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BEYOND MOTIVATION: EXPLORING MATHEMATICAL MODELING AS A CONTEXT FOR DEEPENING STUDENTS' UNDERSTANDINGS OF CURRICULAR MATHEMATICS

ABSTRACT. Views of mathematical modeling in empirical, expository, and curricular references typically capture a relationship between real-world phenomena and mathematical ideas from the perspective that competence in mathematical modeling is a clear goal of the mathematics curriculum. However, we work within a curricular context in which mathematical modeling is treated more as a venue for learning other mathematics than as an instructional goal in its own right. From this perspective, we are compelled to ask *how learning of mathematics beyond modeling may occur* as students generate and validate mathematical models. We consider a diagrammatic model of mathematical modeling as a process that allows us to identify how mathematical understandings may develop or surface while learners engage in modeling tasks. Through examples from prospective teachers' mathematical modeling work, we illustrate how our diagrammatic model serves as a tool to unpack the intricacies of students' mathematical experience while engaging in modeling tasks.

KEY WORDS: learning, mathematical modeling, prospective teachers, theory, understanding

1. INTRODUCTION

What is the point of engaging students in mathematical modeling activities? Several goals may come immediately to mind, including the following: to prepare students to work professionally with mathematical modeling, to motivate students to study mathematics by showing them the real-world applicability of mathematical ideas, and to provide students with opportunities to integrate mathematics with other areas of the curriculum. These reasons, collectively and individually, are appealing. The curricular context of schooling in the USA does not readily admit the opportunity to make mathematical modeling an explicit topic in the K-12 mathematics curriculum. The primary goal of including mathematical modeling activities in students' mathematics experiences within our schools typically is to provide an alternative – and supposedly engaging – setting in which students learn mathematics without the primary goal of becoming proficient modelers. We refer to the mathematics to be learned in these classrooms as "curricular mathematics" to emphasize that this mathematics is the mathematics valued in these schools and does not include mathematical modeling as an

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explicit area of study. Acknowledging this curricular context, we recognize that extensive student engagement in classroom modeling activities is essential in mathematics instruction only if modeling provides our students with significant opportunities to develop deeper and stronger understanding of curricular mathematics. We also suggest this discussion of modeling may be useful to readers who observe students enriching their mathematics understanding in multiple ways in classes for which mathematical modeling is the primary object of instruction.

To unravel mathematical modeling as a venue for mathematical learning requires conceptualizing mathematical modeling in a way that accounts for how learning of curricular mathematics occurs and what curricular mathematics may be learned through modeling activities. Drawing on existing definitions, diagrams, and descriptions of modeling, we focus on subprocesses involved in mathematical modeling as a useful perspective from which to think broadly about how mathematical modeling activities may support learning of curricular mathematics. We describe how mathematical modeling might serve as a venue for mathematical learning in classrooms where learning goals focus on curricular mathematics after we briefly state our underlying assumptions about learning and background issues regarding modeling.

2. BACKGROUND

Our discussion of our conception of the relationship between mathematical modeling and learning of curricular mathematics builds on our assumptions about learning and our understanding of modeling issues.

2.1. *Learning and understanding*

We approached the study from a Neo-Vygotskian sociocultural perspective (Rogoff, 1990; Wertsch, 1991). Our goal is to understand more fully how mathematical modeling activities might serve as contexts in which mathematics learners could encounter ideas that would challenge their previous understandings.

Learning is a discursive activity that involves social and material resources. As Moschkovich (2004) implied, a person who learns mathematics is one who makes new connections between pieces of knowledge, adds new pieces of knowledge, or corrects previous knowledge. Corrected connections, in addition to previous knowledge and connections, corrected and new knowledge, and new connections, constitute understanding that might arise in the mathematical modeling environment. In particular, we explore how modeling activities serve as venues for developing and changing understanding of curricular mathematics as students' existing knowledge is challenged and potential connections become salient through social interaction.

2.2. *Common modeling issues*

Several authors (Blum and Niss, 1991; Doerr, 1996; Galbraith and Clatworthy, 1990; Lesh and Harel, 2003; Preston, 1997; Tanner and Jones, 1994) described mathematical modeling or model development and most used diagrams to convey mathematical modeling as a process. Their accounts collectively identify several issues about mathematical modeling that are central to our work. Mathematical modeling is a non-linear process that involves elements of both a treated-as-real world and a mathematics world. The modeling process involves movement among elements such as the real-world situation, the real-world solution, a mathematical entity, and a mathematical solution. Subprocesses describe how one moves among these elements in a non-linear fashion from a real-world question or task to a solution or description. There should be multiple opportunities for modelers to verify, or at least to monitor and share their progress, including communication with self as well as communication with others. In addition, a conception of modeling should maintain appreciation of the complexity of mathematical modeling rather than stress details of seemingly disconnected subprocesses or oversimplify the complex undertaking.

We underscore the importance of an inclusive conception of modeling to accommodate our curricular mathematics environment. This conception of modeling should acknowledge the differences among a modeling task, an applied problem (Blum and Niss, 1991), and a context-free exercise (Ames, 1980). The mathematics involved should range over various mathematics content areas. Lastly, to be useful in discussing the classroom presence of mathematical modeling, modeling must be conceptualized in a way that allows one to explain how learning occurs. The description of mathematical modeling that follows a comment on mathematical models is intended to meet these aspirations.

