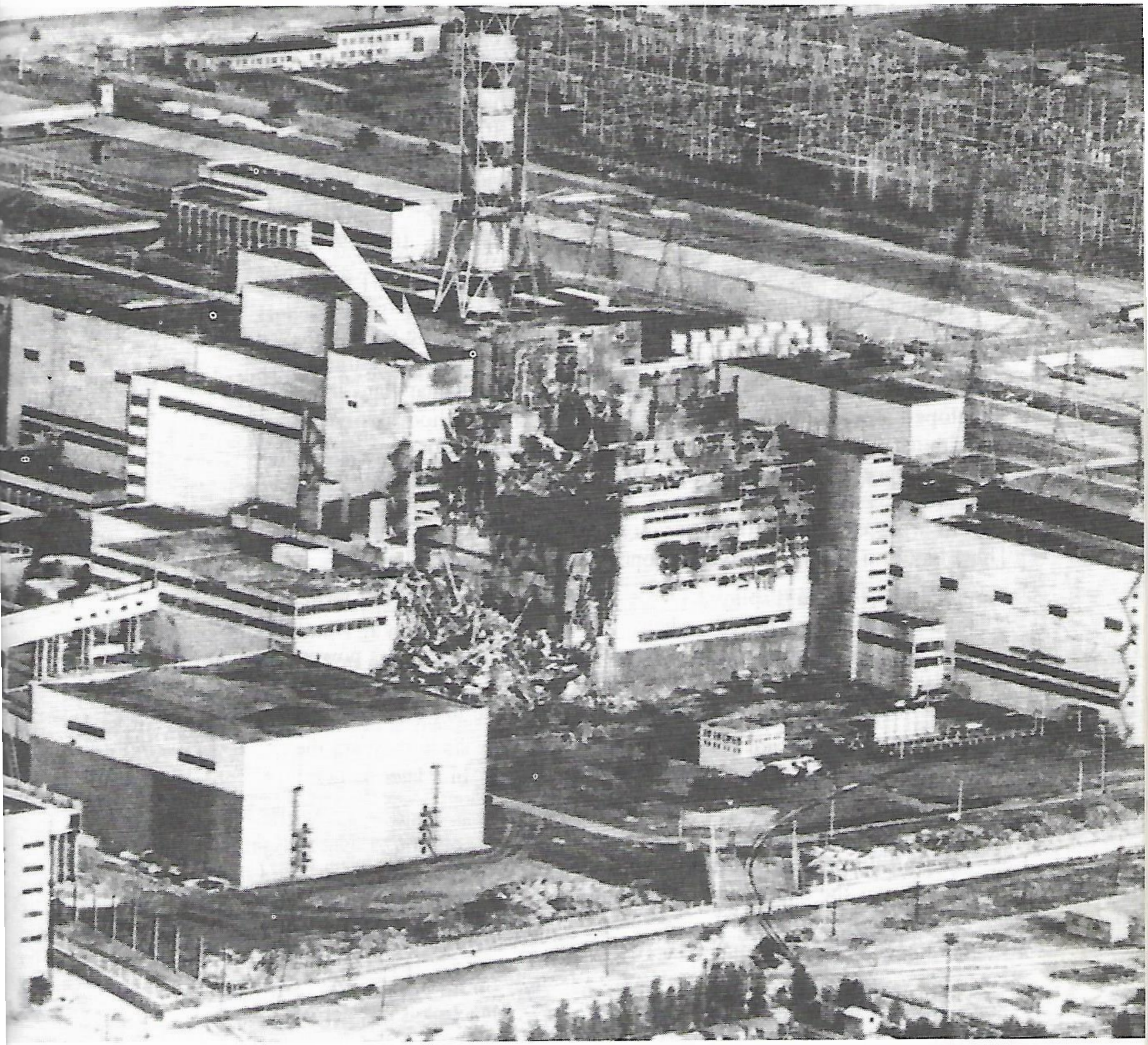
11 Logarithmic Functions



In April 1986 there was a major nuclear accident at the Chernobyl power plant in the Soviet Union. Quantities of radioactive iodine-131 were released into the atmosphere. How long did it take for the level of radiation to reduce to 1% of the level immediately after the accident? (See Section 11-3 *Example 5*.)

11-1 COMMON LOGARITHMS

In *Section 10-4* we encountered problems such as these.

In how many years will an investment double in value at 8% per annum compounded annually?

The answer to this question is the solution of this equation.

$$2 = 1(1.08)^n$$

or $(1.08)^n = 2$

At what depth under water is the light level 10% of the light level at the surface?

The answer to this question is the solution of this equation.

$$10 = 100(0.975)^d$$

or $(0.975)^d = 0.1$

In these equations the variable appears in an exponent. Such equations are called *exponential equations*, and they can be solved to any degree of accuracy by systematic trial. However, exponential equations occur so frequently in applications that mathematicians have developed a more direct method of solution. This method involves logarithms. After defining a logarithm and introducing some of its properties, we will show how the above equations can be solved using logarithms.

In a preceding investigation you may have discovered that the $\boxed{\text{log}}$ key of a calculator gives exponents for powers of 10.

Key in: 100 $\boxed{\text{log}}$ to display 2.
2 is the exponent that 100 has when it is expressed as a power of 10.
Since $100 = 10^2$
we write $\log 100 = 2$

Key in: 0.001 $\boxed{\text{log}}$ to display -3.
-3 is the exponent that 0.001 has when it is expressed as a power of 10.
Since $0.001 = 10^{-3}$
we write $\log 0.001 = -3$

These logarithms are called *common logarithms* since they are the exponents of numbers written as powers with base 10. In a later section we will study logarithms with bases other than 10.

Definition of a Logarithm

- $\log x$ is the exponent that x would have if it were written as a power with base 10.
- $\log x = y$ means that $x = 10^y$.

Since $10^y > 0$ for all real values of y, then $x > 0$. Hence, $\log x$ is defined as a real number only when $x > 0$.

Example 1. Use the definition of a logarithm to evaluate each expression. Check the result with your calculator.

a) $\log 100\,000$ b) $\log 0.01$ c) $\log \sqrt{10}$ d) $\log 1$

Solution.

a) Since $100\,000 = 10^5$, then $\log 100\,000 = 5$
Key in: 100 000 $\boxed{\text{log}}$ to display 5

b) Since $0.01 = 10^{-2}$, then $\log 0.01 = -2$
Key in: 0.01 $\boxed{\text{log}}$ to display -2

c) Since $\sqrt{10} = 10^{0.5}$, then $\log \sqrt{10} = 0.5$
Key in: 10 $\boxed{\sqrt{\ }}$ $\boxed{\text{log}}$ to display 0.5

d) Since $1 = 10^0$, then $\log 1 = 0$
Key in: 1 $\boxed{\log}$ to display 0

We can use the $\boxed{\log}$ key of a calculator to find approximations to the logarithm of any positive number. Hence, we can write any positive number as a power of 10.

Example 2. Use your calculator to evaluate each logarithm. Then write the result in exponential form, and check with the calculator.

a) $\log 7$ b) $\log 500$ c) $\log 0.4$

Solution.

a) Key in: 7 $\boxed{\log}$ to display 0.8450980
Hence, $\log 7 \doteq 0.845\,098$
This means that $10^{0.845098} \doteq 7$
To check, key in: 10 $\boxed{y^x}$ 0.845098 $\boxed{=}$ to display 6.9999993

b) Key in: 500 $\boxed{\log}$ to display 2.6989700
Hence, $\log 500 \doteq 2.698\,97$
This means that $10^{2.69897} \doteq 500$
To check, key in: 10 $\boxed{y^x}$ 2.69897 $\boxed{=}$ to display 499.99999

c) Key in: 0.4 $\boxed{\log}$ to display -0.3979400
Hence, $\log 0.4 \doteq -0.397\,94$
This means that $10^{-0.39794} \doteq 0.4$
To check, key in: 10 $\boxed{y^x}$ 0.39794 $\boxed{+/-}$ $\boxed{=}$ to display 0.4000000

Since $\log x$ is defined as a real number only when $x > 0$, you will get an error message if you attempt to find the logarithm of 0, or of a negative number.

Key in: 0 $\boxed{\log}$ and observe the result.
$\log 0 = y$ means $10^y = 0$, which is impossible.

Key in: 2 $\boxed{+/-}$ $\boxed{\log}$ and observe the result.
$\log(-2) = y$ means $10^y = -2$, which is impossible.

Example 3. Simplify each expression.

a) $\log 10^x$ b) $10^{\log x}$

Solution.

a) $\log 10^x$ is the exponent that 10^x would have if it were written as a power of 10. But, 10^x *is* written as a power of 10, and has exponent x. Hence, $\log 10^x = x$

b) $10^{\log x}$ is 10 raised to the exponent that x would have if x were written as a power of 10. Hence, $10^{\log x} = x$

Example 3 shows that taking a common logarithm of a number and raising the number to a power of 10 are inverse operations, just as squaring a number and taking the square root of the number are inverse operations. If your calculator has a $\boxed{10^x}$ key, you can illustrate this by entering any positive number and then pressing the $\boxed{\log}$ key and the $\boxed{10^x}$ key in either order.

Key in: 4.5 $\boxed{\log}$ $\boxed{10^x}$ to display 4.5

Key in: 4.5 $\boxed{10^x}$ $\boxed{\log}$ to display 4.5

Summary

- A logarithm is an exponent.
- $\log x = y$ means that $x = 10^y$, $x > 0$.
- $\log x$ is defined only when $x > 0$.
- $\log 10^x = x$ and $10^{\log x} = x$

EXERCISES 11-1

(A)

1. Use the definition to evaluate each logarithm.
   a) $\log 100$ b) $\log 1000$ c) $\log 1\,000\,000$ d) $\log 10$
   e) $\log 0.1$ f) $\log 0.001$ g) $\log 1$ h) $\log \sqrt[3]{10}$
   i) $\log 10^5$ j) $\log 10^{\frac{1}{5}}$ k) $\log 10^{\frac{2}{3}}$ l) $\log 10^n$

2. Use your calculator to evaluate each logarithm to 4 decimal places. Then write each result in exponential form, and check it with the calculator.
   a) $\log 5$ b) $\log 18$ c) $\log 62.4$ d) $\log 4877$
   e) $\log 0.25$ f) $\log 0.8$ g) $\log 0.02$ h) $\log 0.006$

3. On February 23, 1987, the Canadian astronomer Ian Shelton discovered a supernova, or exploding star, from an observatory in Chile. State the common logarithm of each number.
   a) The supernova was more than 100 000 light years from the Earth.
   b) One light year is approximately 10^{15} m.
   c) Throughout recorded history only about 10 supernovas have been visible to the unaided eye.
   d) At its brightest, a supernova is about 10^9 times as bright as a star like the sun.
   e) A supernova liberates about 10^{30} times as much energy as would be unleashed by the detonation of all the nuclear weapons on the Earth.

4. On a single optical disk, an amount of data equivalent to all the text appearing in 15 years of daily newspapers can be recorded. State the common logarithm of each number.
   a) More than 10^{12} bytes of data are recorded on each disk.
   b) To avoid errors, a laser beam is focused within 10^{-7} m of dead centre for each pit on the surface of the disk.
   c) The error rate for a typical disk is 10^{-12}.

(B)

5. Write in exponential form.
   a) $\log 10\,000 = 4$ b) $\log 10 = 1$ c) $\log 0.01 = -2$

6. Write in logarithmic form.
   a) $10^3 = 1000$ b) $10^0 = 1$ c) $10^{-3} = 0.001$

7. The centillion is defined as the 100th power of 1 000 000. What is the common logarithm of one centillion?

8. Solve each equation.
   a) $\log x = 2$ b) $\log x = 5$ c) $\log x = -3$
   d) $\log x = 0$ e) $\log x = 1$ f) $\log \log x = 1$

9. a) Simplify each expression.
   i) $\log 10^4$ ii) $\log 10^5$ iii) $\log 10^{-3}$
   iv) $\log 10^{0.5}$ v) $\log 10^{2.4}$ vi) $\log 10^{-1.5}$
   b) Based on the results of part a), state a general result.

10. a) Simplify each expression.
    i) $10^{\log 100}$ ii) $10^{\log 1000}$ iii) $10^{\log 0.01}$
    iv) $10^{\log 20}$ v) $10^{\log 500}$ vi) $10^{\log 0.2}$
    b) Based on the results of part a), state a general result.

11. a) Use your calculator to evaluate each logarithm.
    i) $\log 2$ ii) $\log 20$ iii) $\log 200$ iv) $\log 2000$
    v) $\log 0.2$ vi) $\log 0.02$ vii) $\log 0.002$ viii) $\log 0.0002$
    b) Account for the pattern in the results.

INVESTIGATE

You can use your calculator to find an approximation to the common logarithm of a number *without* using the $\boxed{\log}$ key. The method depends on trying to find a power of that number which is close to a power of 10.

For example, to find an approximation to log 2, try different powers of 2 such as 2^6, 2^7, and 2^8. Notice that $2^{10} = 1024$, which is approximately 10^3. Hence, we may write:

$$2^{10} \doteq 10^3$$

Take the 10th root of both sides.

$$2 \doteq 10^{\frac{3}{10}}$$

Hence, $\log 2 \doteq \frac{3}{10}$, or 0.3

a) Use this method to construct a table of approximations to the logarithms of the integers from 2 to 9.
b) Compare the results with the actual values obtained using the $\boxed{\log}$ key. Which of the approximations is the most accurate?
c) Compare your approximations for log 2, log 4, and log 8. How are they related? Can you explain this relationship?
d) Repeat part c) for log 3 and log 9.

11-2 THE LAWS OF LOGARITHMS FOR MULTIPLICATION AND DIVISION (BASE 10)

A logarithm is an exponent. Hence, it should be possible to write the laws of exponents in logarithmic form.

Consider an example of the law of exponents for multiplication, such as $10^2 \times 10^3 = 10^5$. Since $\log 10^2 = 2$, $\log 10^3 = 3$, and $\log 10^5 = 5$, we can write this equation as:

$\log 10^2 + \log 10^3 = \log 10^5$

or $\quad \log 10^5 = \log 10^2 + \log 10^3$

This example suggests that a possible law of logarithms for multiplication might be $\log xy = \log x + \log y$. This equation states that the exponent that xy would have if it were expressed as a power of 10 is equal to the sum of the exponents that x and y would have if they were expressed as powers of 10. We now prove this law.

Theorem **Law of Logarithms for Multiplication (Base 10)**

If x and y are any positive real numbers, then $\log xy = \log x + \log y$.

Given: Two real numbers x and y

Required to Prove: $\log xy = \log x + \log y$

Proof: Let $\log x = M$ and $\log y = N$

$x = 10^M \qquad y = 10^N$

$$\text{Hence, } xy = (10^M)(10^N) = 10^{M+N}$$

$$\text{Therefore, } \log xy = \log (10^{M+N}) = M + N = \log x + \log y$$

Corollary **Law of Logarithms for Division (Base 10)**

If x and y are any positive real numbers, then $\log \left(\frac{x}{y}\right) = \log x - \log y$.

Example 1. a) Write log 6 as:

i) a sum of two logarithms ii) a difference of two logarithms.

b) Check the results of part a) with a calculator.

Solution. a) i) Since $6 = 2 \times 3$, then by the law of logarithms for multiplication, $\log 6 = \log 2 + \log 3$

ii) Since $6 = 12 \div 2$, then by the law of logarithms for division, $\log 6 = \log 12 - \log 2$

b) i) Key in: 6 $\boxed{\log}$ to display 0.7781512

Key in: 2 $\boxed{\log}$ $\boxed{+}$ 3 $\boxed{\log}$ $\boxed{=}$ to display 0.7781512

ii) Key in: 12 $\boxed{\log}$ $\boxed{-}$ 2 $\boxed{\log}$ $\boxed{=}$ to display 0.7781512

In *Example 1*, 6 can be expressed as a sum or a difference of logarithms in infinitely many other ways, such as:
$\log 6 = \log 1.5 + \log 4$ $\quad$ $\log 6 = \log 18 - \log 3$
$\log 6 = \log 10 + \log 0.6$ $\quad$ $\log 6 = \log 60 - \log 10$
Check these results with your calculator.

