NOTATION. Let $C_c^\infty(U)$ denote the space of infinitely differentiable functions $\phi : U \to \mathbb{R}$, with compact support in U. We will sometimes call a function ϕ belonging to $C_c^\infty(U)$ a *test function*. □

Motivation for definition of weak derivative. Assume we are given a function $u \in C^1(U)$. Then if $\phi \in C_c^\infty(U)$, we see from the integration by parts formula that

$$\int_U u\phi_{x_i}\,dx = -\int_U u_{x_i}\phi\,dx \quad (i = 1, \ldots, n). \tag{1}$$

There are no boundary terms, since ϕ has compact support in U and thus vanishes near ∂U. More generally now, if k is a positive integer, $u \in C^k(U)$, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multiindex of order $|\alpha| = \alpha_1 + \cdots + \alpha_n = k$, then

$$\int_U uD^\alpha\phi\,dx = (-1)^{|\alpha|}\int_U D^\alpha u\phi\,dx. \tag{2}$$

This equality holds since

$$D^\alpha\phi = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}\phi$$

and we can apply formula (1) $|\alpha|$ times.

We next examine formula (2), valid for $u \in C^k(U)$, and ask whether some variant of it might still be true even if u is not k times continuously differentiable. Now the left-hand side of (2) makes sense if u is only locally summable: the problem is rather that if u is not C^k, then the expression "$D^\alpha u$" on the right-hand side of (2) has no obvious meaning. We resolve this difficulty by asking if there exists a locally summable function v for which formula (2) is valid, with v replacing $D^\alpha u$:

DEFINITION. *Suppose $u, v \in L^1_{\text{loc}}(U)$ and α is a multiindex. We say that v is the* α^{th}-weak partial derivative of u, *written*

$$D^\alpha u = v,$$

provided

$$\int_U uD^\alpha\phi\,dx = (-1)^{|\alpha|}\int_U v\phi\,dx \tag{3}$$

for all test functions $\phi \in C_c^\infty(U)$.

In other words, if we are given u and if there happens to exist a function v which verifies (3) for all ϕ, we say that $D^\alpha u = v$ in the weak sense. If there does not exist such a function v, then u does not possess a weak α^{th}-partial derivative.

LEMMA (Uniqueness of weak derivatives). *A weak α^{th}-partial derivative of u, if it exists, is uniquely defined up to a set of measure zero.*

Proof. Assume that $v, \tilde{v} \in L^1_{\text{loc}}(U)$ satisfy

$$\int_U u D^\alpha \phi \, dx = (-1)^{|\alpha|} \int_U v\phi \, dx = (-1)^{|\alpha|} \int_U \tilde{v}\phi \, dx$$

for all $\phi \in C_c^\infty(U)$. Then

$$\int_U (v - \tilde{v})\phi \, dx = 0 \tag{4}$$

for all $\phi \in C_c^\infty(U)$, whence $v - \tilde{v} = 0$ a.e. □

Example 1. Let $n = 1$, $U = (0, 2)$, and

$$u(x) = \begin{cases} x & \text{if } 0 < x \le 1 \\ 1 & \text{if } 1 \le x < 2. \end{cases}$$

Define

$$v(x) = \begin{cases} 1 & \text{if } 0 < x \le 1 \\ 0 & \text{if } 1 < x < 2. \end{cases}$$

Let us show $u' = v$ in the weak sense. To see this, choose any $\phi \in C_c^\infty(U)$. We must demonstrate

$$\int_0^2 u\phi' \, dx = -\int_0^2 v\phi \, dx.$$

But we easily calculate

$$\int_0^2 u\phi' \, dx = \int_0^1 x\phi' \, dx + \int_1^2 \phi' \, dx$$
$$= -\int_0^1 \phi \, dx + \phi(1) - \phi(1) = -\int_0^2 v\phi \, dx,$$

as required. □

Example 2. Let $n = 1$, $U = (0, 2)$, and

$$u(x) = \begin{cases} x & \text{if } 0 < x \le 1 \\ 2 & \text{if } 1 < x < 2. \end{cases}$$

