Ασ

$$\begin{cases} -u'' = \lambda u \ , & x \in (0,\beta) \\ u(0) = 0 \ , & u(\beta) = u'(\beta) \end{cases}$$

α) $\lambda = 0$

$u'' = 0 \Rightarrow u(x) = Ax + B$. $\quad 0 = u(0) = B \Rightarrow u(x) = A$

$u'(\beta) = 0 = A\beta \Rightarrow A = 0$ $\quad$ $\lambda = 0$ δεν ειναι ιδιοτιμη.

β) $\lambda < 0$

$m^2 + \lambda = 0 \Rightarrow u(x) = c_1 e^{\sqrt{-\lambda}\, x} + c_2 e^{-\sqrt{-\lambda}\, x}$ $\quad$ (Γενικη Λυση)

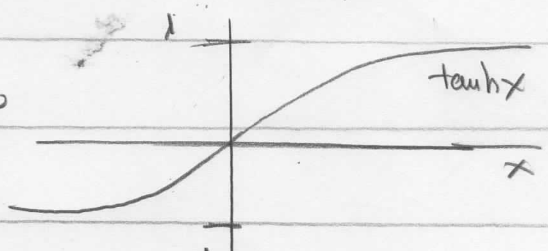
(1) $0 = u(0) = c_1 + c_2$, $\quad u'(x) = c_1\sqrt{-\lambda}\, e^{\sqrt{-\lambda}\, x} - c_2\sqrt{-\lambda}\, e^{-\sqrt{-\lambda}\, x}$

$u'(\beta) = u(\beta) \Leftrightarrow \quad c_1\sqrt{-\lambda}\, e^{\sqrt{-\lambda}\beta} - c_2\sqrt{-\lambda}\, e^{-\sqrt{-\lambda}\beta} = c_1 e^{\sqrt{-\lambda}\beta} + c_2 e^{-\sqrt{-\lambda}\beta}$ $\quad$ (2)

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 e^{\sqrt{-\lambda}\beta}(1 - \sqrt{-\lambda}) + c_2 e^{-\sqrt{-\lambda}\beta}(1 + \sqrt{-\lambda}) = 0. \end{cases}$$

Μη τετριμμενες λυσεις εχουμε μονο εαν $\qquad \dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \tanh x$

$$\begin{vmatrix} 1 & 1 \\ e^{\sqrt{-\lambda}\beta}(1 - \sqrt{-\lambda}) & e^{-\sqrt{-\lambda}\beta}(1 + \sqrt{-\lambda}) \end{vmatrix} = 0$$

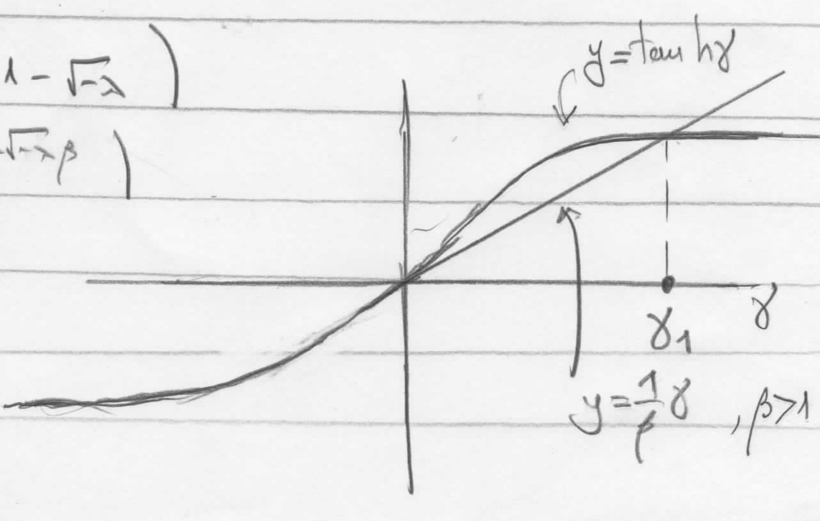


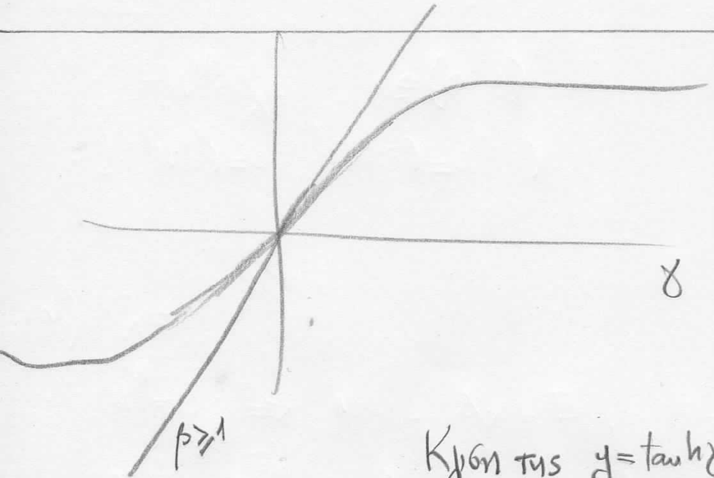
$\Leftrightarrow \quad e^{-\sqrt{-\lambda}\beta}(1 + \sqrt{-\lambda}) = e^{\sqrt{-\lambda}\beta}(1 - \sqrt{-\lambda})$

$\Leftrightarrow \quad e^{\sqrt{-\lambda}\beta} - e^{-\sqrt{-\lambda}\beta} = \sqrt{-\lambda}\left(e^{\sqrt{-\lambda}\beta} + e^{-\sqrt{-\lambda}\beta}\right)$