Example 2. Write each expression as a single logarithm, and check with a calculator.
a) $\log 5 + \log 4$ $\qquad$ b) $\log 21 - \log 3$

Solution. a) $\log 5 + \log 4 = \log (5 \times 4)$
$= \log 20$
Key in: 5 $\boxed{\log}$ $\boxed{+}$ 4 $\boxed{\log}$ $\boxed{=}$ to display 1.3010300
Key in: 20 $\boxed{\log}$ to display 1.3010300

b) $\log 21 - \log 3 = \log \left(\frac{21}{3}\right)$
$= \log 7$
Key in: 21 $\boxed{\log}$ $\boxed{-}$ 3 $\boxed{\log}$ $\boxed{=}$ to display 0.8450980
Key in: 7 $\boxed{\log}$ to display 0.8450980

Example 3. Given that $\log 5 \doteq 0.698\ 97$, find an approximation for each logarithm.
a) $\log 50$ $\qquad$ b) $\log 500$ $\qquad$ c) $\log 0.5$ $\qquad$ d) $\log 0.05$

Solution.
a) $\log 50 = \log 10 + \log 5$
$\doteq 1 + 0.698\ 97$
$\doteq 1.698\ 97$

b) $\log 500 = \log 100 + \log 5$
$\doteq 2 + 0.698\ 97$
$\doteq 2.698\ 97$

c) $\log 0.5 = \log 5 - \log 10$
$\doteq 0.698\ 97 - 1$
$\doteq -0.301\ 03$

d) $\log 0.05 = \log 5 - \log 100$
$\doteq 0.698\ 97 - 2$
$\doteq -1.301\ 03$

Check the results of *Example 3* with your calculator.

Laws of Logarithms for Multiplication and Division (Base 10)
$\log xy = \log x + \log y \qquad x,y > 0$
$\log \left(\frac{x}{y}\right) = \log x - \log y \qquad x,y > 0$

These laws are the laws of exponents for multiplication and division of powers (with base 10) restated in logarithmic form. They allow us to expand logarithms of products and quotients, and to write sums and differences of logarithms as single logarithms.

Example 4. Write $\log \left(\frac{3a}{2b}\right)$ in terms of $\log a$ and $\log b$.

Solution.
$$\begin{aligned}\log \left(\frac{3a}{2b}\right) &= \log (3a) - \log (2b)\\ &= \log 3 + \log a - \log 2 - \log b\\ &= \log 3 - \log 2 + \log a - \log b\\ &= \log \left(\frac{3}{2}\right) + \log a - \log b\\ &= \log 1.5 + \log a - \log b\end{aligned}$$

If desired, the term log 1.5 in the solution of *Example 4* can be evaluated with a calculator, and the result can be approximated as $0.176\ 091\ 3 + \log a - \log b$.

Example 5. Write as a single logarithm. $\log a + \log b - \log c$

Solution. $\log a + \log b - \log c = \log ab - \log c$

$$= \log\left(\frac{ab}{c}\right)$$

EXERCISES 11-2

Ⓐ

1. Write as a single logarithm, and check with your calculator.
   a) log 6 + log 7 b) log 24 − log 6 c) log 3 + log 8
   d) log 35 − log 5 e) log 12 + log 7 f) log 42 − log 3
   g) log 64 − log 2 h) log 1 − log 2 i) log 17 + log 8

2. Write as a sum of logarithms, and check with your calculator.
   a) log 10 b) log 21 c) log 28 d) log 36
   e) log 9 f) log 44 g) log 57 h) log 121

3. Write as a difference of logarithms, and check with your calculator.
   a) log 5 b) log 8 c) log 12 d) log 13
   e) log 10 f) log 21 g) log 17 h) log 40

4. Write as a single logarithm, and check with your calculator.
   a) log 2 + log 3 + log 5 b) log 3 + log 4 + log 7
   c) log 5 + log 8 − log 4 d) log 6 + log 3 + log 5
   e) log 12 − log 4 + log 7 f) log 7 + log 8 − log 2

Ⓑ

5. Given $\log 3 \doteq 0.477\ 12$, find an approximation for each logarithm.
   a) log 30 b) log 3000 c) log 0.3 d) log 0.003

6. If $\log 70 \doteq 1.8451$, find an approximation for each logarithm.
   a) log 7 b) log 700 c) log 0.07
   d) log 0.7 e) log 700 000 f) log 0.007

7. Write in terms of log a and log b.
   a) $\log\left(\frac{2a}{3b}\right)$ b) $\log\left(\frac{7a}{2b}\right)$ c) $\log\left(\frac{5b}{2a}\right)$ d) $\log\left(\frac{12a}{5b}\right)$

8. Write as a single logarithm.
   a) $\log x + \log y - \log z$
   b) $\log m - (\log n + \log p)$
   c) $\log a + \log b - \log c - \log d$
   d) $\log a + \log (a + b) - \log (a - b)$
   e) $\log (m + 3) + \log (m + 7)$
   f) $\log (2x - y) + \log (3x + 2y) - \log (x + y)$

9. Write as a single logarithm. For what values of the variable is each expression not defined?
   a) $\log (x + 3) - \log (x - 1)$ b) $\log (2x - 7) - \log (x + 3)$
   c) $-\log (a - 2) + \log (a + 2)$ d) $\log (8a + 15) - \log (2a + 3)$

10. If $\log 2 = x$ and $\log 3 = y$, write each logarithm as an expression in x and y.
    a) $\log 6$ b) $\log 1.5$ c) $\log 60$ d) $\log 12$
    e) $\log 18$ f) $\log 36$ g) $\log 3.6$ h) $\log \left(\frac{1}{6}\right)$

11. Prove the corollary on page 464 using:
    a) a method similar to the one used to prove the theorem
    b) the result of the theorem.

12. Assume that x and y are natural numbers, and $x > y$. How many solutions does each equation have?
    a) $\log 24 = \log x + \log y$ b) $\log 24 = \log x - \log y$

13. Prove the law of logarithms for three factors. $\log xyz = \log x + \log y + \log z$

Ⓒ

14. Prove each identity, and state the value(s) of x for which the identity is true.
    a) $\log (x - 1) + \log (x - 2) = \log (x^2 - 3x + 2)$
    b) $\log x + \log (x + 3) = \log (x^2 + 3x)$
    c) $\log (x - 5) + \log (x + 5) = \log (x^2 - 25)$

15. Solve and check.
    a) $\log (x + 2) + \log (x - 1) = 1$
    b) $\log (3x + 2) + \log (x - 1) = 2$
    c) $2 \log (x - 1) = 2 + \log 2$

16. Express y as a function of x. What is the domain?
    a) $\log 3 + \log y = \log (x + 2) - \log x$
    b) $\log y - 2 + \log x - \log (x + 1) = 0$
    c) $\log 4y = x + \log 4$

17. a) By writing $x = 10^{\log x}$, $y = 10^{\log y}$, and $xy = 10^{\log xy}$, prove the law of logarithms for multiplication.
    b) Use this method to prove the law of logarithms for division.

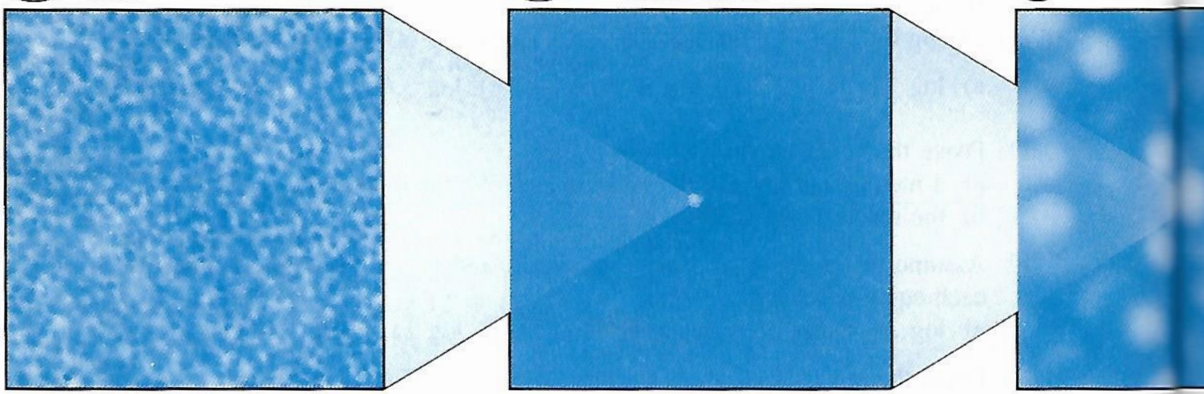
INVESTIGATE

1. Use your calculator to evaluate each logarithm and account for the pattern in the results. $\log 2$, $\log 4$, $\log 8$, $\log 16$
2. Repeat *Question 1*, but round the results to 5 decimal places. Do you see another pattern? Extend the pattern by evaluating these logarithms to 5 decimal places.
   $\log 32$, $\log 64$, $\log 128$, $\log 256$, . . .
   What is the first power of 2 for which the pattern in the rounded values breaks down?

MATHEMATICS AROUND US

Orders of Magnitude

−15 Proton in carbon nucleus −12 Carbon nucleus −9



10^{-15} m 1 fm (femtometre) 10^{-12} m 1 pm (picometre) 10^{-9} m

Scientists have always wanted to extend our range of observation of the world around us, from the microscopic scale to the astronomic scale. What might we see if we could take an imaginary journey along a straight line beginning at the nucleus of an atom and ending at the farthermost reaches of outer space?

The first illustration shows part of the nucleus of a carbon atom. As we get farther and farther away, greater and greater distances are brought into view. The steps we take in this journey are not regular steps, but rather, each step is 1000 times as great as the previous one.

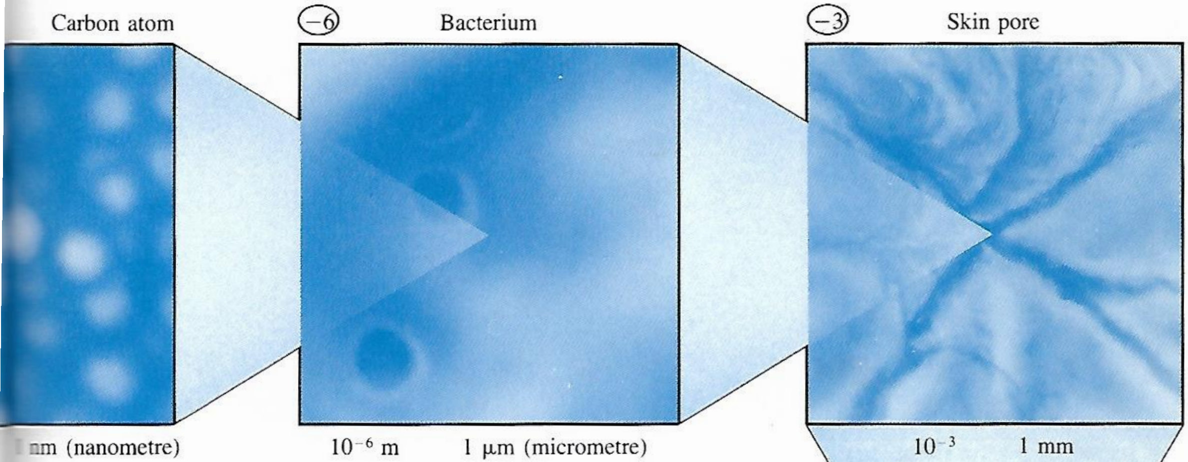
Hence, the dimensions of each illustration represent a distance 1000 times as long as the one before it. And, each illustration shows a 1000× enlargement of a small portion at the centre of the next one. Although it can be seen in only the first illustration, the nucleus of the carbon atom where we started the journey is at the centre of all of them.

The journey covers four pages in this book. Study the illustrations on all four pages before you begin the questions.

QUESTIONS

1. Notice the circled number in the upper left corner of each illustration.
   a) How is this number related to the distance represented by the illustration?
   b) As you move from one illustration to the next, compare the change in the circled number with the change in the distance represented by the illustration.

2. A factor of 10 is called one *order of magnitude*. Hence, a factor of 100, or 10×10, represents two orders of magnitude. How many orders of magnitude are represented by the change from:
   a) any illustration to the next
   b) the first illustration to the last?



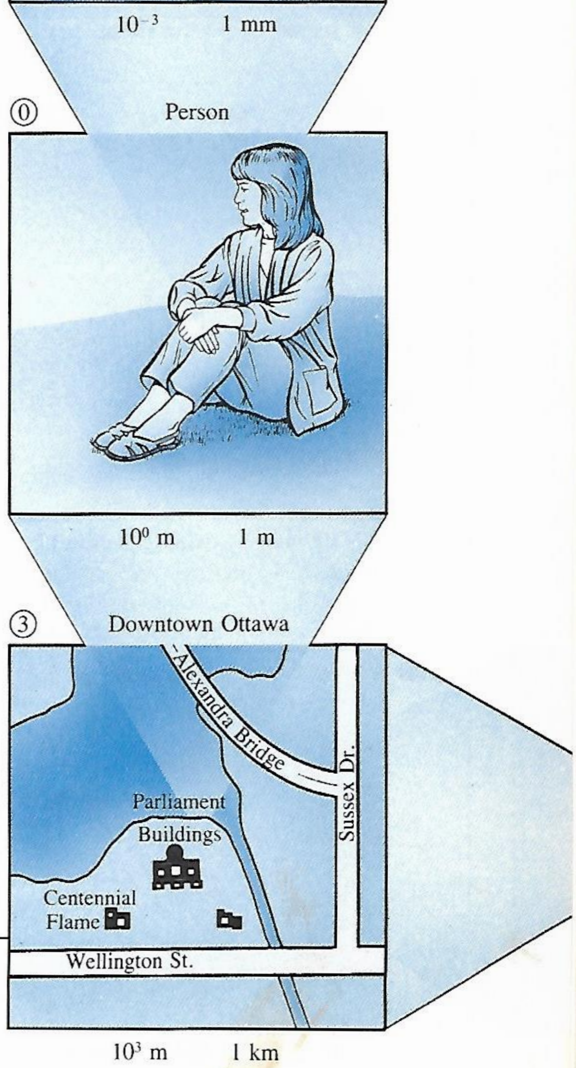
3. Two common units of length are the Ångstrom unit (used for measuring atoms) and the fermi (used for measuring nuclear particles).