We assert u' does not exist in the weak sense. To check this, we must show there does not exist any function $v \in L^1_{\text{loc}}(U)$ satisfying

$$\int_0^2 u\phi'\,dx = -\int_0^2 v\phi\,dx \tag{5}$$

for all $\phi \in C_c^\infty(U)$. Suppose, to the contrary, (5) were valid for some v and all ϕ. Then

$$\begin{aligned} -\int_0^2 v\phi\,dx &= \int_0^2 u\phi'\,dx = \int_0^1 x\phi'\,dx + 2\int_1^2 \phi'\,dx \\ &= -\int_0^1 \phi\,dx - \phi(1). \end{aligned} \tag{6}$$

Choose a sequence $\{\phi_m\}_{m=1}^\infty$ of smooth functions satisfying

$$0 \le \phi_m \le 1, \quad \phi_m(1) = 1, \quad \phi_m(x) \to 0 \text{ for all } x \neq 1.$$

Replacing ϕ by ϕ_m in (6) and sending $m \to \infty$, we discover

$$1 = \lim_{m\to\infty} \phi_m(1) = \lim_{m\to\infty} \left[\int_0^2 v\phi_m\,dx - \int_0^1 \phi_m\,dx\right] = 0,$$

a contradiction. □

More sophisticated examples appear in the next subsection.

5.2.2. Definition of Sobolev spaces.

Fix $1 \le p \le \infty$ and let k be a nonnegative integer. We define now certain function spaces, whose members have weak derivatives of various orders lying in various L^p spaces.

DEFINITION. *The* Sobolev space

$$W^{k,p}(U)$$

consists of all locally summable functions $u : U \to \mathbb{R}$ such that for each multiindex α with $|\alpha| \le k$, $D^\alpha u$ exists in the weak sense and belongs to $L^p(U)$.

Remarks. (i) If $p = 2$, we usually write

$$H^k(U) = W^{k,2}(U) \quad (k = 0, 1, \dots).$$

The letter H is used, since—as we will see—$H^k(U)$ is a Hilbert space. Note that $H^0(U) = L^2(U)$.

(ii) We henceforth identify functions in $W^{k,p}(U)$ which agree a.e.

DEFINITION. *If $u \in W^{k,p}(U)$, we define its* norm *to be*

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\sum_{|\alpha|\le k} \int_U |D^\alpha u|^p \, dx\right)^{1/p} & (1 \le p < \infty) \\ \sum_{|\alpha|\le k} \operatorname{ess\,sup}_U |D^\alpha u| & (p = \infty). \end{cases}$$

DEFINITIONS. (i) *Let $\{u_m\}_{m=1}^\infty$, $u \in W^{k,p}(U)$. We say u_m* converges *to u in $W^{k,p}(U)$, written*

$$u_m \to u \quad \text{in } W^{k,p}(U),$$

provided

$$\lim_{m\to\infty} \|u_m - u\|_{W^{k,p}(U)} = 0.$$

(ii) *We write*

$$u_m \to u \quad \text{in } W^{k,p}_{\text{loc}}(U)$$

to mean

$$u_m \to u \quad \text{in } W^{k,p}(V)$$

for each $V \subset\subset U$.

DEFINITION. *We denote by*

$$W^{k,p}_0(U)$$

the closure of $C^\infty_c(U)$ in $W^{k,p}(U)$.

Thus $u \in W^{k,p}_0(U)$ if and only if there exist functions $u_m \in C^\infty_c(U)$ such that $u_m \to u$ in $W^{k,p}(U)$. We interpret $W^{k,p}_0(U)$ as comprising those functions $u \in W^{k,p}(U)$ such that

$$\text{"}D^\alpha u = 0 \text{ on } \partial U\text{" for all } |\alpha| \le k-1.$$

This will all be made clearer with the discussion of traces in §5.5.

NOTATION. It is customary to write

$$H^k_0(U) = W^{k,2}_0(U).$$

We will see in the exercises that if $n = 1$ and U is an open interval in $\mathbb{R}^1$, then $u \in W^{1,p}(U)$ if and only if u equals a.e. an absolutely continuous function whose ordinary derivative (which exists a.e.) belongs to $L^p(U)$. Such a simple characterization is however only available for $n = 1$. In general a function can belong to a Sobolev space and yet be discontinuous and/or unbounded.