3. MATHEMATICAL MODEL

Associated with any *mathematical model* are a mathematizable situation, a mathematical object, a purpose or question that prompted the modeling activity, and the relationships between these things and the modeler. Our focus on curricular mathematics mandates a few observations about these factors.

The richest modeling experiences may occur when students are "mathematizing authentic situations" (Yerushalmy, 1997, p. 207). Like Trelinski (1983) and Blum and Niss (1991), among others, we note the importance of modeling situations outside of mathematics. In the interests of curricular mathematics, we allow for the inclusion of "applications" of mathematics from one branch of curricular mathematics to another.

To reflect work within mathematics and our interest in how students learn curricular mathematics in general, we assume the mathematical entity to be any mathematical object from any area of curricular mathematics. In the examples in this paper, the mathematical entity often is a geometric figure. We note that a teacher's or a curriculum's focus on learning curricular mathematics often means the situation is narrowly cast to ensure students use a particular type of entity. In this case, the teacher's purpose for a modeling activity is to use the modeling activity to introduce or use a particular type of mathematical entity, regardless of how this purpose and entity match the students' modeling purposes and known or natural entity choices.

Modelers' assumptions about the situation and about the mathematics used to model the situation may be particularly critical in some modeling tasks. In defining mathematical modeling, Galbraith and Clatworthy (1990) required the application of mathematics to unstructured problem situations in real-life situations. In unstructured problem situations, neither the purpose nor the mathematical entity is suggested explicitly. Identifying these is part of the modeler's responsibility, making the modeler a key part of this type of mathematical modeling work. The ways in which students perceive structure in a situation and impose structure in their solutions reflect their underlying assumptions. We note the modeler's assumptions include assumptions about any tools involved in the modeling process. For example, data from Zbiek (1998) evidence the influence of the modeler's knowledge of and assumptions about curve fitting and graphing utilities on the nature of the models they created.

The notion of correcting, adding, or connecting pieces of knowledge about curricular mathematics causes us to underscore the impact of a modeler's assumptions about and awareness of pertinent mathematics on his or her model. Other authors address this component of a mathematical model. For example, Galbraith and Clatworthy's (1990) questionnaire included "choosing the right mathematics" (p. 149). Maki and Thompson (1973) described using appropriate mathematical ideas and techniques as part of the mathematical modeling process. Assumptions play an important role in our thinking and in other authors' treatment of mathematical modeling (e.g., Galbraith and Clatworthy, 1990). We believe the mathematical awareness and assumptions of the modeler influence the extent to which the modeler may see a mathematical structure within a situation as well as his or her choice of mathematical entities and potential success with various mathematizations. Each modeler brings to a situation a unique set of knowledge, intuitions, and conceptions about mathematics and the real world, and this set influences his or her interpretations of the real world situation as well as how he or she will draw upon appropriate mathematical ideas. The particular mathematics on which students draw depends on their knowledge of curricular mathematics and the extent to which they believe the modeling task is designed to evoke a particular familiar mathematical idea.

4. Examples

4.1. *Course and interview setting*

Despite the absence of mathematical modeling as a curricular goal in primary and secondary schools in the USA, secondary mathematics teachers are encouraged to use applications and mathematical modeling tasks as settings for learning curricular mathematics. The setting for our research was a mathematical modeling course for prospective secondary mathematics teachers, the goals of which included both learning mathematical modeling and learning to develop and implement application problems and mathematical modeling tasks within their future classrooms. Students worked on mathematical modeling tasks in group settings. In addition, 17 students participated in individual interviews with a researcher as part of the course requirements. In this paper, we include examples taken from video, audio, and other artifacts of the students' modeling activity in the individual interviews and classroom settings. The purpose of the examples is to underscore how learning curricular mathematics may happen within the subprocesses of mathematical modeling activity, not to recount the case studies of particular students.

4.1.1. *Description of Hospital Problem*

In the interviews, students responded to the Hospital Problem, the statement of which appears in Figure 1. With this prompt, students were given a map of a portion of the northwestern area of the United States. Students had access to a pair of compasses, a protractor, a ruler, and a calculator.

The students were free to solve the problem in any way they wished. The interviewer had supporting materials ready to share if students expressed a need for them. For example, if students asked questions about the populations of the cities, the interviewer gave them the populations of the

Boise (Idaho), Helena (Montana), and Salt Lake City (Utah) are three large cities in the northwestern part of the United States. While each city has medical facilities, imagine the potential of a very high-powered, high-tech, extremely modern medical facility that could be shared by the three cities and their surroundings communities! The map shows the locations of the three cities as well as other large cities in the area. Suppose you are hired to determine the best location for the hospital.

Figure 1. Initial statement of the Hospital Problem.

cities and surrounding counties. The interviewer gave students an atlas if the students inquired about the topography of the area, highways, or other geographical features.

Students often initially approached the situation by noting the medical center should be somewhere "in the middle" of the three cities while motioning to the interior of a triangle whose three vertices represented the three cities. To describe how they would determine a more specific location, most students sketched and sometimes constructed angle bisectors, perpendicular bisectors of sides, or medians of the triangle to locate a particular point. These students often used "centroid," correctly or incorrectly, to name the particular point, regardless of the method they used.