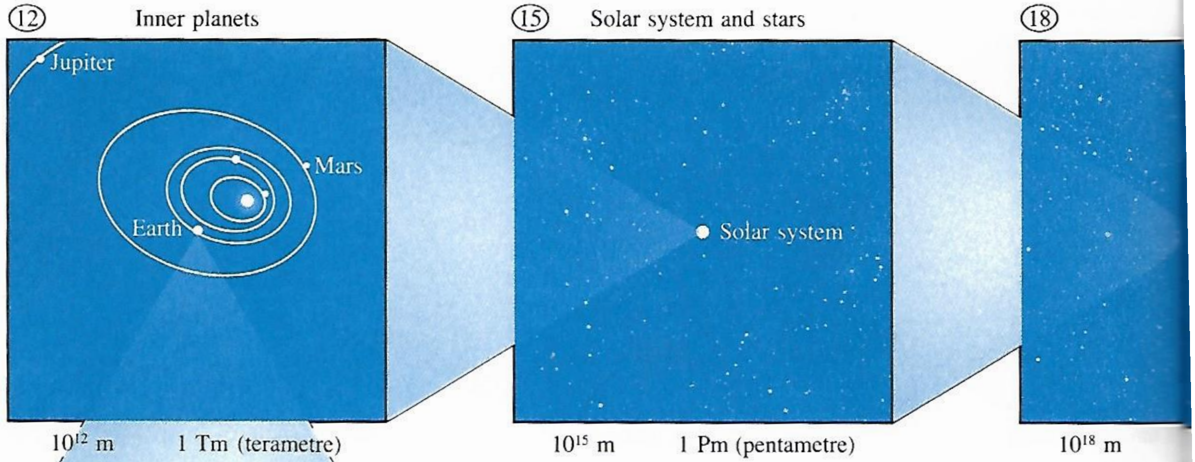
1 Ångstrom unit	10^{-10} m
1 fermi	10^{-15} m

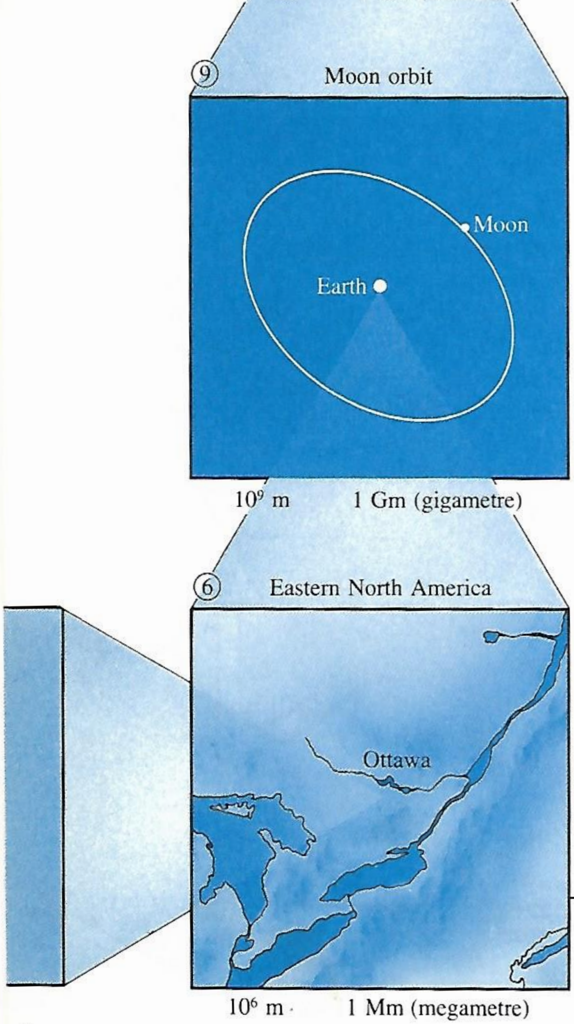
   a) How many orders of magnitude is the Ångstrom unit greater than the fermi?
   b) Name two other units of length that differ by the same order of magnitude.

4. The double-helix strands of a DNA molecule are approximately 2×10^{-9} m apart. If the twisted molecule were stretched out, its length would be 7 orders of magnitude greater. How long is the molecule?

5. What common interval of time is approximately 4 orders of magnitude longer than one minute?







	Diameter (m)
Sun	1.4×10^9
Earth	1.3×10^7
Moon	3.5×10^6

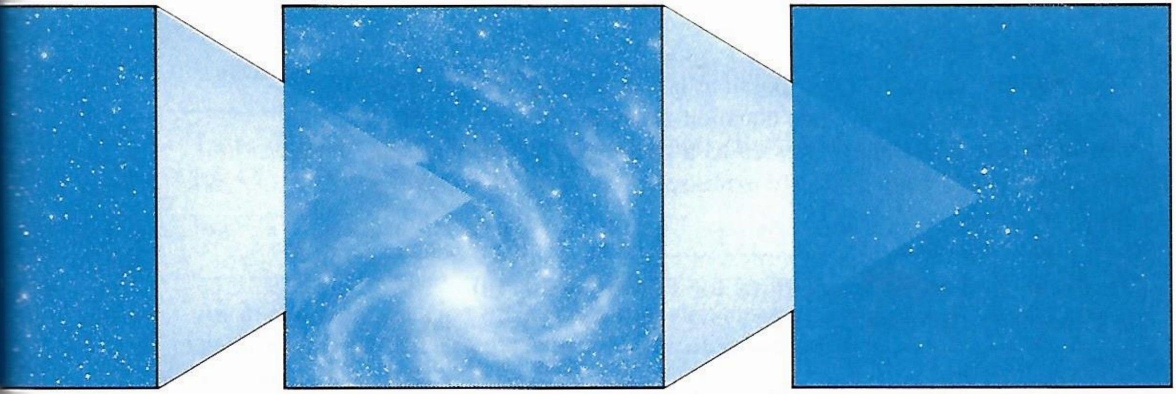
Order of magnitude is a relative term, and refers to a comparison of two numbers using powers of 10. For example, the diameter of the sun is approximately 100, or 10^2 times the diameter of the Earth. We say that the diameter of the sun is approximately 2 orders of magnitude greater than that of the Earth. This statement means that:

$$\frac{\text{diameter of sun}}{\text{diameter of Earth}} \doteq 10^2$$

or, $\log\left(\frac{\text{diameter of sun}}{\text{diameter of Earth}}\right) \doteq 2$

Hence, the difference in their orders of magnitude is the logarithm of their quotient. This example suggests the following definition.

Definition: If a and b are any two positive numbers, the difference in their orders of magnitude is $\log\left(\frac{a}{b}\right)$.



6. Show that the diameter of the sun is approximately 2.6 orders of magnitude greater than that of the moon.

7. Two common units of length are the astronomical unit (used for measuring planetary distances) and the light year (used for measuring stellar and galactic distances).

1 astronomical unit	1.5×10^{11} m
1 light year	9.5×10^{15} m

How many orders of magnitude is the light year greater than the astronomical unit?

8. The planets Neptune and Pluto are approximately 5×10^{12} m from the Earth. How many orders of magnitude greater than this are these distances?
   a) The nearest star, Proxima Centauri, 4×10^{18} m from Earth
   b) The centre of the Milky Way Galaxy, 6.7×10^{20} m from Earth
   c) A chain of galaxies 7×10^{24} m from Earth

9. In 1989, the space probe *Voyager II* will photograph the planet Neptune, about 5×10^{12} m from the Earth. The *Space Telescope* will be able to examine objects 13.4 orders of magnitude farther than this. What is the limit of observation of the Space Telescope?

10. The limit of the known universe is about 2.3 orders of magnitude greater than the distance represented by the last illustration above. How many metres is this?

11. Now that we have finished our journey from the nucleus of the carbon atom to outer space, suppose we reverse our direction and take the return trip back to the nucleus of the carbon atom where we started. What percent of the remaining distance would we cover from one illustration to the next?

11-3 THE LAWS OF LOGARITHMS FOR POWERS AND ROOTS (BASE 10)

The law of logarithms for products may be applied when the factors are equal. For example, if $x = y$, then the law:

$$\log xy = \log x + \log y \quad \text{may be written}$$

$$\log (x)(x) = \log x + \log x$$

or $$\log (x^2) = 2 \log x$$

This example suggests that a possible law of logarithms for powers might be $\log (x^n) = n \log x$. This equation states that the exponent that x^n would have if it were expressed as a power of 10 is n times the exponent that x would have if it were expressed as a power of 10. We now prove this law.

> **Theorem** **Law of Logarithms for Powers (Base 10)**
> If x and n are real numbers, and $x > 0$, then $\log (x^n) = n \log x$

Given: Two real numbers x and n, where $x > 0$
Required to Prove: $\log (x^n) = n \log x$
Proof: Let $\log x = M$

$$x = 10^M$$

Hence, $x^n = (10^M)^n$
$= 10^{nM}$

Therefore, $\log (x^n) = \log (10^{nM})$
$= nM$
$= n \log x$

> **Corollary** **Law of Logarithms for Roots (Base 10)**
> If x and n are real numbers, and $x > 0$, then $\log \sqrt[n]{x} = \frac{1}{n} \log x$

Example 1. a) i) Write log 125 as a product of a whole number and a logarithm.
ii) Write 4 log 3 as a single logarithm.
b) Check the results of part a) with a calculator.

Solution. a) i) Since $125 = 5^3$, then $\log 125 = \log (5^3)$
$= 3 \log 5$

ii) $4 \log 3 = \log (3^4)$, or $\log 81$

b) i) Key in: 125 $\boxed{\log}$ to display 2.0969100
Key in: 5 $\boxed{\log}$ $\boxed{\times}$ 3 $\boxed{=}$ to display 2.0969100
ii) Key in: 3 $\boxed{\log}$ $\boxed{\times}$ 4 $\boxed{=}$ to display 1.9084850
Key in: 3 $\boxed{y^x}$ 4 $\boxed{=}$ $\boxed{\log}$ to display 1.9084850

Example 2. Given that $\log 2 \doteq 0.301\ 03$, find an approximation for each logarithm.
a) $\log 8$ b) $\log \sqrt[3]{2}$

Solution. a) $\log 8 = \log (2^3)$
$= 3 \log 2$
$\doteq 3(0.301\ 03)$
$\doteq 0.903\ 09$

b) $\log \sqrt[3]{2} = \log (2^{\frac{1}{3}})$
$= \frac{1}{3} \log 2$
$\doteq \frac{1}{3}(0.301\ 03)$
$\doteq 0.100\ 34$

Check the results of *Example 2* with your calculator.

Laws of Logarithms for Powers and Roots (Base 10)

$\log (x^n) = n \log x \qquad x > 0$

$\log \sqrt[n]{x} = \frac{1}{n} \log x \qquad x > 0$

An important application of the laws of logarithms is to the problem of expressing any positive number as a power of any other positive number (except 1).

Example 3. Express 19 as a power of 2 and check with a calculator.

Solution. Let $19 = 2^x$
Take the logarithm of each side.
$\log 19 = \log (2^x)$
$\log 19 = x \log 2$
Hence, $x = \dfrac{\log 19}{\log 2}$
Key in: 19 $\boxed{\log}$ $\boxed{\div}$ 2 $\boxed{\log}$ $\boxed{=}$ to display 4.2479275
Therefore, $19 \doteq 2^{4.2479275}$
To check, key in: 2 $\boxed{y^x}$ 4.2479275 $\boxed{=}$ to display 19.000000

Recall that in *Section 11-1* we mentioned that logarithms can be used to solve exponential equations, in which the variable appears as an exponent. Such an equation occurred in *Example 3*. As the solution of *Example 3* indicates, we can solve an exponential equation by taking the logarithm of each side and using the law of logarithms. We can now solve applied problems such as those at the beginning of *Section 11-1*.

Example 4. a) In how many years will an investment double in value at 8% per annum compounded annually?
b) At what depth under water is the light level 10% of the light level at the surface?

Solution. a) The answer is the solution of this equation.
$2 = 1(1.08)^n$
or $(1.08)^n = 2$
Take the logarithm of each side.
$n \log 1.08 = \log 2$
$$n = \frac{\log 2}{\log 1.08}$$

Key in: 2 $\boxed{\log}$ $\boxed{\div}$ 1.08 $\boxed{\log}$ $\boxed{=}$ to display 9.0064683
Hence, $n \doteq 9$
The investment will double in approximately 9 years.

b) The answer is the solution of this equation.

$$10 = 100(0.975)^d$$

or $(0.975)^d = 0.1$

Take the logarithm of each side.

$$d \log 0.975 = \log 0.1$$

$$d = \frac{\log 0.1}{\log 0.975}$$

Key in: 0.1 $\boxed{\log}$ $\boxed{\div}$ 0.975 $\boxed{\log}$ $\boxed{=}$ to display 90.947253
Hence, $d \doteq 91$
At a depth of about 91 m, approximately 10% of surface light is present.

Exponential equations frequently arise in applications involving exponential growth and decay.

Example 5. In April 1986 there was a major nuclear accident at the Chernobyl power plant in the Soviet Union. The atmosphere was contaminated with quantities of radioactive iodine-131, which has a half life of 8.1 days. How long did it take for the level of radiation to reduce to 1% of the level immediately after the accident?

Solution. Let P represent the percent of the original radiation that was present after t days. Then, since the halflife is 8.1 days,

$$P = 100\left(\frac{1}{2}\right)^{\frac{t}{8.1}}$$

Substitute 1 for P and solve for t by taking the logarithm of each side.

$$1 = 100(0.5)^{\frac{t}{8.1}}$$

$$\log 1 = \log 100 + \frac{t}{8.1} \log 0.5$$

$$0 = 2 + \frac{t}{8.1} \log 0.5$$

$$t = -\frac{16.2}{\log 0.5}$$

Key in: 0.5 $\boxed{\log}$ $\boxed{1/x}$ $\boxed{\times}$ 16.2 $\boxed{+/-}$ $\boxed{=}$ to display 53.815235
Hence, $t \doteq 54$
It took about 54 days for the level of radiation to reduce to 1% of the level immediately after the accident.

Another application of the laws of logarithms is to the problem of finding approximations to very large powers.

Example 6. In 1952, Raphael M. Robinson used a SWAC computer to prove that $2^{3217} - 1$ is a prime number.

a) Write an approximation to this number in scientific notation.

b) How many digits does this number have?

Solution. a) The number 2^{3217} is so large that we can ignore the subtraction of 1 from it.

$$\text{Let } 2^{3217} = 10^x$$

Take the logarithm of each side.

$$\log (2^{3217}) = \log (10^x)$$
$$3217 \log 2 = x$$

Key in: 2 $\boxed{\log}$ $\boxed{\times}$ 3217 $\boxed{=}$ to display 968.4135

$$\text{Therefore, } 2^{3217} \doteq 10^{968.4135}$$
$$\doteq 10^{0.4135} \times 10^{968}$$

Key in: 10 $\boxed{y^x}$ 0.4135 $\boxed{=}$ to display 2.5911944

Hence, $2^{3217} - 1 \doteq 2.591\ 194\ 4 \times 10^{968}$

b) Since 10^{968} has 969 digits, then $2^{3217} - 1$ has 969 digits.

Example 7. a) Write $\log (10x^2)$ in terms of $\log x$.

b) Write as a single logarithm. $\log a + 3 \log b - \frac{1}{2} \log c$

Solution. a) $\log (10x^2) = \log 10 + \log (x^2)$
$$= 1 + 2 \log x$$

b) $\log a + 3 \log b - \frac{1}{2} \log c = \log a + \log (b^3) - \log \sqrt{c}$
$$= \log \left(\frac{ab^3}{\sqrt{c}}\right), \text{ or } \log (ab^3c^{-\frac{1}{2}})$$

EXERCISES 11-3

Ⓐ

1. Write as a product of a whole number and a logarithm, and check with your calculator.

a) log 9 b) log 25 c) log 8 d) log 27
e) log 1000 f) log 32 g) log 343 h) log 128

2. Write as a single logarithm, and check with your calculator.

a) 2 log 6 b) 3 log 4 c) 2 log 9 d) 2 log 7
e) 5 log 3 f) 4 log 2 g) 3 log 6 h) 5 log 10

3. Given that $\log 3 \doteq 0.477\ 12$, find an approximation for each logarithm.

a) log 9 b) log 81 c) $\log \sqrt{3}$ d) $\log \sqrt[5]{3}$

4. Given that $\log 5 \doteq 0.698\ 97$, find an approximation for each logarithm.

a) log 625 b) $\log \sqrt[3]{5}$ c) log 0.2 d) log 0.04

(B)

5. Express.
   a) 7 as a power of 3 b) 5 as a power of 2 c) 29 as a power of 2
   d) 77 as a power of 8 e) 3 as a power of 0.5 f) 0.45 as a power of 6

6. Solve to the nearest thousandth.
   a) $2^x = 11$ b) $3^x = 17$ c) $6^x = 5$
   d) $5^{x-1} = 9$ e) $2^{x+3} = 6$ f) $5^{1+x} = 2^{1-x}$

7. Solve.
   a) $3^x = 2$ b) $4^x = 5$ c) $7^{-x} = 3$
   d) $3^{1-x} = 5$ e) $\left(\frac{1}{8}\right)^x = 25$ f) $5^{3x} = 41$

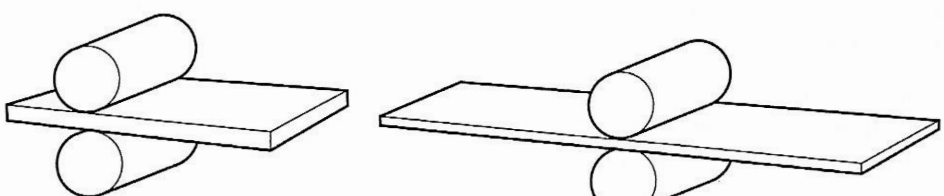
8. Write each expression in terms of $\log x$.
   a) $\log \sqrt{x}$ b) $\log \sqrt{10}x$ c) $\log \sqrt{10x}$ d) $\log 10x$ e) $\log 10\sqrt{x}$

9. Write as a single logarithm.
   a) $2 \log a + 5 \log b$ b) $3 \log x + \frac{1}{2} \log y$
   c) $2 \log m + \log n - 5 \log p$ d) $\frac{1}{2} \log x - 2 \log y - \log z$
   e) $3 \log a + \frac{1}{2} \log b - \frac{5}{4} \log c$ f) $10 \log a - 3 \log b + \frac{1}{2} \log c - \log d$

10. When strontium-90 decays, the percent P remaining is expressed as a function of the time t years by the equation $P \doteq 100(2)^{-0.0357t}$. (See *Example 4*, page 434.) How long is it until the percent remaining is: a) 10% b) 1%?