Example 3. Take $U = B^0(0,1)$, the open unit ball in $\mathbb{R}^n$, and

$$u(x) = |x|^{-\alpha} \quad (x \in U,\ x \neq 0).$$

For which values of $\alpha > 0, n, p$ does u belong to $W^{1,p}(U)$? To answer, note first that u is smooth away from 0, with

$$u_{x_i}(x) = \frac{-\alpha x_i}{|x|^{\alpha+2}} \quad (x \neq 0),$$

and so

$$|Du(x)| = \frac{|\alpha|}{|x|^{\alpha+1}} \quad (x \neq 0).$$

Let $\phi \in C_c^\infty(U)$ and fix $\varepsilon > 0$. Then

$$\int_{U-B(0,\varepsilon)} u\phi_{x_i}\, dx = -\int_{U-B(0,\varepsilon)} u_{x_i}\phi\, dx + \int_{\partial B(0,\varepsilon)} u\phi\nu^i\, dS,$$

$\nu = (\nu^1, \dots, \nu^n)$ denoting the inward pointing normal on $\partial B(0,\varepsilon)$. Now if $\alpha + 1 < n$, $|Du(x)| \in L^1(U)$. In this case

$$\left| \int_{\partial B(0,\varepsilon)} u\phi\nu^i\, dS \right| \leq \|\phi\|_{L^\infty} \int_{\partial B(0,\varepsilon)} \varepsilon^{-\alpha}\, dS \leq C\varepsilon^{n-1-\alpha} \to 0.$$

Thus

$$\int_U u\phi_{x_i}\, dx = -\int_U u_{x_i}\phi\, dx$$

for all $\phi \in C_c^\infty(U)$, provided $0 \leq \alpha < n-1$. Furthermore $|Du(x)| = \frac{|\alpha|}{|x|^{\alpha+1}} \in L^p(U)$ if and only if $(\alpha+1)p < n$. Consequently $u \in W^{1,p}(U)$ if and only if $\alpha < \frac{n-p}{p}$. In particular $u \notin W^{1,p}(U)$ for each $p \geq n$. □

Example 4. Let $\{r_k\}_{k=1}^\infty$ be a countable, dense subset of $U = B^0(0,1)$. Write

$$u(x) = \sum_{k=1}^\infty \frac{1}{2^k}|x - r_k|^{-\alpha} \quad (x \in U).$$

Then $u \in W^{1,p}(U)$ for $\alpha < \frac{n-p}{p}$. If $0 < \alpha < \frac{n-p}{p}$, we see that u belongs to $W^{1,p}(U)$ and yet is unbounded on each open subset of U. □

This last example illustrates a fundamental fact of life, that although a function u belonging to a Sobolev space possesses certain smoothness properties, it can still be rather badly behaved in other ways.

5.2.3. Elementary properties.

Next we verify certain properties of weak derivatives. Note very carefully that whereas these various rules are obviously true for smooth functions, functions in Sobolev space are not necessarily smooth: we must always rely solely upon the definition of weak derivatives.

THEOREM 1 (Properties of weak derivatives). *Assume $u, v \in W^{k,p}(U)$, $|\alpha| \le k$. Then*

(i) *$D^\alpha u \in W^{k-|\alpha|,p}(U)$ and $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta}u$ for all multiindices α, β with $|\alpha| + |\beta| \le k$.*

(ii) *For each $\lambda, \mu \in \mathbb{R}$, $\lambda u + \mu v \in W^{k,p}(U)$ and $D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v$, $|\alpha| \le k$.*

(iii) *If V is an open subset of U, then $u \in W^{k,p}(V)$.*

(iv) *If $\zeta \in C_c^\infty(U)$, then $\zeta u \in W^{k,p}(U)$ and*

$$D^\alpha(\zeta u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta \zeta D^{\alpha-\beta} u \quad \textit{(Leibniz's formula)}, \tag{7}$$

where $\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$.

Proof. 1. To prove (i), first fix $\phi \in C_c^\infty(U)$. Then $D^\beta \phi \in C_c^\infty(U)$, and so

$$\begin{aligned}\int_U D^\alpha u D^\beta \phi \, dx &= (-1)^{|\alpha|} \int_U u D^{\alpha+\beta} \phi \, dx \\ &= (-1)^{|\alpha|}(-1)^{|\alpha+\beta|} \int_U D^{\alpha+\beta} u \phi \, dx \\ &= (-1)^{|\beta|} \int_U D^{\alpha+\beta} u \phi \, dx.\end{aligned}$$

Thus $D^\beta(D^\alpha u) = D^{\alpha+\beta}u$ in the weak sense.