$\Leftrightarrow \quad \dfrac{\gamma}{\beta} = \tanh\gamma \ , \quad \gamma = \sqrt{-\lambda}\,\beta$

$\gamma_1 = \sqrt{-\lambda_1}\,\beta$





Κλίση της $y=\tanh\gamma$ είναι ίση με 1 στο $\gamma=0$

Συμπέρασμα:

$\exists!\ \lambda=\lambda_1<0 \iff \beta>1$

με αντίστοιχη ιδιοσυνάρτηση

$u(x)=c_1\left(e^{\sqrt{-\lambda_1}x}-e^{-\sqrt{-\lambda_1}x}\right)$

$(c_1+c_2=0)$

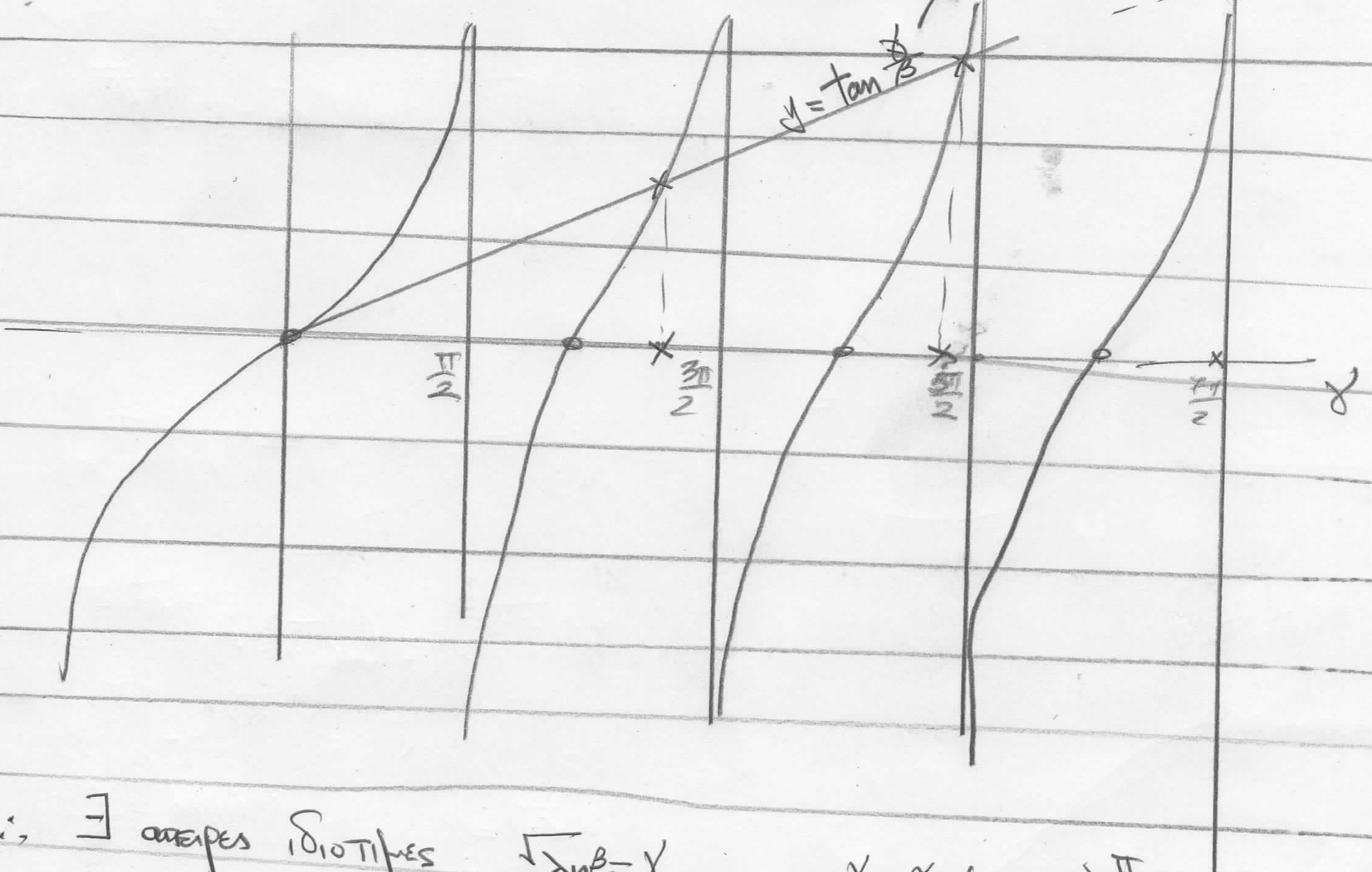
$\therefore\ u(x)=\hat{c}_1\sinh(\sqrt{-\lambda})x$

γ) $\lambda>0$

$m^2+\lambda=0 \implies u(x)=c_1\sin\sqrt{\lambda}x+c_2\cos\sqrt{\lambda}x$

① $0=c_2 \implies u(x)=c_1\sin\sqrt{\lambda}x$

② $\sin\sqrt{\lambda}\beta=\sqrt{\lambda}\cos\sqrt{\lambda}\beta \iff \tan\gamma=\dfrac{\gamma}{\beta}$



$\therefore\ \exists$ άπειρες ιδιοτιμές $\sqrt{\lambda_n}\beta=\gamma_n$, $\gamma_n\simeq(2n+1)\frac{\pi}{2},\ n\to+\infty$

με αντίστοιχες ιδιοσυναρτήσεις $u_n(x)=c_1\sin\left(\dfrac{\gamma_n x}{\beta}\right)$. □