11. The halflives of iodine-131 and cesium-144 were given in *Exercise 11*, page 436. How long is it until the percent remaining of each substance is: a) 10% b) 0.1%?

12. In a steel mill, red-hot slabs of steel are pressed many times between heavy rollers. The drawings show two stages in rolling a slab.



A slab is 2.00 m long and 0.120 m thick. On each pass through the rollers, its length increases by 20%.
   a) Write the equation which expresses the length L metres of the slab as an exponential function of the number of passes n through the rollers.
   b) How many passes are needed to increase the length of the slab to 50 m?

13. a) For the slab in *Exercise 12*, by what factor does the thickness of the slab decrease on each pass through the rollers? Assume the width is constant.
    b) Write an equation which expresses the thickness t metres of the slab as an exponential function of the number of passes n through the rollers.
    c) How many passes are needed to reduce the thickness of the slab to 0.001 m?
    d) How long would the slab be when its thickness is 0.001 m?

14. x and y are two positive numbers. How are $\log x$ and $\log y$ related if:

    a) $y = 10x$ b) $y = \frac{1}{x}$ c) $y = x^2$

    d) $y = \sqrt{x}$ e) $y = 10\sqrt{x}$ f) $y = \sqrt{10x}$?

15. a) What is the first digit in 2^{1000}?
    b) How many digits are there in 2^{1000}?
    c) What is the last digit in 2^{1000}?

16. The table shows some large prime numbers that were discovered using computers. How many digits does each prime number have?

	Prime Number	Year	Computer
a)	$2^{11213} - 1$	1963	ILLIAC-II
b)	$2^{21701} - 1$	1978	CDC-CYBER-174
c)	$2^{132049} - 1$	1983	CRAY-1
d)	$2^{216091} - 1$	1985	CRAY-1

17. In 1938 the physicist Sir Arthur Eddington calculated that the number of particles in the universe is 33×2^{259}. He called this number the *cosmical number*.
    a) Write the cosmical number in scientific notation.
    b) How many digits are there in this number?

18. How many digits are there in the number 9^{9^9}?

19. In 1951 the UNIVAC computer performed approximately 1000 arithmetic operations per second. Since then, the speed of computers has doubled, on the average, about every 2 years.
    a) Express the number of operations per second N as an exponential function of the time n years since 1951.
    b) Predict when computers will be able to perform a billion operations per second. What assumption are you making?

20. After every 10 pages of printing, the ribbon in a dot matrix printer loses about 0.5% of its ink.
    a) What percent of the ink is left in the ribbon after printing:
       i) 100 pages ii) 1000 pages?
    b) Write an equation which expresses the percent P of ink left in the ribbon as a function of the number n of pages printed.
    c) Approximately how many pages can be printed before the ink content is reduced to 75% of its original content?

21. On each bounce a ball rises to 70% of the height from which it fell. Let us agree that, for all practical purposes, the ball stops bouncing when the height to which it rises is only 0.1% of the height from which it was dropped originally. How many bounces will this take?

22. Prove the corollary on page 472 using:
 a) a method similar to the one used to prove the theorem
 b) the result of the theorem.

23. Prior to *Example 3* in this section it was stated that any positive number can be expressed as a power of any other positive number, except 1. Give two reasons why the number 1 is excepted.

24. a) Evaluate each expression without using a calculator.
 i) 2^5 ii) $(-2)^5$ iii) $\sqrt[5]{32}$ iv) $\sqrt[5]{-32}$
 b) Attempt to evaluate each expression in part a) using the $\boxed{y^x}$ key on your calculator. Which expressions result in error messages?
 c) If there was an error message, the chips in your calculator may be programmed to evaluate powers using the law of logarithms $\log(x^n) = n \log x$. How does this account for the results in part b)?

Ⓒ

25. Solve each equation.
 a) $7(2)^{-x} = 5^{2x+3}$ b) $3(4)^x = 13^{3x-1}$ c) $3^x = 4^{x-1}$ d) $10^{2-x} = 4(7)^x$

26. The population of town A is double that of town B, but it is decreasing at the rate of 5% per year. The population of town B is increasing at the rate of 5% per year.
 a) In how many years will the population of town B be double that of town A?
 b) At the end of this time, how will the population of town B compare with the initial population of town A?

27. The total amount of arable land in the world is about 3.2×10^9 ha. At current population rates, about 0.4 ha of land is required to grow food for each person in the world.
 a) Assuming a 1987 world population of 5 billion and a constant growth rate of 1.5%, determine the year when the demand for arable land exceeds the supply.
 b) Compare the effect of each comment on the result of part a).
  i) doubling the productivity of the land so that only 0.2 ha is required to grow food for each person
  ii) reducing the growth rate by one-half, to 0.75%
  iii) doubling the productivity of the land *and* reducing the growth rate by 50%

28. If n is a natural number, find the least value of n such that:
 a) $1.1^n > 10^9$ b) $1.01^n > 10^9$ c) $1.001^n > 10^9$
 d) 1.001^n exceeds the capacity of your calculator's display.

29. Let N be any positive number, no matter how large. Prove that no matter how small the positive number x is, it is always possible to find a value of n such that $(1 + x)^n > N$.

MATHEMATICS AROUND US

The Perception of Sensations

How good are you at distinguishing weight? If you hold an object in each hand, can you tell which is heavier? It has been determined that the mass of an object must be increased by $\frac{1}{50}$ for most people to be able to notice the difference in the weight of the object.

For example, one of the weights for a balance scale has a mass of 100 g. An increase of $\frac{1}{50}$ results in a mass of $100\left(1 + \frac{1}{50}\right)$ g, or 102 g. Hence, most people notice the difference between the weights of masses of 100 g and 102 g. We can apply these increases in succession to produce a sequence of masses. Most people would just be able to tell the difference between the weight of each mass and the next.

$100\left(1 + \frac{1}{50}\right)^2 \text{ g} = 104.04 \text{ g} \qquad 100\left(1 + \frac{1}{50}\right)^3 \text{ g} \doteq 106.12 \text{ g}$

$100\left(1 + \frac{1}{50}\right)^4 \text{ g} \doteq 108.24 \text{ g} \; \ldots$

QUESTIONS

1. Another weight for a balance scale has a mass of 200 g. How many weights, each slightly heavier than the previous one, could be placed between weights with masses of 100 g and 200 g such that most people would just be able to tell the difference between each weight and the next?

The table shows the fractional increase which is just enough so that most people would notice a difference for certain sensations. Use the data in the table to answer Questions 2 and 3.

Sensation	Fraction
Distinguishing weights	$\frac{1}{50}$
Brightness	$\frac{1}{60}$
Taste, saline	$\frac{1}{5}$

2. Suppose one light is twice as bright as another. How many lights, each slightly brighter than the previous one, could be placed between them such that most people would just be able to tell the difference in brightness between each light and the next?

3. Suppose one food is twice as salty as another. How many foods, each slightly saltier than the previous one, could be placed between them such that most people would just be able to tell the difference in saltiness between each food and the next?

11-4 INTRODUCTION TO LOGARITHMIC FUNCTIONS

Many examples of exponential functions were given in the previous chapter. Associated with each of these functions there is a corresponding function whose equation we can obtain by solving for the variable in the exponent.

Growth of Populations

In 1987 the world population reached 5 billion. At the time, the population was increasing at the rate of approximately 1.6% per year. If the rate of growth remains constant, then the population P billion is expressed as an exponential function of the number of years n relative to 1987 by this equation.

$$P = 5(1.016)^n \ldots ①$$

Suppose we ask in how many years will the population reach P billion? We express the number of years n as a function of P by solving equation ① for n. Hence, we take the logarithm of each side.

$$\log P = \log 5 + n \log 1.016$$

Solve for n.

$$n \log 1.016 = \log P - \log 5$$

$$n = \frac{\log\left(\frac{P}{5}\right)}{\log 1.016}$$

The coefficient of the expression on the right side is $\frac{1}{\log 1.016}$.

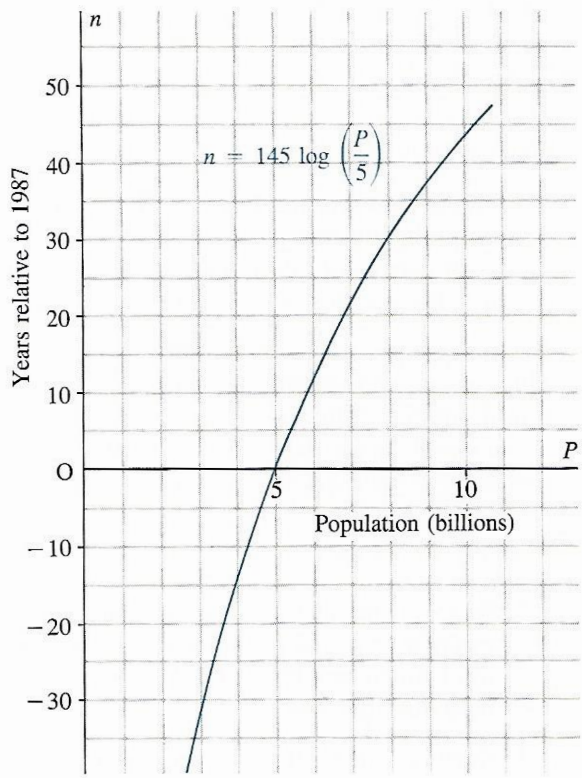
Key in: 1.016 $\boxed{\log}$ $\boxed{1/x}$ to display 145.05982

Hence, the equation for n becomes

$$n \doteq 145 \log\left(\frac{P}{5}\right) \ldots ②$$

Equation ② expresses the number of years n as a logarithmic function of the population P. The graph shows the values of n for $3 \leq P \leq 10$.

Compare this graph with the one on page 426.



In this example, notice that n is not defined if P is 0, or if P is negative. This is reasonable, since the population P must be a positive number. Hence, the domain of the function is the set of positive integers.

Light Penetration Under Water

For every metre a diver descends below the surface, the light intensity is reduced by 2.5%. The percent P of surface light present is expressed as an exponential function of the depth d metres by this equation.

$$P = 100(0.975)^d \ldots ①$$

Suppose we ask at what depth is the light intensity P%? We express d as a function of P by solving equation ① for d. Take the logarithm of each side.

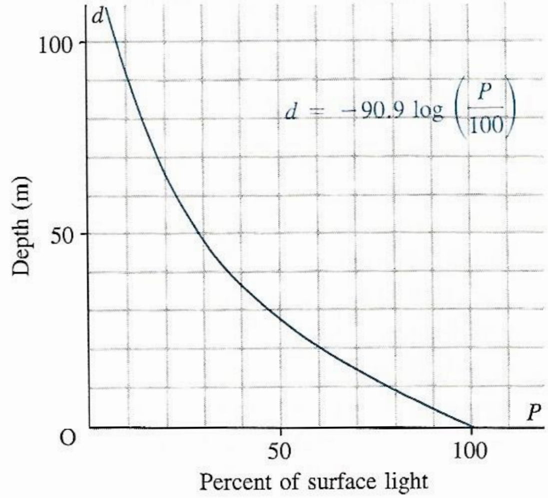
$\log P = \log 100 + d \log 0.975$

Solve for d.

$$d = \frac{\log\left(\frac{P}{100}\right)}{\log 0.975}$$

$$d \doteq -90.9 \log\left(\frac{P}{100}\right) \ldots ②$$

Equation ② expresses the depth d metres as a logarithmic function of the light intensity P. The graph shows the values of d for $0 < P \leqslant 100$. Compare this graph with the one on page 427.



Acid Rain

Acid rain has become a major environmental problem. The acidity of rainwater is measured on a special scale called a *pH scale*. Each 1 unit decrease in pH represents a 10-fold *increase* in acidity. For example, the pH of vinegar is 2 units less than that of tomatoes. Hence, vinegar is 10^2, or 100 times more acidic than tomatoes.

Let A represent the acid content of a substance with a pH of P. Then, since each increase of 1 unit in P represents a 10-fold decrease in A,

$$A = A_0(0.1)^P \ldots ①$$

where A_0 represents the acid content of a substance with pH 0.
To express P as a function of A, solve equation ① for P.

$$\log A = \log A_0 + P \log 0.1$$

$$P \log 0.1 = \log A - \log A_0$$

$$P = \frac{\log\left(\frac{A}{A_0}\right)}{\log 0.1}$$

or $$P = -\log\left(\frac{A}{A_0}\right) \ldots ②$$

Equation ② expresses the pH of a substance as a logarithmic function of its acid content.

Summary

Consider the similarities in the equations of the above examples.

$$n \doteq 145 \log\left(\frac{P}{5}\right) \qquad d \doteq -90.9 \log\left(\frac{P}{100}\right) \qquad P = -\log\left(\frac{A}{A_0}\right)$$

Initial population — Initial intensity — Acid content when $P = 0$

In each equation the expression on the right side involves the logarithm of the variable. Also, each coefficient is the reciprocal of the logarithm of the base of the corresponding exponential function.

> A *logarithmic function* has an equation which can be written in the form $y = k \log\left(\frac{x}{c}\right)$ where k and c are constants, and $c > 0$.

In the equation for pH, $P = -\log\left(\frac{A}{A_0}\right)$, notice that the value of A_0 is not given. Despite this, we can still use this equation to obtain useful information. This involves a comparison of the acid content, or pH of two substances.

Example 1. A lake in the Muskoka region of Ontario has a pH of 4.0. How many times as acidic as clean rain water, which has a pH of 5.6, is the water in this lake?

Solution. Use the equation developed above. $P = -\log\left(\frac{A}{A_0}\right)$

Let P_1 and A_1 represent the pH and acid content of clean rain water, and let P_2 and A_2 represent the pH and acid content of the lake. Then,

$$P_1 = -\log\left(\frac{A_1}{A_0}\right)$$

$$P_2 = -\log\left(\frac{A_2}{A_0}\right)$$

Subtract and then use the law of logarithms for division.

$$\begin{aligned} P_1 - P_2 &= -\log\left(\frac{A_1}{A_0}\right) + \log\left(\frac{A_2}{A_0}\right) \\ &= \log\left(\frac{A_2}{A_0}\right) - \log\left(\frac{A_1}{A_0}\right) \\ &= \log\left(\frac{A_2}{A_1}\right) \end{aligned}$$

Substitute 5.6 for P_1 and 4.0 for P_2.

$$5.6 - 4.0 = \log\left(\frac{A_2}{A_1}\right)$$

$$1.6 = \log\left(\frac{A_2}{A_1}\right)$$

By the definition of a logarithm

$$\frac{A_2}{A_1} = 10^{1.6}$$

$$\doteq 39.8$$

Hence, the lake is about 40 times as acidic as clean rain water.

Example 2. Given the exponential function $f(x) = 10^x$

a) Determine the inverse function $y = f^{-1}(x)$.

b) Graph $y = f(x)$ and $y = f^{-1}(x)$ on the same grid.