After the students concluded that they had completed the original task, the interviewer followed with questions that asked students who did not use population information how, if at all, the population information would influence their solution. If any students had not used the notion of distance from each city, the interviewer would have asked about it at this time. During the final stage of the interview, students reacted to any of three prepared alternative models that involved curricular mathematics that did not arise in the students' original work. In the spirit of curricular mathematics, the alternative solutions were opportunities to introduce previously unmentioned mathematical ideas into the conversation. These alternatives reflected the circumcenter of the triangle whose vertices represented the three cities, the centroid of this triangle, and a point whose distances from the three cities were related to the populations of the cities. This third point was found by weighting each vertex with the corresponding population value, and then finding the point representing the weighted mean of the three vertices. The populations of the cities were used to determine ratios into which to divide the lengths of segments. The first step was to divide the segment from Salt Lake City to Helena in a ratio of 898,387 to 55,716, or approximately 15 to 1 to locate a point. That point was weighted with the sum of the populations of Salt Lake City and Helena, 954,103. The point marked in the third alternative solution is the point that divides the segment from this new point to Boise in a ratio of 954,103 to 300,904, or approximately 16 to 5. These alternatives were offered as objects to critique, not as exemplary solutions.

Students were presented with completed solutions; the interviewer did not reveal how the solutions were derived.

4.1.2. *Description of Hospital Quiz*
Approximately two weeks after all students participated in individual interviews, an instructor shared a previously unseen potential solution method in the whole-class setting. The process of viewing the solution and commenting on it constituted the Hospital Quiz. The solution involved constructing segments representing the distances between two cities and proportionally dividing the segments based on the city populations. This solution was an alternative to student work in that students looked for ways to involve the relative values of the populations but no student divided segments proportionally. It is a variation of the third alternative solution. The instructor revealed the solution in steps using the Geometer's Sketchpad. Figure 2 captures the two main subsets of those steps. Figure 2a shows the results of constructing point X, the location of a hospital between Salt Lake City and Helena; Figure 2b shows the results of locating the hospital between X and Boise. This proposed solution included mathematically questionable moves, such as the modeler's accidental use of 5/8 rather than the correct 5/16 as the ratio of the population of Boise to the combined population of Helena and Salt Lake City. Students commented in writing on the assumptions, simplifications, mathematics used, strengths and weaknesses, and possible improvements of the process and results of this modeling work.

4.2. *Summary of three students' approaches to the Hospital Problem*

In the next section, we draw on the work of three students: Summer, Charmaigne, and Carl. (All student names are pseudonyms.) Their approaches are representative of the variety of student response to the initial task. We describe their general approaches here before referring to aspects of their work in our subsequent discussion of modeling as a venue for learning mathematics.

Summer began by stating that the centroid of the triangle would give the required location. She used a ruler to sketch two medians of the triangle and marked their intersection. She then used a ruler to measure the distances from the vertices to the centroid. She concluded that this point was not equidistant from the vertices and thus was not the point she wanted. After trying several other nearby points, she (inappropriately) concluded that there was no point equidistant from all three vertices. Summer then asked for more information, saying, "If I were given more information . . . about maybe about which city . . . is more populated maybe" (Summer, 27 March 2003, lines 187–192). When given the population information, she noted

