# Time Series

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# Difference Equations

A Difference Equation is an expression relating a variable $y_t$ to its previous values.

Linear First-order Difference Equation:
(only the first lag of $y_t$ appears in the equation)

$$y_t = \phi y_{t-1} + w_t,$$

where $w_t$ is called the input variable and $y_t$ is called the output variable.

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Linear Second-order Difference Equation:

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t.$$

# Solving a Difference Equation by Recursive Substitution

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By recursive substitution

$$y_t = \phi^t y_0 + \phi^{t-1} w_1 + \phi_{t-2} w_2 + \ldots + \phi w_{t-1} + w_t$$

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The effect of $w_t$ on $y_{t+j}$ is given by

$$\frac{\partial y_{t+j}}{\partial w_t} = \phi^j,$$

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Different values of $\phi$ can produce a variety of dynamic responses of $y$ to $w$.

# The Effect of the Input Variable

- If $0 < \phi < 1$, the multiplier $\phi^j$ decays geometrically towards zero. In this case, the system is stable.
- If $-1 < \phi < 0$, $\phi^j$ alternates signs, with $\mid \phi^j \mid$ decaying geometrically towards zero. The system is still stable.
- If $\phi > 1$, $\phi^j$ increases exponentially over time.
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Note that, for $\mid \phi \mid < 1$, the system is stable. For $\mid \phi \mid > 1$, the system is explosive. An interesting possibility is the borderline case, $\phi = 1$:

$$y_{t+j} = y_t + w_t + w_{t-1} + \ldots + w_{t+j-1} + w_{t+j}.$$

Hence, $\frac{\partial y_{t+j}}{\partial w_t} = 1$, for $j = 0, 1, 2, \ldots$

$$y_t = \phi y_{t-1} + w_t,$$

or, equivalently,

$$y_t = \phi B y_t + w_t,$$

$$(1 - \phi B) y_t = w_t,$$

where "1" denotes the identity operator, i.e $1 y_t = y_t$, and $(1 - \phi B)^{-1}(1 - \phi B) = 1$.

# First-order Difference Equations - The Backward Operator

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Therefore,

$$y_t = (1 - \phi B)^{-1} w_t,$$

In order for the output variable to be bounded, the right hand side of this equation has to converge.

# Second-order Difference Equations

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + w_t,$$

or, equivalently,

$$(1 - \phi_1 B - \phi_2 B^2) y_t = w_t,$$

$$(1 - \lambda_1 B)(1 - \lambda_2 B) y_t = w_t,$$

where $\lambda_1 + \lambda_2 = \phi 1$ and $\lambda_1 \lambda_2 = -\phi_2$.

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Solving the system: $\lambda_1 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2}$ and $\lambda_2 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}$.

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Therefore,

$$y_t = (1 - \lambda_1 B)^{-1} (1 - \lambda_2 B)^{-1} w_t,$$

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# *p*th Order Difference Equations - AR(p) Models

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- The previous calculations and observations generalise to *p*th Order Difference Equations.
- AR models are *p*th Order Stochastic Difference Equations. These models are used to describe dynamic relationships observed in discrete-time data.
- Similarly, Stochastic Differential Equations are used to model continuous-time data.

# Autoregressive Models

AR(1) Model: $(1 - \phi_1 B)y_t = \Phi(B)y_t = \epsilon_t$
where $\Phi(B) = (1 - \phi_1 B)$ and the $\Phi(B)^{-1}$ is such that
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The polynomial $\Phi(z) = (1-\phi_1 z)$ is called the characteristic polynomial of the AR(1) model. We have that

$$(1-\phi_1 z)^{-1} = 1 + \phi_1 z + \phi_1^2 z^2 + \dots$$

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The AR(1) model is stationary if the root of its characteristic polynomial lies outside of the unit circle, that is if $\mid \frac{1}{\phi_1} \mid > 1$. Solving for $\phi_1$ we get the stationarity condition $\mid \phi_1 \mid < 1$.

# Autoregressive Models

AR(2) Model: $(1 - \phi_1 B - \phi_2 B^2) y_t = \Phi(B) y_t = \epsilon_t$
where $\Phi(B) = (1 - \phi_1 B - \phi_2 B^2) = (1 - \lambda_1 B)(1 - \lambda_2 B)$, with
$\lambda_1 = \frac{\phi_1 + \sqrt{\phi_1^2 + 4\phi_2}}{2}$ and $\lambda_2 = \frac{\phi_1 - \sqrt{\phi_1^2 + 4\phi_2}}{2}$.

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Then,

$$y_t = (1-\lambda_1 B)^{-1}(1-\lambda_2 B)^{-1}\epsilon_t,$$

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AR(2) Model: $(1-\phi_1 B-\phi_2 B^2)y_t=\Phi(B)y_t=\epsilon_t$
where $\Phi(B)=(1-\phi_1 B-\phi_2 B^2)=(1-\lambda_1 B)(1-\lambda_2 B)$, with
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Then,

$$y_t=(1-\lambda_1 B)^{-1}(1-\lambda_2 B)^{-1}\epsilon_t,$$

The time series, $y_t$ is stationary if the righthand side of this equation converges.

The polynomial $\Phi(z)=(1-\phi_1 z-phi_2 z^2)$ is called the characteristic polynomial of the AR(2) model. The AR(2) model is stationary if the roots of its characteristic polynomial lie outside of the unit circle, that is if $\mid \frac{1}{\lambda_1} \mid > 1$ and $\mid \frac{1}{\lambda_2} \mid > 1$.

**Find the stationarity conditions of the AR(2) model.**