Solution. a) Recall that to obtain the inverse of a function from its equation, we interchange x and y in the equation and solve for y. Hence, to find the inverse of $y = 10^x$:

Step 1. Interchange x and y. $x = 10^y$

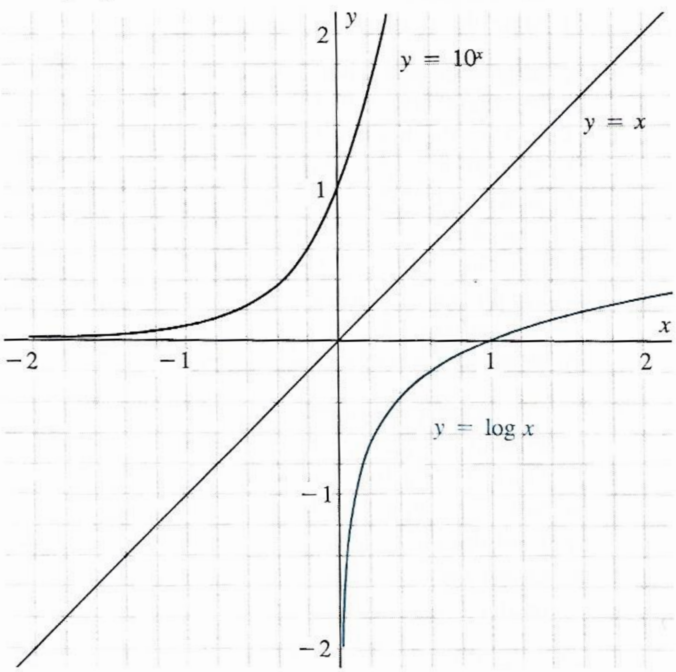
Step 2. Solve for y. $y = \log x$

Hence, the inverse of the exponential function $f(x) = 10^x$ is the logarithmic function $f^{-1}(x) = \log x$.

b) We graph $f(x) = 10^x$ using a table of values. Recall that we can graph the inverse by reflecting the graph of $y = 10^x$ in the line $y = x$. This is equivalent to interchanging the ordered pairs in the table of values for $y = f(x)$.

$y = 10^x$

x	y
−2	0.01
−1.5	0.03
−1	0.10
−0.5	0.32
−0.2	0.63
0	1.00
0.1	1.26
0.2	1.58
0.3	2.00



Recall that in *Section 11-1* we observed that taking a common logarithm of a number and raising the number to a power of 10 are inverse operations. This is consistent with *Example 2* above, which shows that the logarithmic function $y = \log x$ can be defined as the inverse of the exponential function $y = 10^x$.

EXERCISES 11-4

(A)

1. Solve each equation for x, thus expressing x as a logarithmic function of y.
   a) $y = 5(2)^x$ b) $y = 1.3(10)^x$ c) $y = 8.2(1.03)^x$
   d) $y = 6.4\left(\frac{1}{2}\right)^x$ e) $y = 3.5(2.7)^x$ f) $y = 2.75\left(\frac{2}{3}\right)^x$

2. Find the values of x for the given y values. Give the answers to 3 decimal places where necessary.
   a) $y = 3(2)^x$ and $y = 12; 48; 3072$ b) $y = 7(3)^x$ and $y = 189; 75; 462$
   c) $y = 2(5)^x$ and $y = 0.08; 12; 32$ d) $y = 5(7)^x$ and $y = 5; 275; 675$

(B)

3. An investment of \$500 at 8% per annum compounded annually grows to A dollars in n years. In Chapter 10, page 426, we showed that an equation expressing the amount A dollars as an exponential function of the time n years is $A = 500(1.08)^n$.
   a) Solve this equation for n, thus expressing n as a logarithmic function of A.
   b) Calculate the value of n for each value of A and interpret the result.
      i) $A = 1250$ ii) $A = 350$
   c) Graph the function in part a) for $0 < A \leq 1250$. Compare your graph with the one on page 426.
   d) State the domain and the range of the function.

4. A ball is dropped from a height of 2.00 m. On each bounce the ball rises to 70% of the height from which it fell. In Chapter 10, page 427, we showed that an equation expressing the bounce height h metres as an exponential function of the number of bounces n is $h = 2.00(0.7)^n$.
   a) Solve this equation for n, thus expressing n as a logarithmic function of h.
   b) Calculate the value of n for each value of h and interpret the result.
      i) 0.7 m ii) 0.12 m
   c) Graph the function in part a) for $0 < h \leq 2.00$. Compare your graph with the one on page 427.
   d) What is the range of the function?

5. a) The population of the town of Elmira was 6800 in 1987. If the population is growing at the rate of 1.8% per annum, write an equation expressing the population P as a function of n, the number of years relative to 1987.
   b) Solve this equation for n.
   c) Find the value of n if P is: i) 9200 ii) 5500.
   d) Graph the functions in parts a) and b) on the same grid. How are these functions related?

6. On bright sunny days, the amount of bromine in a municipal swimming pool decreases by 10% each hour. If there was 145 g of bromine in the pool at noon on a sunny day, when would the pool contain: a) 102 g b) 85 g c) 200 g?

7. Between 1956 and 1976 the annual average pH of precipitation at Sault Ste. Marie, Ontario, dropped from 5.6 to 4.3. How many times as acidic as the precipitation in 1956 was the precipitation in 1976?

8. In the spring, the pH of a stream dropped from 6.5 to 5.5 during a 3-week period in April.
   a) How many times as acidic did the stream become?
   b) Why would this happen in April?
   c) The mean pH of Lake Huron is 8.2. How many times as acidic was the stream:
      i) before the 3-week period ii) after the 3-week period?

9. When the pH of the water in a lake falls below 4.7, nearly all species of fish in the lake are deformed or killed. How many times as acidic as clean rainwater, which has a pH of 5.6, is such a lake?

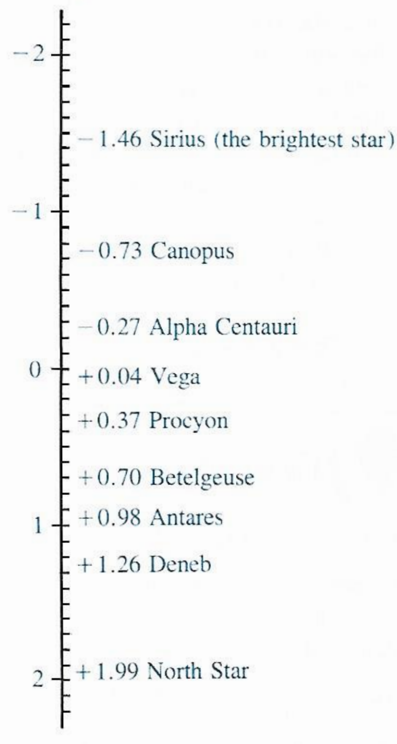
10. Given the exponential function $f(x) = 3^x$, graph $y = f(x)$ and $y = f^{-1}(x)$ on the same grid.

11. Graph each function and its inverse on the same grid.
    a) $y = 2^x$ b) $y = \left(\frac{2}{3}\right)^x$

C

12. In astronomy, the brightnesses of stars are compared using a scale of magnitudes defined by a logarithmic function. If a star has brightness b, then its *magnitude* m is defined by this equation.
    $m = -2.5 \log \left(\frac{b}{b_0}\right)$, where b_0 is the brightness of a star with magnitude 0.
    a) Write this equation in exponential form.
    b) If the magnitude increases by 2.5 units, by what factor does the brightness change? Does it increase or decrease?
    c) i) How many times as bright as the North Star is Vega?
       ii) How many times as bright as Vega is Sirius?
       iii) How many times as bright as the North Star is Sirius?

Stellar magnitudes



Magnitudes in the Solar System

Sun	−26.8
Full moon	−12.7
Venus	−4.4*
Uranus	+5.5
Pluto	+15

*maximum brightness

13. Refer to *Exercise 12*. How many times as bright as:
    a) the full moon is the sun b) Venus is the sun c) Pluto is Uranus?

14. In 1987, the Canadian astronomer Ian Shelton became the first person to observe an exploding star, or supernova, in our galaxy since the invention of the telescope four hundred years ago. Supernova 1987A, as it is now called, increased in brightness at least 200 times in the first day, and almost 1000 times in the first two days. What change occurred in its magnitude:
    a) in the first day
    b) in the first two days?

15. Our star, the sun, appears billions of times brighter than the other stars because it is relatively near to us. Hence, astronomers are interested in comparing the brightnesses of stars if they could all be viewed from the same distance. The *luminosity* of a star refers to its brightness at a distance of 32.6 light years. By allowing for the sun's magnitude, and its distance from us, astronomers have established the following formula for the luminosity L of a star relative to the sun.
    $\log L = 0.4(5 \log d - m) - 1.1$,
    where m is the magnitude of the star, and d is its distance in light years.
    a) How many times as luminous as the sun are the stars in the table?
    b) The distance to the sun is 1.55×10^{-5} light years. Check that the luminosity of the sun, as defined by the above equation, is 1.

Some Stellar Distances

Star	**Distance** (light years)
Sirius	8.7
Vega	26.5
North Star	680
Deneb	1600

INVESTIGATE

At the beginning of this section, the equation $n \doteq 145 \log\left(\frac{P}{5}\right)$ was derived to represent the number of years for the world population to grow to P billion, assuming a constant growth rate of 1.6% per year. Notice that the coefficient 145 is the reciprocal of the logarithm of the base of the corresponding exponential function; that is, $\frac{1}{\log 1.016} \doteq 145$. This suggests that the form of the equation of the logarithmic function will be simpler if the base of the corresponding exponential function is 10, for then that coefficient will be 1, since $\log 10 = 1$.

Investigate whether this is true by first changing the base of the corresponding exponential function, $P = 5(1.016)^n$, to base 10, and then solving for n to obtain the corresponding logarithmic function.

11-5 DEFINING AND GRAPHING LOGARITHMIC FUNCTIONS

In *Example 2* of the preceding section we saw that the logarithmic function $y = \log x$ can be defined as the inverse of the exponential function $y = 10^x$. This suggests that other logarithmic functions can be defined as inverses of exponential functions with bases other than 10. In fact, for each choice of base for the exponential function $y = a^x$, $a > 0$, there is an associated logarithmic function. Hence, we define the function $y = \log_a x$, $a > 0$, as follows.

> The logarithmic function $y = \log_a x$ ($a > 0$, $a \neq 1$) is the inverse of the exponential function $y = a^x$.

We say, "y equals log to the base a of x".

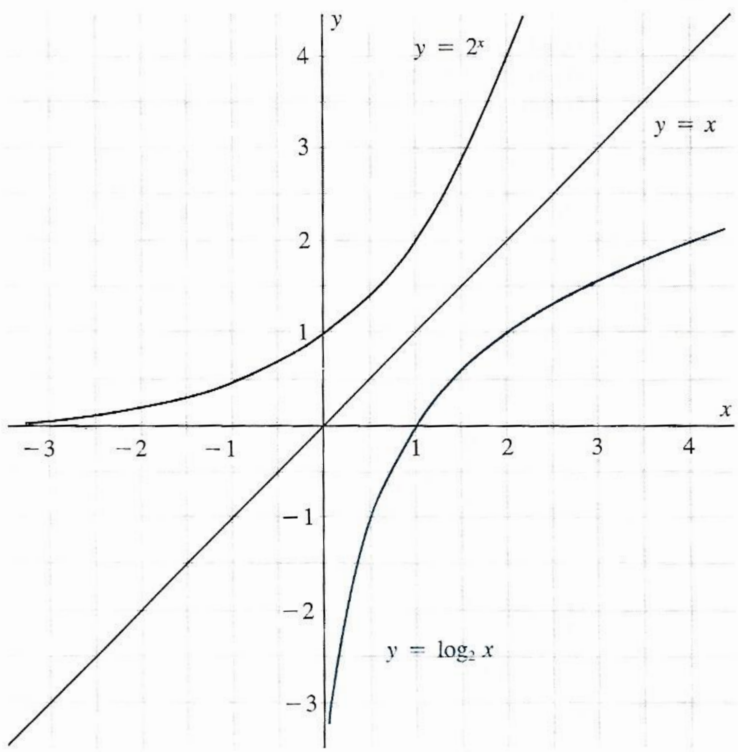
Recall that we can graph the inverse of any function by reflecting its graph in the line $y = x$. This is equivalent to interchanging the ordered pairs in the table of values of the function. For example, the graph below shows the function $y = 2^x$ and its inverse $y = \log_2 x$. Compare this graph with the one on page 483.

$y = 2^x$

x	y
−3	0.13
−2	0.25
−1	0.50
−0.5	0.71
0	1.00
0.5	1.41
1	2.00
1.5	2.83
2	4.00

$y = \log_2 x$

x	y
0.13	−3
0.25	−2
0.50	−1
0.71	−0.5
1.00	0
1.41	0.5
2.00	1
2.83	1.5
4.00	2



The graph illustrates the following properties of the function $y = \log_2 x$. These properties are consequences of the corresponding properties of $y = 2^x$.

y-intercept
There is no y-intercept since the function $y = 2^x$ has no x-intercept.

x-intercept
The x-intercept is 1, since the y-intercept of $y = 2^x$ is 1. Hence, $\log_2 1 = 0$

Domain
The domain of $y = \log_2 x$ is the set of positive real numbers, since this is the range of $y = 2^x$.

Range
The range of $y = \log_2 x$ is the set of all real numbers, since this is the domain of $y = 2^x$.

If any exponential function is given, we can sketch its graph. The graph of the inverse is then the graph of the corresponding logarithmic function.

Example. a) Sketch the graph of the exponential function $y = \left(\frac{1}{3}\right)^x$.

b) Sketch the graph of the inverse of the function in part a) on the same grid.

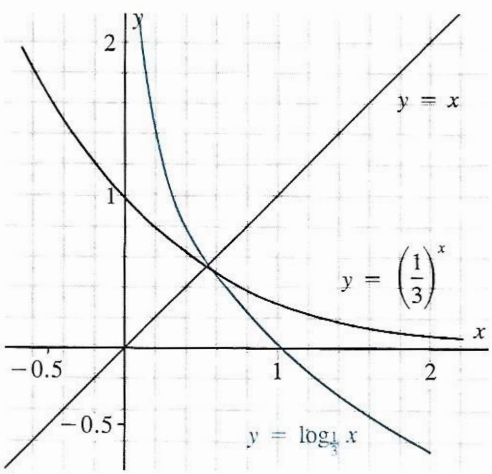
c) Write the equation of the inverse function.

Solution. a) $y = \left(\frac{1}{3}\right)^x$

When x is very large and positive, y is very small and positive.
When $x = 0$, $y = 1$
When x is negative and has a large absolute value, y is very large.

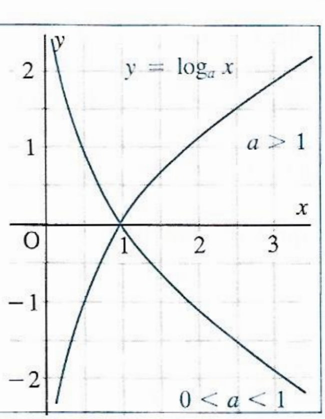
b) Reflect $y = \left(\frac{1}{3}\right)^x$ in the line $y = x$. The image is $y = \log_{\frac{1}{3}} x$.

c) The equation of the inverse function is $y = \log_{\frac{1}{3}} x$.



The graphs in the above examples illustrate properties of the logarithmic function $y = \log_a x$.

Properties of the function $y = \log_a x$
y-intercept: none
x-intercept: 1
Domain: all positive real numbers
Range: all real numbers



EXERCISES 11-5

Ⓐ

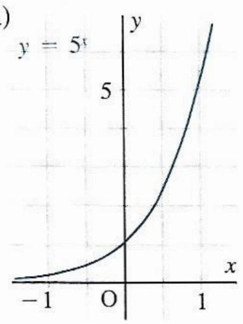
1. Write the inverse of each exponential function.
   a) $y = 10^x$ b) $y = 3^x$ c) $y = 7^x$
   d) $y = (0.4)^x$ e) $y = \left(\frac{3}{2}\right)^x$ f) $y = 15^x$

2. Write the inverse of each logarithmic function.
   a) $y = \log x$ b) $y = \log_2 x$ c) $y = \log_6 x$
   d) $y = \log_{\frac{1}{2}} x$ e) $y = \log_{\frac{5}{4}} x$ f) $y = \log_{21} x$

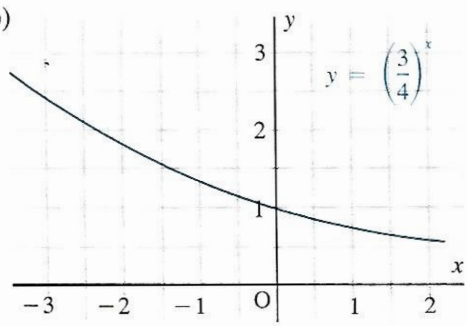
Ⓑ

3. Copy each graph and sketch the graph of the inverse of the given function on the same grid. Then write the equation of the inverse function.