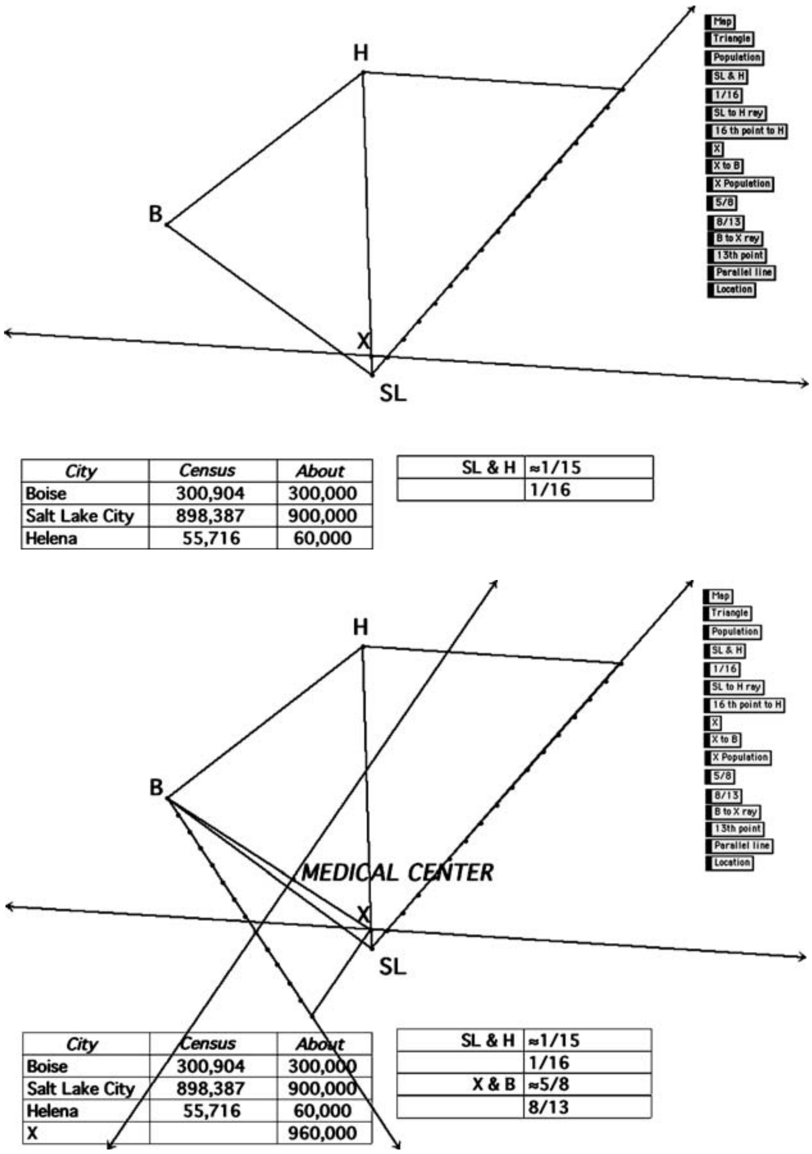

Figure 2. Diagrams students viewed and then evaluated in the Hospital Quiz.

that Boise and Salt Lake City were more heavily populated than Helena. She tried a variety of strategies to locate the hospital closer to the larger cities. She attempted to locate the hospital approximately midway between Boise and Salt Lake City and slightly towards the small city, Helena. She loosely quantified distances in different ways as she identified several possible locations. However, she thought geographic features made travel to her preferred location from the larger cities difficult. Her final location for the hospital was the intersection of a ray from Salt Lake City and a ray from Boise that passed as closely as possible to but avoided these geographic difficulties.

Charmaigne's first answer was to put the hospital in the "middle" of the triangle formed by the three cities. She then asked for the population information, but decided to disregard it and to find a location equidistant from the three cities. She mentioned several segments that she could draw: "I'm trying to figure out what would be in the middle [as she taps the end of the ruler in the interior of the triangle]. If I drew the perpendicular bisector of each line [pause] If I took the median of each line [pause] Think I'm going to take the perpendicular of each line" (Charmaigne, 28 March 2003, lines 154–169). She proceeded to draw segments from each vertex of the triangle to the opposite side, but she did not measure or use any other mathematical tool to determine the placement of the three segments. Her segments appeared to be altitudes but she called them "perpendicular bisectors." She expected the segments to intersect in a point equidistant from the vertices, but they did not. She attributed this result to her lack of precision in drawing. Upon being given more information about the fastest growing cities in the U. S., Charmaigne decided that Las Vegas should also have access to the hospital. However, she eventually concluded that her original location was the best location for the hospital.

Carl began by sketching the perpendicular bisectors and calling the point where they coincide the "centroid." He then discarded this point as a location for the hospital and drew the angle bisectors, claiming that the point where they coincide would be the balance point. He discarded this point as well, and spent several minutes measuring distances from various points to the cities, finally settling on a location for the hospital approximately at the base of the median from Boise.

5. Mathematical modeling as process with diagram

Several authors describe mathematical modeling as a cyclic or iterative process involving revisions before one arrives at an acceptable conclusion (Lamon, 1997; Lesh and Harel, 2003; Trelinski, 1983; Webb, 1994). We too find a diagram (Figure 3) a useful tool to convey aspects of the mathematical modeling process and relationships among its components to our readers, although we did not share the diagram with the students. We represent each subprocess with a bi-directional arrow in the diagram. Some subprocesses also have subprocesses, which are represented by circular arrows.

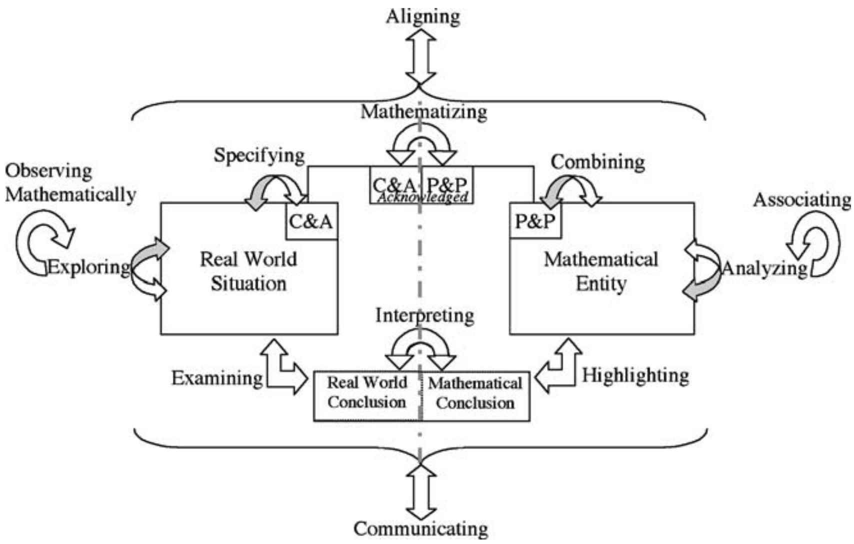

Figure 3. Modeling process diagram.

