *intervention modes*, also and particularly *strategic* interventions.

Implication 4:

⇒ Teachers have to know ways how to support adequate student *strategies* for solving modelling tasks.

A few more remarks on implication 4: For modelling tasks, a specific strategic tool is available, the *modelling cycle*. The *seven* step schema (presented in ①) is appropriate and sometimes even indispensable for research and teaching purposes. For students’, the following *four* step schema (also developed in the DISUM project; compare Blum 2007) seems to be more appropriate.

Four steps to solve a modelling task (“**Solution Plan**”)

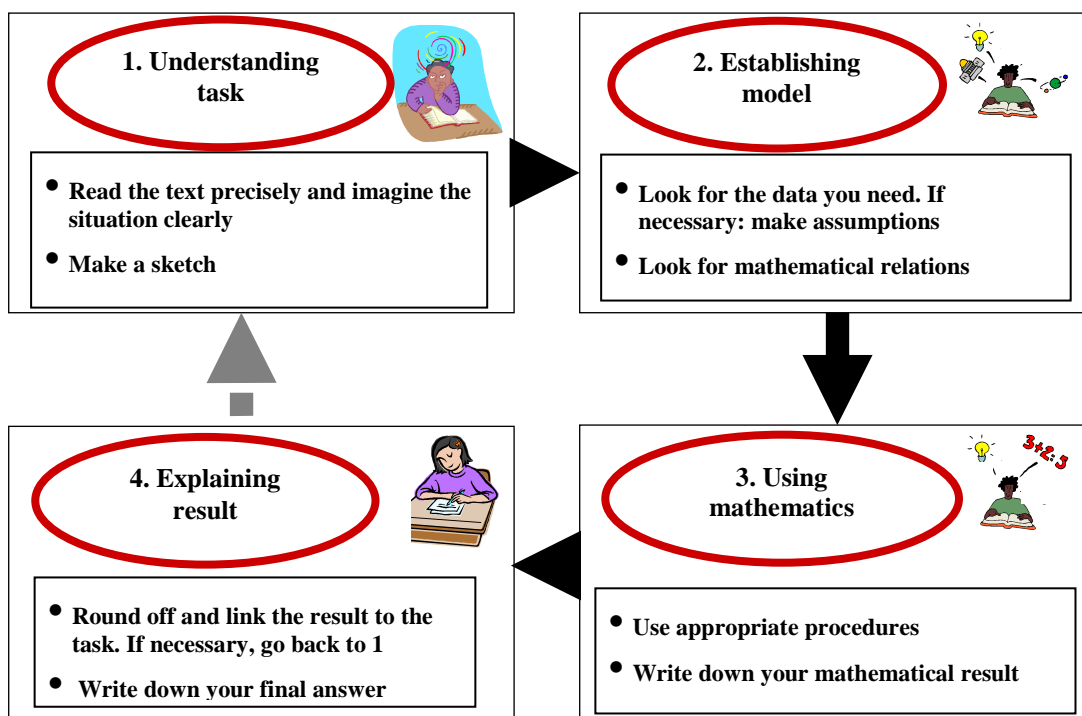


Figure 4 – The “Solution Plan” for modelling tasks

Here, steps 2 and 3 from the seven step schema (fig. 1) are united to one step (“establishing”), as well as steps 5, 6 and 7 (“explaining”). As can be seen, there are some similarities of this “*Solution Plan*” for modelling tasks to George Polya’s general problem solving cycle (compare Polya 1957). This Solution Plan is not meant as a schema that has to be used by students’ but as an aid for difficulties that might occur in the course of the solution process. The goal is that students’ learn to use this plan independently whenever appropriate. Experiences show that a careful and stepwise introduction of this plan is necessary, as well as repeated exercises how to use it. If this is taken into account, even students’ from Hauptschule (low ability track) are able to successfully handle this plan.

4.2. Some encouraging empirical results

We will close by presenting some more encouraging empirical results. In the DISUM project, we have developed a so-called “*operative-strategic*” teaching unit for modelling (to be used in grades 8/9, embedded in the unit on the Pythagorean theorem). The most important guiding principles for this teaching unit were:

- Teaching aiming at students’ active and independent constructions and individual solutions (realising permanently the aspired balance between students’ independence and teacher’s guidance)
- Systematic change between independent work in groups (coached by the teacher) and whole-class activities (especially for comparison of different solutions and retrospective reflections)
- Teacher’s coaching based on the modelling cycle and on individual diagnoses.

In autumn 2006 (with 4 Realschule classes) and in autumn 2007 (with 17 Realschule classes) we have compared the effects of this “operative-strategic” teaching with a so-called “directive” teaching and with students’ working totally alone, both concerning students’ achievement and attitudes. The most important guiding principles for “directive” teaching were:

- Development of common solution patterns by the teacher
- Systematic change between whole-class teaching, oriented towards a fictive “average student”, and students’ individual work in exercises

Both “operative-strategic” and “directive” teaching were conceived as optimised teaching styles and realised by experienced teachers from a reform project (“SINUS”, see Blum/Leiß 2008). All teachers were particularly trained for this purpose. Our study had a classical design:

Ability test / Pre-test / Treatment (10 lessons with various modelling tasks) with accompanying questionnaires / Post-test / Follow-up-test (3 months later)

The tests comprised both modelling tasks and classical “Pythagorean” tasks. According to our knowledge, this study was unique insofar it was a quasi-experimental study with more than 600 students’, yielding both quantitative (tests and questionnaires) and qualitative (videos) data. Since two optimised teaching styles were implemented, one could possibly expect no differences between the two treatments concerning students’ achievement and attitudes. However, there were remarkable differences. Here are some results:

- Both students’ in “operative-strategic” and in “directive” classes made significant progress (not so students’ working alone); the progress of students’ in “operative-strategic” classes was significantly higher and more enduring than for students’ in “directive” classes.
- The progress of “directive” students’ was essentially due to their progress in the “Pythagorean” tasks. Only “operative-strategic” students’ made significant progress in their modelling competency.
- The best results were achieved in those classes where, according to our ratings, the balance between students’ independence and teacher’s guidance was realised best, with a mixture of different kinds of adaptive interventions.

We will report about our study into more detail in another context.

Altogether, these and other results suggest the following answer to the question in the title of this paper: *Mathematical modelling seems to be actually teachable and learnable!* The aim must be, of course, to implement all these insights and ideas into *everyday teaching*. For that, it is necessary to implement these insights into *teacher education*, both in-service and pre-service.

We have reported here on some findings concerning the learning and teaching of mathematical modelling at the lower secondary level. There are, of course, still lot of open questions (compare DaPonte 1993; Niss 2001; ICMI Study 14 Discussion Document Blum et al. 2002); here are two examples of important questions left to answer:

- We know that modelling competency has to be built up in *longterm* learning processes (over years). What is actually achievable regarding long-term competency development?
- Modelling is an important competency, but the goal is a *comprehensive* mathematical education of the students'. How can the interplay between *different* competencies be advanced systematically?

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