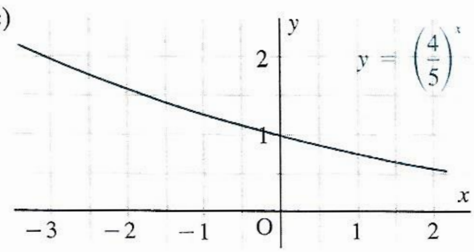
a)



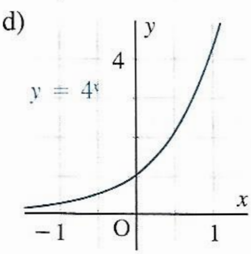
b)



c)



d)



4. a) Sketch the graph of the exponential function $y = 3^x$.
   b) Sketch the graph of the inverse of the function in part a) on the same grid.
   c) Write the equation of the inverse function.

5. Repeat *Exercise 4*, starting with the function $y = \left(\frac{1}{2}\right)^x$

6. Graph each function.

   a) $y = \log_4 x$ b) $y = \log_6 x$ c) $y = \log_{\frac{1}{2}} x$

   d) $y = \log_{0.8} x$ e) $y = \log_{1.5} x$ f) $y = \log_{\frac{2}{5}} x$

7. Prove that if $f(x) = \log_a x$, then $f(xy) = f(x) + f(y)$.

©

8. In the *Example*, the graphs of $y = \left(\frac{1}{3}\right)^x$, $y = \log_{\frac{1}{3}} x$, and $y = x$ are shown.
   Determine the coordinates of their point of intersection to 3 decimal places.

9. Given the exponential function $f(x) = a^x$ and its inverse $f^{-1}(x) = \log_a x$, where $a > 0$
   a) For what values of a do the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect?
   b) Find out as much as you can about the point of intersection of the graphs in part a).

INVESTIGATE

Changing the Base of Logarithms

We can solve an equation such as $a^x = y$ in two ways.

- Using base a logarithms
  $x = \log_a y$
- Using base 10 logarithms
  $x \log a = \log y$
  $x = \dfrac{\log y}{\log a}$

Since the results must be equal, we have the following formula.

$$\boxed{\log_a y = \frac{\log y}{\log a}}$$

This formula can be used to change from base 10 logarithms to base a logarithms without having to solve an exponential equation each time.

1. Use the formula and your calculator to find an approximation to each logarithm.
   a) $\log_3 4$ b) $\log_3 6$ c) $\log_3 40$ d) $\log_3 225$

2. If a and b represent two different bases, prove that $\log_a y = \dfrac{\log_b y}{\log_b a}$.

11-6 LOGARITHMS AS EXPONENTS

Recall that to find the inverse of a function from its equation, we interchange x and y in the equation and solve for y. Hence, to find the inverse of $y = a^x$:

Step 1. Interchange x and y. $x = a^y$ . . . ①

Step 2. Solve for y. We can do this using common logarithms, but it is preferable to use the definition on page 487. According to the definition, the inverse is $y = \log_a x$ . . . ②

Hence, this is the equation that results when equation ① is solved for y. Comparing equations ① and ②, we see that

$$\underbrace{\log_a}_{\text{base}} x = \underbrace{y}_{\text{exponent}} \quad \text{means that} \quad x = a^y,\ x > 0.$$

Hence, $\log_a x$ is an exponent. It is the exponent that x would have if it were written in power form with base a ($a > 0$, $a \neq 1$). If the base is omitted, it is understood to be base 10.

Example 1. Evaluate each logarithm.

a) $\log_5 25$ b) $\log_7 \sqrt{7}$ c) $\log_{\frac{1}{3}} 9$ d) $\log_a a$

Solution. a) Since $25 = 5^2$, then $\log_5 25 = 2$

b) Since $\sqrt{7} = 7^{\frac{1}{2}}$, then $\log_7 \sqrt{7} = \frac{1}{2}$

c) Write 9 as a power of $\frac{1}{3}$. Since $\left(\frac{1}{3}\right)^2 = \frac{1}{9}$, then $\left(\frac{1}{3}\right)^{-2} = 9$

Hence, $\log_{\frac{1}{3}} 9 = -2$

d) Since $a = a^1$, then $\log_a a = 1$

Since any positive number can be expressed as a power of any other positive number (except 1), we can find approximations to the logarithm of any positive number to any positive base (except 1).

Example 2. Find $\log_5 9$ to the nearest thousandth.

Solution. To find $\log_5 9$ means to find the exponent that 9 would have if it were expressed as a power of 5.

Let $9 = 5^x$

Take the logarithm of each side to base 10.

$\log 9 = \log (5^x)$

$\log 9 = x \log 5$

$$x = \frac{\log 9}{\log 5}$$

Key in: 9 $\boxed{\log}$ $\boxed{\div}$ 5 $\boxed{\log}$ $\boxed{=}$ to display 1.3652124

Hence, $x \doteq 1.365\,212\,4$

To the nearest thousandth, $9 \doteq 5^{1.365}$

Therefore, $\log_5 9 \doteq 1.365$

Example 2 illustrates why a calculator does not have $\boxed{\log}$ keys for many different bases. Logarithms to any base can be found by converting to base 10 and then using the $\boxed{\log}$ key.

Example 3. Simplify each expression.

a) $\log_a a^x$ b) $a^{\log_a x}$

Solution. a) $\log_a a^x$ is the exponent that a^x would have if it were written as a power of a. This exponent is x. Hence, $\log_a a^x = x$

b) $a^{\log_a x}$ is a raised to the exponent that x would have if x were written as a power of a. Hence, $a^{\log_a x} = x$

Summary

- $\log_a x = y$ means that $x = a^y$, where $a > 0$, $a \neq 1$, and $x > 0$
- $\log_a a^x = x$ and $a^{\log_a x} = x$
- $\log_a a = 1$

Example 4. Write each expression in exponential form.

a) $\log_2 16 = 4$ b) $\log_2 0.5 = -1$

Solution. a) $\log_2 16 = 4$
$16 = 2^4$

b) $\log_2 0.5 = -1$
$0.5 = 2^{-1}$

Example 5. Write each expression in logarithmic form.

a) $3^5 = 243$ b) $a^b = c$

Solution. a) $3^5 = 243$
$5 = \log_3 243$

b) $a^b = c$
$b = \log_a c$

Example 6. If $\log_8 5 = m$ and $\log_4 3 = n$, find an expression for $\log_2 15$ in terms of m and n.

Solution. Since $\log_8 5 = m$, then $8^m = 5 \ldots$ ①
Since $\log_4 3 = n$, then $4^n = 3 \ldots$ ②
Let $\log_2 15 = x$, so that $2^x = 15$.
Since $15 = 5 \times 3$, we can use equations ① and ②.

$$\begin{aligned} 2^x &= 5 \times 3 \\ &= 8^m \times 4^n \\ &= 2^{3m} \times 2^{2n} \\ &= 2^{3m + 2n} \end{aligned}$$

Hence, $x = 3m + 2n$, or $\log_2 15 = 3m + 2n$

EXERCISES 11-6

(A)

1. Write in exponential form.
   a) $\log_2 8 = 3$ b) $\log_2 32 = 5$ c) $\log_2\left(\frac{1}{4}\right) = -2$
   d) $\log_5 625 = 4$ e) $\log_3 9 = 2$ f) $\log_9 3 = \frac{1}{2}$

2. Evaluate each logarithm.
   a) $\log_2 16$ b) $\log_2 4$ c) $\log_3 27$ d) $\log_5 25$
   e) $\log_5\left(\frac{1}{5}\right)$ f) $\log_7 7$ g) $\log_3 1$ h) $\log_3 3^4$

3. Evaluate each logarithm.
   a) $\log_5 \sqrt{5}$ b) $\log_{\frac{1}{2}}\left(\frac{1}{16}\right)$ c) $\log_{\frac{3}{2}}\left(\frac{9}{4}\right)$ d) $\log_{\sqrt{3}} 9$
   e) $\log_{\frac{1}{2}} 8$ f) $\log_{\frac{2}{5}}\left(\frac{25}{4}\right)$ g) $\log_3 (\sqrt{3})^3$ h) $\log_{\sqrt{5}} 125$

4. In geography, sediments are classified by particle size, as shown.
   a) Write the logarithm to base 2 of each number.
   b) Write the logarithm to base 4 of each number.

Type of sediment	Size (mm)
Boulder	256
Cobble	64
Pebble	4
Granule	2
Sand	$\frac{1}{16}$
Silt	$\frac{1}{256}$

5. Simplify each expression.
   a) $\log_2 2^5$ b) $\log_7 7^{13}$ c) $\log_{0.4} (0.4)^9$
   d) $3^{\log_3 7}$ e) $14^{\log_{14} 5}$ f) $0.23^{\log_{0.23} 11}$

6. Write in logarithmic form.
   a) $6^2 = 36$ b) $4^{-2} = \frac{1}{16}$ c) $3^5 = 243$
   d) $7^3 = 343$ e) $8^{\frac{1}{3}} = 2$ f) $2^0 = 1$

(B)

7. Write in logarithmic form.
   a) $5^{-2} = 0.04$ b) $4^{-\frac{1}{2}} = \frac{1}{2}$ c) $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$
   d) $\left(\frac{2}{3}\right)^{-1} = \frac{3}{2}$ e) $\left(\frac{1}{9}\right)^2 = \frac{1}{81}$ f) $x^y = z$

8. Write in exponential form.
   a) $\log_{20} 400 = 2$ b) $\log_7\left(\frac{1}{49}\right) = -2$ c) $\log_8 4 = \frac{2}{3}$
   d) $\log_6 36^2 = 4$ e) $\log_{0.5} 8 = -3$ f) $\log_r s = t$

9. Evaluate each logarithm to the nearest thousandth.
 a) $\log_3 5$ b) $\log_7 4$ c) $\log_2 50$ d) $\log_5 12$ e) $\log_4 27$ f) $\log_{16} 8$

10. Evaluate each logarithm to the nearest thousandth.
 a) $\log_{2.5} 6$ b) $\log_5 9.3$ c) $\log_{\frac{1}{2}} \left(\frac{1}{3}\right)$
 d) $\log_{2.57} 3.68$ e) $\log_{\frac{3}{2}} 19.76$ f) $\log_{4.9} 35.27$

11. If $\log_8 3 = x$ and $\log_4 7 = y$, find an expression in terms of x and y for:
 a) $\log_2 21$ b) $\log_2 63$.

12. If $\log_9 4 = a$ and $\log_{27} 5 = b$, find an expression in terms of a and b for:
 a) $\log_3 20$ b) $\log_3 80$ c) $\log_{\sqrt{3}} 100$ d) $\log_{\sqrt{3}} 40$.

13. Solve for x.
 a) $\log_2 x = 9$ b) $\log_4 1 = x$ c) $\log_x 16 = 2$
 d) $\log_x 125 = -3$ e) $\log_3 x = -4$ f) $\log_{\sqrt{2}} 32 = x$

14. Given that $f(x) = x - \log_2 x$ and $g(x) = 2^x$, find: a) $f(g(x))$ b) $g(f(x))$.

15. If a telephone network is designed to carry N telephone calls simultaneously, then the number of switches needed per call must be at least $\log_2 N$. If the network can carry 10 000 calls simultaneously, how many switches would be needed:
 a) for one call b) for 10 000 simultaneous calls?

Ⓒ

16. a) Evaluate each logarithm. i) $\log_2 8$ and $\log_8 2$ ii) $\log_5 25$ and $\log_{25} 5$
 b) On the basis of the results of part a), make a conjecture about how $\log_a b$ and $\log_b a$ are related, where $a, b > 0$. Prove your conjecture.

17. a) Evaluate each logarithm.
 i) $\log_2 4$ ii) $\log_{\frac{1}{2}} 4$ iii) $\log_4 2$ iv) $\log_{\frac{1}{4}} 2$
 v) $\log_2 \left(\frac{1}{4}\right)$ vi) $\log_{\frac{1}{2}} \left(\frac{1}{4}\right)$ vii) $\log_4 \left(\frac{1}{2}\right)$ viii) $\log_{\frac{1}{4}} \left(\frac{1}{2}\right)$
 b) Using the results of part a) as a guide, make a conjecture about how you can tell, given the values of x and y, if:
 i) $\log_y x > 0$ ii) $\log_y x < 0$ iii) $\log_y x > 1$
 iv) $\log_y x < -1$ v) $0 < \log_y x < 1$ vi) $-1 < \log_y x < 0$.
 c) Illustrate the results of part b) on a grid.
 d) Compare the results in parts a), b), and c) with *Exercise 16*, page 437.

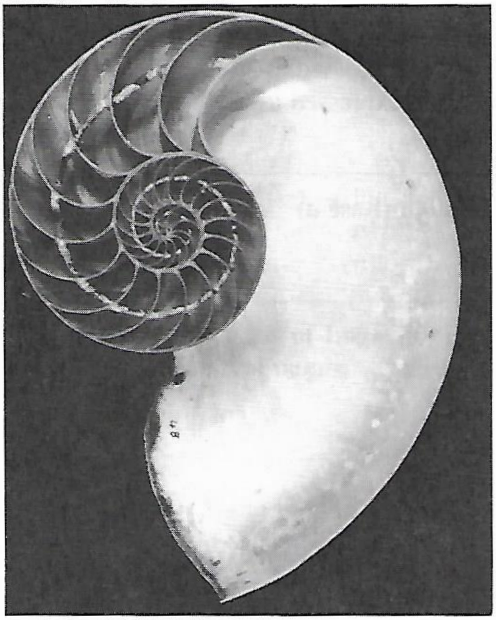
18. Let a and b be any two positive numbers. Prove that for all positive values of x, $\log_b x$ is directly proportional to $\log_a x$.

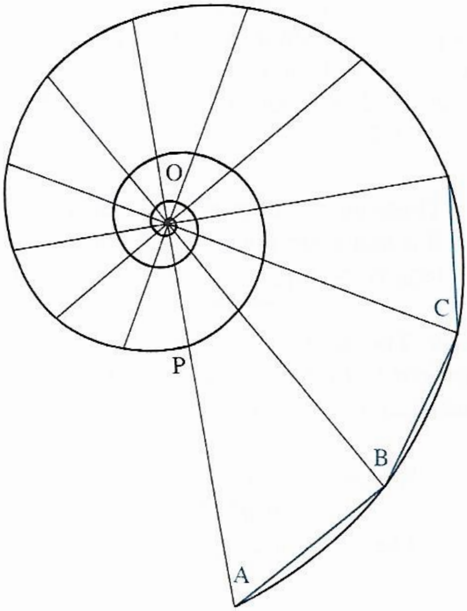
INVESTIGATE

Your calculator should have a key marked $\boxed{\ln}$. This key calculates logarithms of numbers to a base different from 10. Find the base of these logarithms as accurately as you can.

MATHEMATICS AROUND US

The Logarithmic Spiral





Some living creatures exhibit exponential growth in their dimensions. A well-known example is the chambered nautilus of the Indian and Pacific Oceans. As it grows, the shell extends continuously, generating a natural spiral.

QUESTIONS

1. The diagram shows a series of equally-spaced radii drawn from the centre of the spiral. The radii are spaced every 30°.
   a) Measure and record the length of each radius, starting with OP and proceeding clockwise around the spiral.
   b) Verify that the length of the radius L centimetres satisfies the equation $L = 1.5(1.0034)^{\theta}$, where θ degrees is the angle of rotation measured clockwise starting at OP.
2. a) Measure the angles represented by A, B, C, . . ., on the diagram.
   b) Prove that if the length of the radius is an exponential function of the angle of rotation, then the angles A, B, C, . . ., are all equal.
3. a) Suggest why the spiral is called a *logarithmic spiral*.
   b) The spiral is also referred to as an *equiangular spiral*. Suggest why.

11-7 THE LAWS OF LOGARITHMS (BASE a)

In *Sections 11-2* and *11-3* we developed the laws of logarithms for logarithms with base 10. The restriction to base 10 is not necessary, and the laws can be extended to logarithms with any positive base (except 1).