Each modeler's work traces a path through the diagram. Different modelers may take different paths based on different purposes as well as different background knowledge, intuitions, and beliefs. It is possible for the modeler to omit some subprocesses and to focus on other subprocesses. In trivial cases, such as when a modeling task fades into an overly structured task or an applied problem, a modeler may follow a linear path. Confined to a static document, we order the subprocesses as they arise in a stereotypical path while using excerpts from students' work with the Hospital Problem to illustrate each subprocess. In the next section, we will connect these subprocesses to learning curricular mathematics.

5.1. *Working within the real-world situation*

Exploring is obtaining more information about things within the situation. The modeler may be questioning, clarifying, or attending more carefully to given, known, or remembered information about the situation. For example, a modeler may request or seek the populations of two cities involved in the situation. Carl, as he explored the situation, used a ruler to measure the distances from one city to another. The process of exploring begins with what Dunne (1998) called "specify the problem" (p. 30). However, exploring goes beyond a simple statement of the problem, allowing the modeler to find out more information about a situation than was originally apparent, and allowing for the phenomenon of rereading a problem and perceiving different information from what was first noted. *Observing mathematically* occurs when, during the process of exploring, a modeler uses a mathematical idea to describe situational information. A simple example is observing that one population is twice as large as another. Exploring and observing mathematically potentially allow the modeler to transcend his or her preconceptions of the situation, particularly when discussing the situation with others.

Specifying is identifying the *conditions* and *assumptions* (C & A) of the real-world context to which the modeler will attend as he or she mathematizes the situation. When a modeler is specifying, he or she is identifying variables within the situation and constraints upon the situation. One should be able to state explicitly the conditions and assumptions being specified. For example, a modeler may say that the time required for a real-world action in the situation should be small. In his Hospital Problem work, Carl stated two conditions and assumptions when he began working on the problem: there are three cities and “the cities and their surround (sic) areas can all use the hospital efficiently” (Carl, 31 March 2003, lines 19–20). The action of specifying is included in other diagrams of the modeling process as “listing assumptions” (Dunne, 1998, p. 30). However, we believe a key aspect of specifying is that what is specified is the collection of conditions and assumptions that the modeler has at this point stated or acknowledged. The modeler may also specify that some information is not important, in this way simplifying the situation in the sense that other authors use “simplify” or “simplification” (NCTM, 1989, p. 138; Giordano, Weir, and Fox, 2003, p. 54). Charmaigne decided that the population information was not important in her solution to the Hospital Problem, instead focusing on the condition of equidistance to the cities. Similar to Lester and Kehle’s (2003) Simplifying/Problem Posing phase of problem solving, specifying includes moving from the original situation to the particular (real-world) problem that is to be solved.

5.2. *Linking real-world situation with mathematical entity*

Mathematizing includes introducing mathematical ideas that eventually relate to the mathematical entity. When mathematizing, a modeler creates or acknowledges mathematical *properties* and *parameters* (P & P) that correspond to the situational conditions and assumptions that have been specified. Mathematizing is a bridge between the real world of the situation and the mathematical world of the model. An example of mathematizing is a student creating a variable expression (a property or part of the mathematical entity) to represent the time required to complete a real-world action (a condition or key piece of the real-world situation). In the Hospital Problem, Carl used points to represent the cities. It is possible for a modeler to state properties or parameters which he or she deems necessary for the situation, but to which he or she cannot connect a mathematical property. The properties and parameters are aspects of what will eventually be deemed the mathematical object or entity. The mathematical pieces generated in mathematizing may not be perceived from the modeler's view as a single mathematical object. Too many pieces may be present or key pieces may be missing.

The modeler may already be very familiar with a desired connection between the identified conditions/assumptions and the desired properties/parameters. In this case, mathematizing may happen so quickly or so implicitly that the subprocess appears trivial. However, we contend that mathematizing is not always a rapid or implicit event. In fact, the duration and nature of mathematizing may be related to the epistemic act in which the modeler engages. In the terminology used by Hershkowitz et al. (2001) for epistemic acts, seemingly trivial or less intense mathematizing may happen if the modeler is able to "recognize" or "build-with" familiar constructs. More intense mathematizing may happen when the modeler needs to "construct" a novel idea to serve as a property or parameter. Hershkowitz and colleagues note that it seems more rare to observe constructing actions than to observe recognizing and building-with actions. We suspect that many modeling tasks in our schooling context are fundamentally applied problems in disguise and are presented to use existing mathematical knowledge rather than to evoke new mathematical knowledge. Thus, we expect that these school experiences often involve mathematizing that does not foster new mathematical knowledge and that may seem trivial or implicit.