2. Assertions (ii) and (iii) are easy, and the proofs are omitted.

3. We prove (7) by induction on $|\alpha|$. Suppose first $|\alpha| = 1$. Choose any $\phi \in C_c^\infty(U)$. Then

$$\begin{aligned}\int_U \zeta u D^\alpha \phi \, dx &= \int_U u D^\alpha(\zeta\phi) - u(D^\alpha \zeta)\phi \, dx \\ &= -\int_U (\zeta D^\alpha u + u D^\alpha \zeta)\phi \, dx.\end{aligned}$$

Thus $D^\alpha(\zeta u) = \zeta D^\alpha u + u D^\alpha \zeta$, as required.

Next assume $l < k$ and formula (7) is valid for all $|\alpha| \le l$ and all functions ζ. Choose a multiindex α with $|\alpha| = l+1$. Then $\alpha = \beta + \gamma$ for some $|\beta| = l$, $|\gamma| = 1$. Then for ϕ as above,

$$\int_U \zeta u D^\alpha \phi \, dx = \int_U \zeta u D^\beta (D^\gamma \phi) \, dx$$

$$= (-1)^{|\beta|} \int_U \sum_{\sigma \le \beta} \binom{\beta}{\sigma} D^\sigma \zeta D^{\beta-\sigma} u D^\gamma \phi \, dx$$

(by the induction assumption)

$$= (-1)^{|\beta|+|\gamma|} \int_U \sum_{\sigma \le \beta} \binom{\beta}{\sigma} D^\gamma (D^\sigma \zeta D^{\beta-\sigma} u) \phi \, dx$$

(by the induction assumption again)

$$= (-1)^{|\alpha|} \int_U \sum_{\sigma \le \beta} \binom{\beta}{\sigma} [D^\rho \zeta D^{\alpha-\rho} u + D^\sigma \zeta D^{\alpha-\sigma} u] \phi \; dx$$

(where $\rho = \sigma + \gamma$)

$$= (-1)^{|\alpha|} \int_U \left[\sum_{\sigma \le \alpha} \binom{\alpha}{\sigma} D^\sigma \zeta D^{\alpha-\sigma} u \right] \phi \; dx,$$

since

$$\binom{\beta}{\sigma - \gamma} + \binom{\beta}{\sigma} = \binom{\alpha}{\sigma}. \qquad \square$$

Not only do many of the usual rules of calculus apply to weak derivatives, but the Sobolev spaces themselves have a good mathematical structure:

THEOREM 2 (Sobolev spaces as function spaces). *For each $k = 1, \dots$ and $1 \le p \le \infty$, the Sobolev space $W^{k,p}(U)$ is a Banach space.*

Proof. 1. Let us first of all check that $\|u\|_{W^{k,p}(U)}$ is a norm. (See the discussion at the end of §5.1, or refer to §D.1, for definitions.) Clearly

$$\|\lambda u\|_{W^{k,p}(U)} = |\lambda| \|u\|_{W^{k,p}(U)},$$

and

$$\|u\|_{W^{k,p}(U)} = 0 \text{ if and only if } u = 0 \text{ a.e.}$$

Next assume $u, v \in W^{k,p}(U)$. Then if $1 \leq p < \infty$, Minkowski's inequality (§B.2) implies

$$\begin{aligned}
\|u+v\|_{W^{k,p}(U)} &= \left(\sum_{|\alpha|\leq k} \|D^\alpha u + D^\alpha v\|^p_{L^p(U)} \right)^{1/p} \\
&\leq \left(\sum_{|\alpha|\leq k} (\|D^\alpha u\|_{L^p(U)} + \|D^\alpha v\|_{L^p(U)})^p \right)^{1/p} \\
&\leq \left(\sum_{|\alpha|\leq k} \|D^\alpha u\|^p_{L^p(U)} \right)^{1/p} + \left(\sum_{|\alpha|\leq k} \|D^\alpha v\|^p_{L^p(U)} \right)^{1/p} \\
&= \|u\|_{W^{k,p}(U)} + \|v\|_{W^{k,p}(U)}.
\end{aligned}$$