For example, an equation such as $2^3 \times 2^4 = 2^7$ can be written in logarithmic form as $\log_2 8 + \log_2 16 = \log_2 128$. This equation states that the sum of the exponents that 8 and 16 have when expressed as powers of 2 is equal to the exponent that 128 has when expressed as a power of 2.

> **Theorem** **Law of Logarithms for Multiplication (Base a)**
> If x and y are positive real numbers, then
> $\log_a xy = \log_a x + \log_a y \qquad a > 0,\ a \neq 1$

Given: Two positive real numbers x and y

Required to Prove: $\log_a xy = \log_a x + \log_a y \qquad a > 0,\ a \neq 1$

Proof: Let $\log_a x = M$ and $\log_a y = N$

$$x = a^M \qquad y = a^N$$

Hence, $xy = (a^M)(a^N)$

$= a^{M+N}$

Therefore, $\log_a xy = \log_a (a^{M+N})$

$= M + N$

$= \log_a x + \log_a y$

> **Corollary** **Law of Logarithms for Division (Base a)**
> If x and y are positive real numbers, then
> $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y \qquad a > 0,\ a \neq 1$

Example 1. Write $\log_2 15$ as:

a) a sum of two logarithms b) a difference of two logarithms.

Solution. a) Since $15 = 5 \times 3$, then $\log_2 15 = \log_2 5 + \log_2 3$

b) Since $15 = 30 \div 2$, then $\log_2 15 = \log_2 30 - \log_2 2$

What other answers can you find for *Example 1*?

Example 2. Write each expression as a single logarithm and simplify it.

a) $\log_3 6 + \log_3 1.5$ b) $\log_5 50 - \log_5 0.4$

Solution. a) $\log_3 6 + \log_3 1.5 = \log_3 (6 \times 1.5)$

$= \log_3 9$

$= 2$

b) $\log_5 50 - \log_5 0.4 = \log_5 \left(\frac{50}{0.4}\right)$

$= \log_5 125$

$= 3$

Theorem **Law of Logarithms for Powers (Base a)**
If x and n are real numbers, and $x > 0$, then
$\log_a (x^n) = n \log_a x \qquad a > 0, a \neq 1$

Given: Two real numbers x and n, where $x > 0$
Required to Prove: $\log_a (x^n) = n \log_a x$
Proof: Let $\log_a x = M$

$$x = a^M$$

Hence, $x^n = (a^M)^n$
$= a^{nM}$

Therefore, $\log_a (x^n) = \log_a (a^{nM})$
$= nM$
$= n \log_a x$

Corollary **Law of Logarithms for Roots (Base a)**
If x and n are real numbers, and $x > 0$, then
$\log_a \sqrt[n]{x} = \frac{1}{n} \log_a x \qquad a > 0, a \neq 1$

Example 3. a) Write $\log_5 16$ as a product of a whole number and a logarithm.
b) Write as a single logarithm. i) $2 \log_6 5$ ii) $\frac{1}{3} \log_4 125$

Solution. a) Since $16 = 2^4$, then $\log_5 16 = \log_5 (2^4)$
$= 4 \log_5 2$

b) i) $2 \log_6 5 = \log_6 (5^2)$
$= \log_6 25$

ii) $\frac{1}{3} \log_4 125 = \log_4 (\sqrt[3]{125})$
$= \log_4 5$

Example 4. Given that $\log_2 7 \doteq 2.8074$, find an approximation for each logarithm.

a) $\log_2 14$ b) $\log_2 49$ c) $\log_2 \left(\frac{4}{7}\right)$ d) $\log_2 \sqrt[3]{7}$

Solution. a) $\log_2 14 = \log_2 (7 \times 2)$
$= \log_2 7 + \log_2 2$
$\doteq 2.8074 + 1$
$\doteq 3.8074$

b) $\log_2 49 = \log_2 (7^2)$
$= 2 \log_2 7$
$\doteq 2(2.8074)$
$\doteq 5.6148$

c) $\log_2 \left(\frac{4}{7}\right) = \log_2 4 - \log_2 7$
$\doteq 2 - 2.8074$
$\doteq -0.8074$

d) $\log_2 \sqrt[3]{7} = \frac{1}{3} \log_2 7$
$\doteq \frac{1}{3} (2.8074)$
$\doteq 0.9358$

Laws of Logarithms (Base a) $a > 0, a \neq 1$

- Multiplication $\log_a xy = \log_a x + \log_a y$ $x, y > 0$
- Division $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$ $x, y > 0$
- Powers $\log_a (x^n) = n \log_a x$ $x > 0$
- Roots $\log_a \sqrt[n]{x} = \frac{1}{n} \log_a x$ $x > 0$

Example 5. Prove that if (x_1, y_1) and (x_2, y_2) are points on the graph of $y = \log_a x$, then $(x_1x_2, y_1 + y_2)$ is also a point on the graph.

Solution. Since (x_1, y_1) is on the graph of $y = \log_a x$, its coordinates satisfy the equation.

Hence, $y_1 = \log_a x_1$ . . . ①

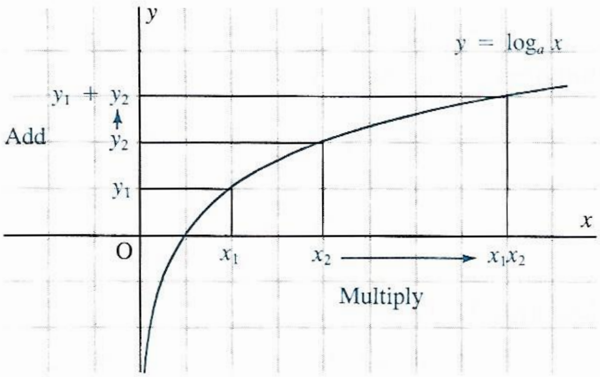
Similarly, $y_2 = \log_a x_2$ . . . ②

Add equations ① and ②, and use the law of logarithms.

$$\begin{aligned} y_1 + y_2 &= \log_a x_1 + \log_a x_2 \\ &= \log_a (x_1x_2) \end{aligned}$$

Hence, the coordinates of the point $(x_1x_2, y_1 + y_2)$ also satisfy the equation $y = \log_a x$. That is, $(x_1x_2, y_1 + y_2)$ is also a point on the graph.

The result of *Example 5* is a consequence of the law of logarithms for multiplication. Multiplying the x-coordinates of points on the graph of $y = \log_a x$ corresponds to adding their y-coordinates. Compare this example with *Example 2*, page 455.



We can solve equations involving logarithms by using the laws of logarithms and the definition of a logarithm.

Example 6. Solve each equation, and check.

a) $2 \log x = \log 8 + \log 2$ b) $\log_8 (2 - x) + \log_8 (4 - x) = 1$

Solution. a)

$$\begin{aligned} 2 \log x &= \log 8 + \log 2 \\ \log x^2 &= \log 16 \\ \text{Hence, } x^2 &= 16 \\ x &= \pm 4 \end{aligned}$$

To check, substitute each value of x into the original equation.

When $x = 4$,

$$\begin{aligned} \text{L.S.} &= 2\log 4 & \text{R.S.} &= \log 8 + \log 2 \\ &= \log 16 & &= \log 16 \end{aligned}$$

4 is a root.

When $x = -4$, the left side is not defined since $\log x$ is defined only when $x > 0$. Hence, -4 is an extraneous root.

b) $\log_8 (2 - x) + \log_8 (4 - x) = 1$

Simplify the left side using the law of logarithms for multiplication.

$$\log_8 (2 - x)(4 - x) = 1$$

Use the definition of a logarithm.

$$\begin{aligned} (2 - x)(4 - x) &= 8^1 \\ 8 - 6x + x^2 &= 8 \\ -6x + x^2 &= 0 \\ x &= 0 \text{ or } x = 6 \end{aligned}$$

When $x = 0$,

$$\begin{aligned} \text{L.S.} &= \log_8 2 + \log_8 4 & \text{R.S.} &= 1 \\ &= \log_8 8 \\ &= 1 \end{aligned}$$

0 is a root.

When $x = 6$, the left side is not defined. Hence, 6 is an extraneous root.

In *Example 6b* we may ask, if 6 is an extraneous root, then where did it come from? The key to the answer is the quadratic equation $(2 - x)(4 - x) = 8$ which occurred in the solution. This equation may also be written as $(x - 2)(x - 4) = 8$ without changing the two roots. But then the associated logarithmic equation would be $\log_8 (x - 2) + \log_8 (x - 4) = 1$. Hence, if this equation were solved using the method of *Example 6b* the same two roots 0 and 6 would result, but this time 6 would be the root and 0 would be extraneous.

EXERCISES 11-7

A

1. Write each expression as a single logarithm, and simplify it.
   a) $\log_6 9 + \log_6 4$ b) $\log_5 15 - \log_5 3$ c) $\log_4 2 + \log_4 32$
   d) $\log_2 48 - \log_2 6$ e) $\log_3 54 - \log_3 2$ f) $\log_3 9 + \log_3 9$

2. Write as a sum of logarithms.
   a) $\log_3 20$ b) $\log_7 45$ c) $\log_5 90$ d) $\log_{12} 6$ e) $\log_8 75$ f) $\log_{20} 39$

3. Write as a difference of logarithms.
   a) $\log_4 11$ b) $\log_3 12$ c) $\log_9 5$ d) $\log_6 7$ e) $\log_{11} 21$ f) $\log_2 13$

4. Write each expression as a single logarithm and simplify it.
   a) $\log_6 4 + \log_6 3 + \log_6 3$ b) $\log_4 8 + \log_4 6 + \log_4 \left(\frac{4}{3}\right)$
   c) $\log_3 18 + \log_3 5 - \log_3 10$ d) $\log_2 20 - \log_2 5 + \log_2 8$
   e) $\log_5 50 + \log_5 10 - \log_5 4$ f) $\log_3 45 - \log_3 5 + \log_3 3$

5. a) Simplify.
   i) $\log_2 (8 \times 16)$ ii) $\log_3 (27 \times 81)$ iii) $\log_5 (625 \times 25)$
   b) Simplify.
   i) $\log_2 \left(\frac{32}{4}\right)$ ii) $\log_3 \left(\frac{27}{3}\right)$ iii) $\log_5 \left(\frac{125}{25}\right)$

6. Write each logarithm as a product of a whole number and a logarithm.
   a) $\log_3 8$ b) $\log_5 36$ c) $\log_2 27$ d) $\log_6 32$ e) $\log_{12} 81$ f) $\log_4 125$

7. Write as a single logarithm.
   a) $3 \log_2 5$ b) $2 \log_7 4$ c) $6 \log_3 8$
   d) $5 \log_{12} 4$ e) $15 \log_2 3$ f) $2 \log_5 9$

(B)

8. Write as a single logarithm and simplify it.
   a) $\log_4 48 + \log_4 \left(\frac{2}{3}\right) + \log_4 8$ b) $\log_8 24 + \log_8 4 - \log_8 3$
   c) $\log_9 36 + \log_9 18 - \log_9 24$ d) $\log_4 20 - \log_4 5 + \log_4 8$

9. Write as a single logarithm and simplify it.
   a) $\log_3 \sqrt{45} - \log_3 \sqrt{5}$ b) $\log_2 \sqrt{5} - \log_2 \sqrt{40}$
   c) $\log_5 \sqrt{10} + \log_5 \sqrt{\frac{25}{2}}$ d) $\log_4 \sqrt{40} + \log_4 \sqrt{48} - \log_4 \sqrt{15}$

10. Simplify.
   a) $\log_2 24 - \log_2 \left(\frac{3}{4}\right)$ b) $\log_2 20 + \log_2 0.4$
   c) $\log_8 48 + \log_8 4 - \log_8 3$ d) $\log_{21} 7 + \log_{21} 9 + \log_{21} \left(\frac{1}{3}\right)$

11. Given $\log_2 5 \doteq 2.3219$, find an approximation for each logarithm.
   a) $\log_2 20$ b) $\log_2 25$ c) $\log_2 2.5$ d) $\log_2 \sqrt{5}$

12. Given $\log_3 10 \doteq 2.0959$, find an approximation for each logarithm.
   a) $\log_3 1000$ b) $\log_3 30$ c) $\log_3 \sqrt{0.3}$ d) $\log_3 \left(\frac{100}{9}\right)$

13. Express y as a function of x. What is the domain?
   a) $\log_2 xy = 3 \log_2 x$
   b) $\log_5 y = 2 \log_5 (x + 1) + \log_5 (x - 1)$
   c) $\log_3 (y - 3) = 1 + 2 \log_3 (x + 3)$

14. Use your calculator to evaluate each expression.
   a) i) $\log_2 3000$ ii) $\log_2 300$ iii) $\log_2 30$ iv) $\log_2 3$
   v) $\log_2 0.3$ vi) $\log_2 0.03$ vii) $\log_2 0.003$ viii) $\log_2 0.0003$
   b) Can you find a pattern in the results of part a)? Account for the pattern.

15. Prove that if (x_1,y_1) and (x_2,y_2) are two points on the graph of $y = \log_a x$, then $\left(\frac{x_1}{x_2}, y_1 - y_2\right)$ is also a point on the graph.

16. Prove that if (x_1,y_1) is a point on the graph of $y = \log_a x$, then $(x_1^2, 2y_1)$ and $\left(\frac{1}{x_1}, -y_1\right)$ are also points on the graph.

17. If $\log_3 2 = x$, simplify each logarithm.
a) $\log_3 8$ b) $\log_3 24$ c) $\log_3 \sqrt{2}$ d) $\log_3 6\sqrt{2}$

18. If $\log_2 5 = x$, simplify each logarithm.
a) $\log_2 20$ b) $\log_2 100$ c) $\log_2 10\sqrt{5}$ d) $\log_2 \left(\frac{\sqrt[3]{5}}{2}\right)$

19. Given that $\log_2 x = 5$, evaluate each logarithm.
a) $\log_2 2x$ b) $\log_2 \left(\frac{x}{2}\right)$ c) $\log_2 (x^2)$ d) $\log_2 (4x^2)$

20. Given that $\log_3 x = 2$ and $\log_3 y = 5$, evaluate each logarithm.
a) $\log_3 xy$ b) $\log_3 (9x^2y)$ c) $\log_3 \left(\frac{3x^2}{y}\right)$ d) $\log_3 (27x^{-2}y)$

21. Solve and check.
a) $2 \log x = \log 32 + \log 2$
b) $2 \log x = \log 3 + \log 27$
c) $\log_4 (x + 2) + \log_4 (x - 1) = 1$
d) $\log_2 (x - 5) + \log_2 (x - 2) = 2$
e) $\log_2 x + \log_2 (x + 2) = 3$
f) $\log_6 (x - 1) + \log_6 (x + 4) = 2$

22. Solve and check.
a) $2 \log m + 3 \log m = 10$
b) $\log_3 x^2 - \log_3 2x = 2$
c) $\log_3 s + \log_3 (s - 2) = 1$
d) $\log (x - 2) + \log (x + 1) = 1$
e) $\log_7 (x + 4) + \log_7 (x - 2) = 1$
f) $\log_2 (2m + 4) - \log_2 (m - 1) = 3$

Ⓒ

23. Solve each equation to the nearest thousandth.
a) $\log_2 x + \log_4 x = 5$
b) $\log_5 x + \log_{10} x = 5$

24. Prove.
a) $(\log_a b)(\log_b x) = \log_a x$
b) $(\log_a b)(\log_b c)(\log_c x) = \log_a x$

25. Let a represent any positive number other than 1. Prove that:
a) if $(x + 1)^2 = 4x$, then $\log_a \left(\frac{x+1}{2}\right) = \frac{1}{2} \log_a x$
b) if $(x + y)^2 = 4xy$, then $\log_a \left(\frac{x+y}{2}\right) = \frac{1}{2} (\log_a x + \log_a y)$.

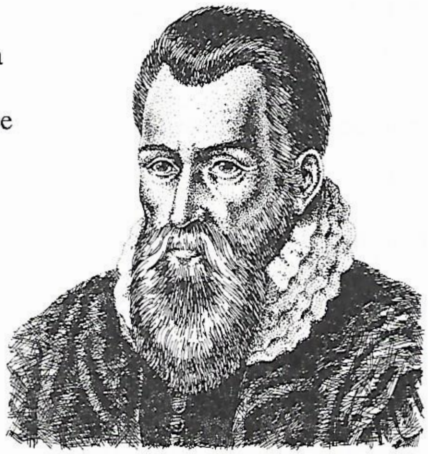
26. Determine how $\log_a x$ and $\log_x a$ are related.