As one delves more deeply into the real-world situation, more conditions and assumptions may become apparent, or the modeler may choose to narrow or broaden the scope of his or her modeling. In addition, mathematizing one condition or assumption may require several properties or parameters, or several conditions or assumptions may be incorporated into one property or parameter. In Figure 4a, we start with the two small rectangles at the top. Points in the small rectangle to the left (labeled C & A) represent individual conditions and assumptions while points in the small rectangle to the right (labeled P & P) represent individual properties and parameters. We can then "zoom-in" on a particular condition or assumption. One condition or assumption from the rectangle on the left can be seen as a collection of smaller conditions and assumptions, the large circle on the left. Its associated property or parameter from the rectangle on the right can be seen as a collection of smaller properties and parameters, the large circle on the right. Figure 4b shows how one assumption or condition may embody several 'smaller' assumptions or conditions in the case of Hospital Problem. The figure also shows how "three cities" as one condition or

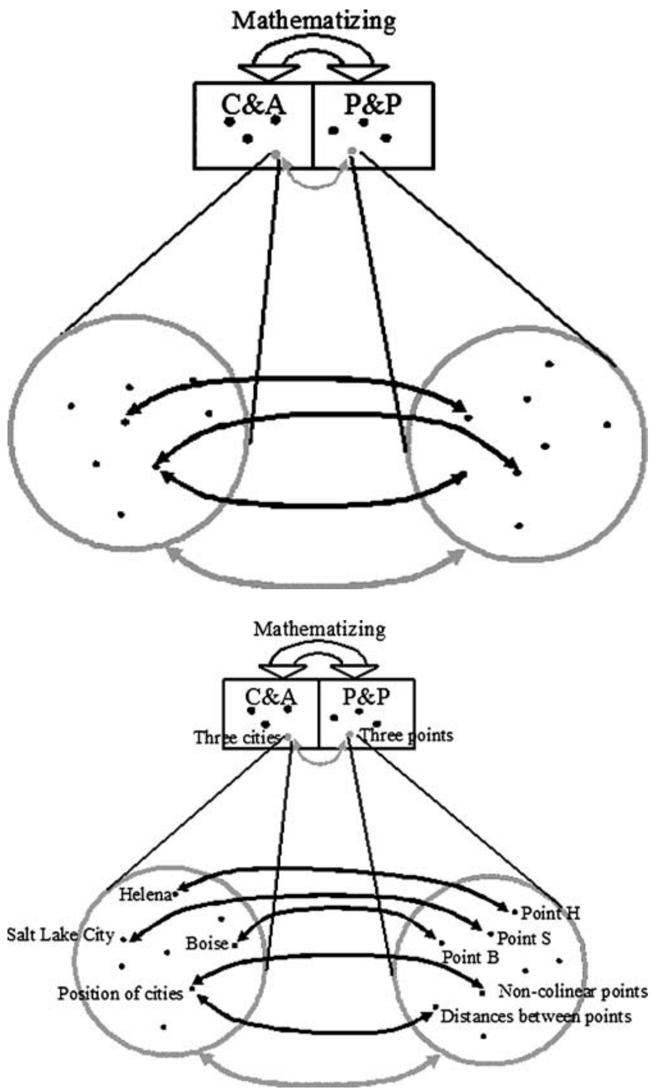

Figure 4. Layers of relationships between conditions/assumptions and properties/ parameters.

assumption of the situation matches with “three points” as one property or parameter. The three-cities condition includes Helena, Boise, Salt Lake City, and the position of the three cities. The three-points property may include the related ’smaller’ properties or parameters of non-collinearity and particular distances between points. As noted in Figure 4b, the condition “location of the cities” may be connected to two (or more) properties, such as the non-collinearity of the points and relative distances between points. Properties, parameters, conditions, or assumptions may be embedded in several layers. The modeler may not be aware of some layers. Zooming in on any of the conditions, assumptions, properties, or parameters in the larger circles might result in a modeler making more conditions, assumptions, properties, and parameters apparent to himself or herself. Zooming out might result in the modeler combining properties and parameters into more manageable pieces. Zooming processes may continue until the modeler attains a desirable or useful level of specificity. Mathematizing may require multiple journeys between conditions and assumptions and properties and parameters, a pattern that contributes to a non-linear modeling path.

5.3. *Working with the mathematical entity*

Combining includes identifying a mathematical entity that has the properties and parameters that have been introduced or identified. It also includes verifying that the mathematical entity matches the identified properties and parameters. The word 'combining' refers to combining mathematical objects, properties, and parameters (that may have been introduced as the modeler was mathematizing) into a single mathematical entity. 'Combining' does not refer to mathematical activity such as combining like terms – an act that most likely falls under our analyzing label. Combining may be seen by some to be similar to reifying or encapsulating. However, combining refers to creating a mathematical object from other mathematical objects or identifying a mathematical object as containing all the required properties and parameters rather than conceiving of something as an object rather than a process. An example of combining in our view is combining properties of vertex and intercept to create a particular quadratic relationship (e.g., if the x-intercepts are -1 and 1 and the vertex is $(0, -1)$, the quadratic relationship could be given by $f(x) = x^2 - 1$). In Carl's first instance of combining while working on the Hospital Problem, he used the three points (the mathematization of the three cities) to create a triangle. Blending the mathematizing and combining under one label ("abstracting"), Lester and Kehle (2003) explicitly note the importance of "selection of mathematical concepts to represent the essential features of the realistic model" (our mathematizing) and blending the selected elements into a "mathematical representation of both the setting and the problem" (our combining) (p. 514). To us, mathematizing and combining are two separate subprocesses, during each of which different forms of learning may occur.