2. It remains to show that $W^{k,p}(U)$ is complete. So assume $\{u_m\}_{m=1}^\infty$ is a Cauchy sequence in $W^{k,p}(U)$. Then for each $|\alpha| \leq k$, $\{D^\alpha u_m\}_{m=1}^\infty$ is a Cauchy sequence in $L^p(U)$. Since $L^p(U)$ is complete, there exist functions $u_\alpha \in L^p(U)$ such that

$$D^\alpha u_m \to u_\alpha \quad \text{in } L^p(U)$$

for each $|\alpha| \leq k$. In particular,

$$u_m \to u_{(0,\ldots,0)} =: u \quad \text{in } L^p(U).$$

3. We now claim

$$u \in W^{k,p}(U), \; D^\alpha u = u_\alpha \quad (|\alpha| \leq k). \tag{8}$$

To verify this assertion, fix $\phi \in C_c^\infty(U)$. Then

$$\begin{aligned}
\int_U u D^\alpha \phi \, dx &= \lim_{m\to\infty} \int_U u_m D^\alpha \phi \, dx \\
&= \lim_{m\to\infty} (-1)^{|\alpha|} \int_U D^\alpha u_m \phi \, dx \\
&= (-1)^{|\alpha|} \int_U u_\alpha \phi \, dx.
\end{aligned}$$

Thus (8) is valid. Since therefore $D^\alpha u_m \to D^\alpha u$ in $L^p(U)$ for all $|\alpha| \leq k$, we see that $u_m \to u$ in $W^{k,p}(U)$, as required. □

5.3. APPROXIMATION

5.3.1. Interior approximation by smooth functions.

It is awkward to return continually to the definition of weak derivatives. In order to study the deeper properties of Sobolev spaces, we therefore need to develop some systematic procedures for approximating a function in a Sobolev space by smooth functions. The method of mollifiers, set forth in §C.4, provides the tool.

Fix a positive integer k and $1 \leq p < \infty$. Remember that $U_\varepsilon = \{x \in U \mid \operatorname{dist}(x, \partial U) > \varepsilon\}$.

THEOREM 1 (Local approximation by smooth functions). *Assume $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$, and set*

$$u^\varepsilon = \eta_\varepsilon * u \quad \text{in } U_\varepsilon.$$

Then

(i) $u^\varepsilon \in C^\infty(U_\varepsilon)$ *for each* $\varepsilon > 0$,

and

(ii) $u^\varepsilon \to u$ *in* $W^{k,p}_{\mathrm{loc}}(U)$, *as* $\varepsilon \to 0$.

Proof. 1. Assertion (i) is proved in §C.4.

2. We next claim that if $|\alpha| \leq k$, then

$$D^\alpha u^\varepsilon = \eta_\varepsilon * D^\alpha u \quad \text{in } U_\varepsilon; \tag{1}$$

that is, *the ordinary α^{th}-partial derivative of the smooth function u^ε is the ε-mollification of the α^{th}-weak partial derivative of u.* To confirm this, we compute for $x \in U_\varepsilon$

$$\begin{aligned} D^\alpha u^\varepsilon(x) &= D^\alpha \int_U \eta_\varepsilon(x-y)u(y)\,dy \\ &= \int_U D^\alpha_x \eta_\varepsilon(x-y)u(y)\,dy \\ &= (-1)^{|\alpha|}\int_U D^\alpha_y \eta_\varepsilon(x-y)u(y)\,dy. \end{aligned}$$

Now for fixed $x \in U_\varepsilon$ the function $\phi(y) := \eta_\varepsilon(x-y)$ belongs to $C^\infty_c(U)$. Consequently the definition of the α^{th}-weak partial derivative implies:

$$\int_U D^\alpha_y \eta_\varepsilon(x-y)u(y)\,dy = (-1)^{|\alpha|}\int_U \eta_\varepsilon(x-y)D^\alpha u(y)\,dy.$$

Thus

$$\begin{aligned} D^\alpha u^\varepsilon(x) &= (-1)^{|\alpha|+|\alpha|} \int_U \eta_\varepsilon(x-y) D^\alpha u(y)\, dy \\ &= [\eta_\varepsilon * D^\alpha u]\,(x). \end{aligned}$$

This establishes (1).

3. Now choose an open set $V \subset\subset U$. In view of (1) and §C.4, $D^\alpha u^\varepsilon \to D^\alpha u$ in $L^p(V)$ as $\varepsilon \to 0$, for each $|\alpha| \leq k$. Consequently

$$\|u^\varepsilon - u\|^p_{W^{k,p}(V)} = \sum_{|\alpha|\leq k} \|D^\alpha u^\varepsilon - D^\alpha u\|^p_{L^p(V)} \to 0$$

as $\varepsilon \to 0$. This proves assertion (ii). □

5.3.2. Approximation by smooth functions.

Next we show that we can find smooth functions which approximate in $W^{k,p}(U)$ and not just in $W^{k,p}_{\text{loc}}(U)$. Notice in the following that we make no assumptions about the smoothness of ∂U.