27. a) Show that: i) $\frac{1}{\log_3 10} + \frac{1}{\log_4 10} = \frac{1}{\log_{12} 10}$ ii) $\frac{1}{\log_3 x} + \frac{1}{\log_4 x} = \frac{1}{\log_{12} x}$.
b) Using the results of part a) as a guide, state a general result and prove it.

THE MATHEMATICAL MIND

Natural Logarithms

Logarithms were introduced into mathematics almost four hundred years ago by the Scotsman, John Napier. The invention was enthusiastically hailed throughout Europe as a great breakthrough in computation. This was because logarithms can be used to reduce multiplication and division to the simpler operations of addition and subtraction. For example, the law of logarithms, $\log xy = \log x + \log y$, can be applied to multiply two numbers x and y by adding their logarithms. In the past, extensive tables of logarithms were prepared for this purpose. Of course, modern technology has rendered this method of computation obsolete.



Originally, Napier's logarithms had a certain base which was different from 10. These logarithms are called *natural logarithms*.

You can evaluate natural logarithms using the $\boxed{\ln}$ key on your calculator. For example, key in: 3 $\boxed{\ln}$ to display 1.0986123. We write $\ln 3 \doteq 1.098\ 612\ 3$, and we say "lawn 3 is approximately 1.098 612 3". To explain what this means, we need to know the base of the logarithms. The base of the natural logarithms is always represented by the letter e.

You can use your calculator to find the value of e.
Key in: 1 $\boxed{e^x}$ or 1 $\boxed{\text{INV}}$ $\boxed{\ln}$ to display 2.7182818.
Hence, $e \doteq 2.718\ 281\ 8$
Therefore, $\ln 3 \doteq 1.098\ 612\ 3$ means that $e^{1.0986123} \doteq 3$, where $e \doteq 2.718\ 281\ 8$.

Natural logarithms are a particular case of logarithms to base a, which were studied earlier in this chapter. Hence, natural logarithms have all the properties of logarithms to base a. This means that we can use natural logarithms to solve problems like those solved earlier.

For example, to solve the equation $e^x = 3.5$ for x, take the natural logarithm of both sides, and write $x \ln e = \ln 3.5$. Since $\ln e = 1$, then $x = \ln 3.5$. Key in: 3.5 $\boxed{\ln}$ to display 1.2527630. Hence, $x \doteq 1.252\ 763$.

QUESTIONS

1. Use your calculator to evaluate each logarithm. Then write the result in exponential form and check with the calculator.
   a) $\ln 2$ b) $\ln 4$ c) $\ln 30$ d) $\ln 100$
   e) $\ln 8750$ f) $\ln 0.5$ g) $\ln 0.1$ h) $\ln 0.000\,44$

2. Solve for x.
   a) $e^x = 5$ b) $e^x = 15$ c) $e^x = 53.9$ d) $e^x = 266$
   e) $e^x = 1$ f) $e^x = 0.25$ g) $e^x = 0.092$ h) $e^x = 0.0003$

3. Solve for x.
   a) $\ln x = 1$ b) $\ln x = 1.6$ c) $\ln x = 3$ d) $\ln x = 4.5$
   e) $\ln x = 0.33$ f) $\ln x = -1$ g) $\ln x = -1.4$ h) $\ln x = -2.2$

4. Write as a single logarithm, and check with your calculator.
   a) $\ln 5 + \ln 3$ b) $\ln 2 + \ln 10$ c) $2 \ln 6$
   d) $\ln 18 - \ln 2$ e) $\ln 21 - \ln 3$ f) $\frac{1}{2} \ln 25$

5. a) Simplify each expression.
   i) $\ln e$ ii) $\ln e^2$ iii) $\ln e^{-3}$ iv) $\ln e^{0.2}$
   b) Based on the results of part a), state a general result.

6. About 200 years ago, at age 15, Carl Friedrich Gauss noticed that the number of primes less than a given natural number n can be approximated by $\frac{n}{\ln n}$.
   Use this expression to approximate the number of primes less than:
   a) 10 b) 100 c) 1000 d) 10^6 e) 10^9.

7. Although it has never been proved, mathematicians have observed that the number of twin primes less than a given number n is approximately equal to $\frac{2n}{(\ln n)^2}$. Use this result to approximate the number of twin primes less than:
   a) 10 b) 100 c) 1000 d) 10^6 e) 10^9.

8. It has been proved that the average spacing of the prime numbers near a given natural number n is approximately equal to $\ln n$. For example, the six prime numbers closest to 50, and the successive differences between them are:
   $$\begin{array}{ccccccccccc} 41 & & 43 & & 47 & & 53 & & 59 & & 61 \\ & 2 & & 4 & & 6 & & 6 & & 2 & \end{array}$$
   The average spacing is $\frac{2 + 4 + 6 + 6 + 2}{5} = 4$.
   a) Find $\ln 50$, and compare it with the above result.
   b) Check that the average spacing of the six primes closest to:
   i) 100 is approximately $\ln 100$ ii) 150 is approximately $\ln 150$.

MATHEMATICS AROUND US

Applications of Natural Logarithms

In the applications of exponential and logarithmic functions studied in *Chapters 10* and *11*, many different bases were used. For example, in compound interest applications the base depended on the interest rate. In other applications we used bases 2, $\frac{1}{2}$, and 10. It would simplify matters to use the same base every time, and mathematicians have found that there is an advantage to using base e.

For example, consider population growth. In 1987 the world population reached 5 billion, and was increasing at about 1.6% per annum. Hence, an equation expressing the population P billion as a function of time t years relative to 1987 is

$$P = 5(1.016)^t \; . \; . \; . \; ①$$

Let's investigate what would happen if we express this equation with base e instead of base 10. To do this, we must write 1.016 as a power of e.

Let $1.016 = e^k$. Then, by definition, $k = \ln 1.016$

Key in: 1.016 $\boxed{\ln}$ to display 0.0158733

To two significant figures, $k \doteq 0.016$

Hence, $1.016 \doteq e^{0.016}$, and equation ① can be written as follows.

$$P = \underbrace{5}_{\text{Initial population}} e^{\underbrace{0.016t}_{\text{Growth rate}}}$$

Look at that! The constant in the exponent is 0.016, which is the growth rate. We now see an advantage of using base e. When an exponential function is expressed with base e, the constant in the exponent is the rate of growth. e is the only number with this property. Hence, it is the natural base to use in problems involving exponential growth and decay.

There is another advantage. Notice that the value of k obtained was not exactly 0.016. This slight discrepancy is caused by the way in which e is defined in higher mathematics. The definition assumes that the population grows continuously, and that the new members are not added all at once at the end of the year. In this case, the growth rate is called *instantaneous*. In the above example, the instantaneous rate of growth is 0.015 873 3, whereas the annual rate is 0.016. In some applications the difference may not be significant. Since a rigorous development of instantaneous rates of growth requires calculus, we will ignore its effect.

Example 1. In 1986 the population of Canada was 25.5 million, and was growing at the rate of approximately 1.0% per annum.

a) Write an equation for the population P million after t years.

b) Assuming that the growth rate remains constant, use the equation to determine:

i) the predicted population in the year 2000

ii) the number of years required for the population to reach 40 million.

Solution. a) The equation is $P = 25.5e^{0.01t}$.

b) i) The year 2000 is 14 years later than 1986. Hence, substitute 14 for t.

$$\begin{aligned} P &= 25.5e^{0.01(14)} \\ &= 25.5e^{0.14} \\ &\doteq 29.331\ 982 \end{aligned}$$

The population will be approximately 29.3 million in the year 2000.

ii) Substitute 40 for P.

$$40 = 25.5e^{0.01t}$$

To solve for t, take the natural logarithm of each side.

$$\begin{aligned} \ln 40 &= \ln 25.5 + 0.01t \\ t &= \frac{\ln 40 - \ln 25.5}{0.01} \\ &\doteq 45.020\ 100 \end{aligned}$$

The population will reach 40 million 45 years after 1986, or in the year 2031.

Example 2. In 1987 the world population reached 5 billion. According to United Nations forecasts, the population will reach 6.1 billion in the year 2001. Calculate the average annual rate of growth from 1987 to 2001.

Solution. Let $P = P_0e^{kt}$

Substitute 5 for P_0, 6.1 for P, and 14 for t.

$$6.1 = 5e^{14k}$$

Take the natural logarithm of each side.

$$\begin{aligned} \ln 6.1 &= \ln 5 + 14k \\ k &= \frac{\ln 6.1 - \ln 5}{14} \\ &\doteq 0.014\ 203\ 6 \end{aligned}$$

Hence, the average annual rate of growth is about 1.42%.

The conventions of writing $\log x$ to mean the logarithm to base 10 of x, and $\ln x$ to mean the logarithm to base e of x are by no means universal. In higher mathematics, natural logarithms are usually the only logarithms that are used, and $\log x$ often refers to the natural logarithm of x. Also, many computer languages use LOG(X) for the natural logarithm function.

QUESTIONS

1. Each equation represents the population P million of a country t years after 1985. State the 1985 population and the growth rate for each country.
   a) Italy $P = 57e^{0.007t}$
   b) Kenya $P = 20e^{0.030t}$
   c) Costa Rica $P = 2.6e^{0.038t}$

2. In 1985 the population of India was 770 million, and was growing at approximately 1.6% per annum.
   a) Write an equation for the population P million after t years, using an exponential function with base e.
   b) Assuming that the growth rate is constant, determine:
      i) the predicted population in 1995
      ii) when the population will reach 1 billion
      iii) when the population was 500 million.

3. When uranium-238 decays, the percent P remaining after t years is given by the equation $P = 100e^{-1.53 \times 10^{-10}t}$.
   a) What percent remains after 10 million years?
   b) Determine the halflife of uranium-238.

4. The altitude of an aircraft can be determined by measuring the air pressure. In the stratosphere (between 12 000 m and 30 000 m) the pressure P kilopascals is expressed as an exponential function of the altitude h metres by the equation $P = 130e^{-0.000155h}$.
   a) What is the altitude if the pressure is 8.5 kPa; 2.5 kPa?
   b) What is the pressure at an altitude of 20 000 m?
   c) Solve the equation for h to obtain an equation expressing the altitude as a logarithmic function of the pressure.

5. A rule of thumb which is used to approximate the time required for an investment to double in value is to divide 70 by the interest rate. For example, if the interest rate is 8%, then an investment will double in approximately $\frac{70}{8}$, or 9 years. Explain why the rule of thumb works.

1. Evaluate.
   a) $\log 100\,000$ b) $\log 0.001$ c) $\log \sqrt[3]{10}$ d) $\log_2 8$

2. Evaluate to 4 decimal places.
   a) $\log 6$ b) $\log 7.4$ c) $\log 19$ d) $\log 27.1$

3. Write in exponential form.
   a) $\log 1000 = 3$ b) $\log \sqrt{10} = \frac{1}{2}$ c) $\log_3 81 = 4$

4. Write in logarithmic form.
   a) $10^4 = 10\,000$ b) $10^{-3} = 0.001$ c) $5^4 = 625$

5. Solve for x.
   a) $\log x = 2$ b) $\log x = -5$ c) $\log_x 64 = 2$
   d) $\log_3 x = 3$ e) $\log_5 0.04 = x$ f) $\log_2 x = 5$

6. Simplify.
   a) $\log 10^5$ b) $\log 10^{3.1}$ c) $\log 10^{-1.5}$ d) $\log_3 3^7$ e) $\log_{11} 121$

7. Simplify.
   a) $10^{\log 7}$ b) $10^{\log 2.8}$ c) $10^{\log 0.09}$ d) $6^{\log_6 2}$ e) $17^{\log_{17} 11}$

8. Write as a single logarithm.
   a) $\log 7 + \log 4 - \log 5$ b) $\log_5 142 - \log_5 19 - \log_5 3$
   c) $\log p - \log q + \log r$ d) $\log (2a - 3) + \log (a + 5)$
   e) $2 \log_4 m + 5 \log_4 n - 3 \log_4 p$ f) $\frac{2}{3} \log_a x - \frac{1}{4} \log_a y - \log_a z$

9. If $\log 6 = m$ and $\log 5 = n$, write each logarithm as an expression in m and n.
   a) $\log 30$ b) $\log 1.2$ c) $\log 7.2$ d) $\log 0.24$

10. Given that $\log 7 \doteq 0.8451$ find an approximation for each logarithm.
    a) $\log 7^{\frac{1}{2}}$ b) $\log 343$ c) $\log \sqrt[3]{7}$ d) $\log \left(\frac{1}{49}\right)$

11. Express.
    a) 8 as a power of 3 b) 24 as a power of 6
    c) 12 as a power of 1.3 d) 0.78 as a power of 2

12. Solve for x. Give the answers to 4 decimal places.
    a) $5^x = 9$ b) $14^x = 8$ c) $3^{2x-1} = 25$ d) $4^{5-x} = 45$
    e) $7^{3-x} = 4$ f) $8^{5x-2} = 69$ g) $2^{1-x} = 9^{x+1}$ h) $5^{3x+1} = 12^{x+4}$

13. m and n are two positive numbers. How are $\log m$ and $\log n$ related if:
    a) $m = 100n$ b) $m = n^3$ c) $m = \sqrt{10n}$ d) $m = \frac{1}{\sqrt{n}}$?

14. Express x as a logarithmic function of y.
    a) $y = 2^x$ b) $y = 3(5)^x$ c) $y = 2.7(8)^x$

15. Evaluate.

a) $\log 10\,000$ b) $\log_2 16$ c) $\log_3 243$ d) $\log_2\left(\frac{1}{8}\right)$

e) $\log_{\frac{1}{3}} 27$ f) $\log_{\sqrt{2}} 32$ g) $\log_5 0.008$ h) $\log_7 343$

16. Evaluate to the nearest thousandth.

a) $\log_2 7$ b) $\log_{12} 8$ c) $\log_{3.5} 19.1$ d) $\log_{\frac{1}{4}} 0.65$

e) $\log_5 42$ f) $\log_{2.1} 78$ g) $\log_{1.8} 27.3$ h) $\log_{0.4} 0.21$

17. Solve for x.

a) $\log_5 x = -3$ b) $\log_{17} 1 = x$ c) $\log_x 64 = -3$

18. Simplify.

a) $\log_6 90 + \log_6 12 - \log_6 5$ b) $\log_3 24 - \log_3 16 + \log_3 6\sqrt{3}$

c) $\log_8 16 + \log_8 5 - \log_8 2.5$ d) $\log_5 8 - \log_5 40 - \log_5 50 + \log_5 10$

19. If $\log_3 4 \doteq 1.2619$ find an approximation for each logarithm.

a) $\log_3 12$ b) $\log_3 64$ c) $\log_3\left(\frac{16}{3}\right)$ d) $\log_3 2$

20. Solve and check.

a) $3 \log x = \log 512 - \log 8$ b) $\log_2 x + \log_2 (x - 3) = 2$

c) $\log_{\sqrt{2}} (x - 2) + \log_{\sqrt{2}} (x + 1) = 4$ d) $\log_6 (x + 3) + \log_6 (x - 2) = 1$

21. Graph each function and its inverse on the same grid.

a) $y = 3^x$ b) $y = \log_5 x$ c) $y = \log_{\frac{1}{3}} x$

22. The halflife of a radioactive substance is 23 days. How long is it until the percent remaining is:

a) 10% b) 3% ?

23. a) An air filter loses about 0.3% of its effectiveness each day. What is its effectiveness after 145 days as a percent of its initial effectiveness?

b) The filter should be replaced when its effectiveness has decreased to 20% of its initial value. After how long should it be replaced?

24. If the population of a city is 178 500 and it is growing at the rate of 2.1% per annum, in how many years will the population be 210 000?

25. The pH of water in a small lake in northern Quebec has dropped from 5.4 to 4.8 in the last three years. How many times as acidic as it was three years ago, is the lake now?

26. If a coil spring is stretched 1.5 m beyond its resting point and then released it will return to a point which is 90% of the previous distance from the resting point. How many vibrations are required before the spring moves less than 10 cm from its resting point?