Analyzing includes mathematically manipulating or interpreting the mathematical entity to derive one or more new properties or parameters of the mathematical entity. Analyzing may involve solving (as in solving an equation or a system of equations), but analyzing is not limited to solving. For example, one might differentiate a function, solve an equation, produce the angle bisectors of a triangle, or find the coordinates of the vertex of a parabola. Analyzing is influenced by the modeler's preconceptions of the mathematical entity and his or her shared work with others. Carl, as he modeled the hospital situation, drew by hand the perpendicular bisectors of the sides of the triangle to find what he called its "centroid." Carl was analyzing since his work produced what he called a "centroid." With the triangle entity present, he created his "centroid" as a property of the triangle that was not yet introduced into his modeling work. Carl's preconceived notions led to his interpretation of his point (the circumcenter) as the centroid.

During the process of analyzing, a modeler may engage in *associating* if he or she refers or connects to real-world issues outside of the situation. This implies that the modeler is drawing on some real-world knowledge in order to think about mathematical ideas. This act produces, elicits, or depends on a metaphor or other connection between a mathematical entity and a real-world object. One might say that the graph of the function looks like a snake and we are interested in where the snake bends (when the situation has nothing to do with snakes). Carl engaged in associating when he compared what he called the "centroid" (his desired location of the hospital) to the balance point of a sheet of metal.

Highlighting is making obvious any previously unacknowledged properties or parameters of the mathematical entity that will serve as a mathematical conclusion that may be interpreted as the real-world conclusion. Highlighting is not simply emphasizing a mathematical idea. An example is, for a quadratic function, explicitly acknowledging that the root of the equation formed by setting the function's derivative's expression equal to 0 is the input value where the vertex of the graph occurs. When Carl drew a dot to show where his "centroid" was located, he was highlighting that point as a mathematical conclusion. Highlighting is a way to make the mathematical conclusion clear. It may be included in other descriptions of the modeling processes within steps such as "solve" (Barnes, 1991, p. 11) or "transformations" (NCTM, 1989, p. 138).

5.4. *Connecting with the real-world situation*

Interpreting is putting the mathematical conclusion in context. It includes but is not limited to adding labels. Like mathematizing, interpreting may be seen as a bridge between the real world and the mathematical world. Both involve connecting real-world ideas with mathematical ideas. However, interpreting differs from mathematizing in that mathematizing relates to conditions and assumptions and properties and parameters while interpreting relates to conclusions. Interpreting occurs when the modeler, for example, looks at the graph of a fitted function and concludes that revenue increases and then decreases as price increases. Carl interprets his "centroid" as he identifies it as a location for the hospital. In some descriptions of the modeling process, interpreting is grouped together with other processes as in "interpret and validate the solution" (Dunne, 1998, p. 30) or "interpret the solution, explain and predict" (Barnes, 1991, p. 11). We note that interpreting may appear to be a subliminal act (i.e., the modeler may not explicitly state both a mathematical conclusion and a real-world conclusion).

Examining is comparing the real-world conclusion with the situation while considering the modeling purpose to ensure the real-world conclusion aligns with the realistic situation in light of the modeling goal. Examining includes acknowledging the presence or absence of some characteristics of the situation in the mathematical entity. For example, the student may say that the average height of a person predicted by the model makes sense because that height falls within the range of heights in the data set. Carl examined (and abandoned) his first location for the hospital because he did not think it was equidistant from all three cities.

5.5. *Subprocesses that permeate modeling process*

The last two subprocesses, aligning and communicating, transcend any one position in the diagram. Reflection on the appropriateness of any other part of the modeling work, reconciling inconsistent results from different subproblems or from different modelers, or verifying that one's modeling work fits with one's purpose is *aligning*. Aligning may occur at any time in the modeling process, and it is part of what makes a modeler's path non-linear. An example of aligning is considering (perhaps during mathematizing or analyzing) whether the desired outcome is a single value or a range of values, which may determine whether the modeler should work in that modeling task with an equation or an inequality or any other mathematical object. Aligning includes the ongoing metacognitive activity of comparing the current state to earlier states valued by Lester and Kehle (2003).

Communicating is putting forth ideas, information, or details about the mathematical entity, the solution, or the process. Communicating is not limited to giving a final account of one's work, and it may occur at any time during the modeling processes. For example, in an interview setting, the interviewer or the student reiterating the student's prior work is communicating. Communicating includes but is not limited to Galbraith and Clatworthy's (1990) description of communicating as the last step in their diagram: “communicate: use model to explain, predict, decide, design, desist...” (p. 139). Communication may be intended for others or for one’s own benefit. It may be the expression of ideas verbally, or through motions, writing, or pictures, with the crucial feature of intention to share familiar thoughts rather than to introduce new information.

6. Learning, understanding and motivation

As students engage with purpose, mathematical entities, and situations in various mathematical modeling subprocesses, there are opportunities for them to grow as knowers and doers of curricular mathematics. Mathematical-modeling activity influences student learning through effects on motivation and through changes in understanding. We will briefly address issues of motivation before delving more deeply into examples of learning and understanding.

6.1. *Effects on motivation*

In our schooling context, there is a common perception that mathematical modeling tasks motivate students to engage in mathematics. We contend engagement in mathematical modeling activities supports three different types of motivation. The first type of motivation is confirmation that real-world situations appeal to (some) learners. For example, during *exploring*, students may become excited by the appeal of the context but not be driven to learn particular mathematics or to engage further in the task. A different kind of excitement may come with *associating,* as the student sees a connection between some mathematics and some real-world issue and believes that mathematics may be useful to (other) people. The motivation in each of these cases does not necessarily develop further as the student continues engaging in the modeling task or in studying mathematics.