THEOREM 2 (Global approximation by smooth functions). *Assume U is bounded, and suppose as well that $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Then there exist functions $u_m \in C^\infty(U) \cap W^{k,p}(U)$ such that*

$$u_m \to u \quad \text{in } W^{k,p}(U).$$

Note carefully that we do not assert $u_m \in C^\infty(\bar{U})$ (but see Theorem 3 below).

Proof. 1. We have $U = \bigcup_{i=1}^\infty U_i$, where

$$U_i := \{x \in U \mid \operatorname{dist}(x, \partial U) > 1/i\} \quad (i = 1, 2, \dots).$$

Write $V_i := U_{i+3} - \bar{U}_{i+1}$.

Choose also any open set $V_0 \subset\subset U$ so that $U = \bigcup_{i=0}^\infty V_i$. Now let $\{\zeta_i\}_{i=0}^\infty$ be a smooth partition of unity subordinate to the open sets $\{V_i\}_{i=0}^\infty$; that is, suppose

$$\text{(2)} \qquad \begin{cases} 0 \leq \zeta_i \leq 1, & \zeta_i \in C_c^\infty(V_i) \\ \sum_{i=0}^\infty \zeta_i = 1 & \text{on } U. \end{cases}$$

Next, choose any function $u \in W^{k,p}(U)$. According to Theorem 1(iv) in §5.2, $\zeta_i u \in W^{k,p}(U)$ and $\operatorname{spt}(\zeta_i u) \subset V_i$.

2. Fix $\delta > 0$. Choose then $\varepsilon_i > 0$ so small that $u^i := \eta_{\varepsilon_i} * (\zeta_i u)$ satisfies

$$
(3) \qquad \begin{cases} \|u^i - \zeta_i u\|_{W^{k,p}(U)} \leq \frac{\delta}{2^{i+1}} & (i = 0, 1, \dots) \\ \operatorname{spt} u^i \subset W_i & (i = 1, \dots), \end{cases}
$$

for $W_i := U_{i+4} - \bar{U}_i \supset V_i$ $(i = 1, \dots)$.

3. Write $v := \sum_{i=0}^{\infty} u^i$. This function belongs to $C^\infty(U)$, since for each open set $V \subset\subset U$ there are at most finitely many nonzero terms in the sum. Since $u = \sum_{i=0}^{\infty} \zeta_i u$, we have for each $V \subset\subset U$

$$
\begin{aligned}
\|v - u\|_{W^{k,p}(V)} &\leq \sum_{i=0}^{\infty} \|u^i - \zeta_i u\|_{W^{k,p}(U)} \\
&\leq \delta \sum_{i=0}^{\infty} \frac{1}{2^{i+1}} \quad \text{by (3)} \\
&= \delta.
\end{aligned}
$$

Take the supremum over sets $V \subset\subset U$, to conclude $\|v - u\|_{W^{k,p}(U)} \leq \delta$. □

5.3.3. Global approximation by smooth functions.

We now ask when it is possible to approximate a function $u \in W^{k,p}(U)$ by functions belonging to $C^\infty(\bar{U})$, rather than only $C^\infty(U)$. Such an approximation requires some condition to exclude ∂U being wild geometrically.

THEOREM 3 (Global approximation by functions smooth up to the boundary). *Assume U is bounded and ∂U is C^1. Suppose $u \in W^{k,p}(U)$ for some $1 \leq p < \infty$. Then there exist functions $u_m \in C^\infty(\bar{U})$ such that*

$$
u_m \to u \quad \text{in } W^{k,p}(U).
$$

Proof. 1. Fix any point $x^0 \in \partial U$. As ∂U is C^1, there exist, according to §C.1, a radius $r > 0$ and a C^1 function $\gamma : \mathbb{R}^{n-1} \to \mathbb{R}$ such that—upon relabeling the coordinate axes if necessary—we have

$$
U \cap B(x^0, r) = \{x \in B(x^0, r) \mid x_n > \gamma(x_1, \dots, x_{n-1})\}.
$$

Set $V := U \cap B(x^0, r/2)$.

2. Define the shifted point

$$
x^\varepsilon := x + \lambda \varepsilon e_n \quad (x \in V,\ \varepsilon > 0),
$$