A second type of motivation is simply motivation to (or to continue to) study mathematics in general. In *specifying*, a student may note the complexity of real-world phenomena and believe that mathematics may be a tool to unravel that complexity. Students who are *interpreting* may have confirmation that mathematics is applicable to the real world. Modeling in general may suggest to students that they need to study a variety of mathematics in order to address a range of real-world problems.

Motivation to learn new mathematics, the third type of motivation, emerges when a student modeler embraces a purpose that, in that modeler’s opinion, is not sufficiently met by the mathematics the modeler knows, and the modeler actively seeks understanding of the needed mathematics and thus is motivated to add a new piece of knowledge or new connections among known pieces of knowledge. The modeler may experience this need within several subprocesses. In *mathematizing*, the student's need to include a particular condition or assumption as a property or parameter can motivate learning about a new entity or about new properties of known entities. For example, consider a modeling task within which students are developing a function to match a data set. Students may have "constant rate of change" and "decreasing output" as properties. Students who are not familiar with negative rates of change may be compelled to blend the two properties into a conception of "negative rate of change". The *combining* need to merge several properties and parameters can motivate learning more about a known entity or learning about a related entity. Students engaged in generating a function to match a data set may know that a linear function may have "negative slope" and that a linear function may have "only positive dependent variable values". However, the students may not understand that no one linear function can have both of these properties simultaneously. Attempts to use technology to fit a linear model in combining these two properties invariably fail, leading the students to try alternative technology options. As they test these options, they may learn that exponential functions of the form $f(x) = \mathrm{C}a^x$, $\mathrm{C} > 0$, $0 < a < 1$, can have both of the properties.

The need to derive a new property or parameter can motivate learning a new procedure or learning new characteristics of a known entity in *analyzing*. Students who fit an exponential function to the data set may develop a (crude?) method to identify the constant ratio between function values of arguments with unit difference. Students thus learn a process for writing an explicit form, $f(x) = \mathrm{C}a^x$, as a recursive form, $f(x) = a \cdot f(x - 1)$. Realization during *examining* that an answer does not completely satisfy the purpose may provide motivation to revisit and learn what mathematics, if any, would lead to the solution. Needing words to describe procedures, results, or other aspects of the modeling work may inspire the need to learn terminology during *communicating*. It is important in our curricular context to note that motivation to learn particular mathematics may arise as students work with an application problem in which they engage in only one or two of the modeling subprocesses (e.g. analyzing).

6.2. *Changes in understanding*

6.2.1. *Changes while working in the real world*

Modeling work provides not only *motivation* to learn mathematics but also *opportunities* to learn mathematics. Some of the learning involves mathematics not directly related to either curricular mathematics or the mathematics needed for the modeling task. For example, learning may occur when *observing mathematically* if new connections are made. *Specifying* offers opportunities to develop mathematical insight through real-world insight. This occurs when the student sees something that matters in the real world and uses this sense of what matters in the real world to elicit the importance of something in mathematics. For example, a student engaged in the Hospital Problem may place importance on the populations of the cities in the real world. This attention then may elicit the mathematical importance of weighted means.

6.2.2. *Changes while working in the mathematics world*

Attempts in *combining* a set of properties and parameters into an entity may yield a combination of only a subset of these properties and parameters into a familiar mathematical entity. For example, embedded in most solutions of the Hospital Problem is a conception of *weighted mean*. One geometric interpretation of weighted mean a modeler might have is a point within a triangle whose distance from each of the vertices reflects a relationship among three positive numbers and their relative positions. In the Hospital Quiz, students saw one attempt (albeit flawed) to construct a point that would reflect the weighted mean: the hospital is placed at the key point, proportionally distant from the vertices that are weighted by the populations of the cities. The student must then resolve how, if at all, the remaining properties and parameters, such as requiring the point to be equidistant from the vertices, also relate to that known entity. If the student figures out the relationship, the resolution leads to deeper understanding of the known entity. In the problem-solving sessions, students expressed these emerging ideas as they struggled to find mathematical ways to express how to place the hospital so that it would be closer to the larger cities (Boise and Salt Lake City) and farther from the small city of Helena.

However, if the remaining properties and parameters (the weights, for example) cannot be mathematically linked to a known entity, the student has the need for a new entity that embodies these properties and parameters. The student may associate these properties and parameters with a different entity with which the student is familiar but with which the student did not previously associate these features. If no familiar-to-the-student mathematical entity has all of the desired properties and parameters, there is provocation for the student to develop understanding of a related but previously unfamiliar entity. When more properties and parameters need to be combined than the student is *able* to combine, a student may see a need for knowing how to combine more properties and parameters into a single entity. As students realized that the solution present in the Hospital Quiz had all of the properties they desired, the students struggled to make sense of the solution. The students verbalized the extent to which they thought the solution captured the conditions and assumptions that mattered, although they did not immediately follow each mathematical step or claim in the presenter's solution process. They saw that the presented solution method related to their methods; for example, their methods also used lines through the vertices but assigned the same weight to each vertex. The developed understanding involved "centroid as a weighted mean when the weights are equal."

The learning that occurs within *combining* seems to lead to conceptual understanding of mathematical entities. From the curricular mathematics standpoint, this opportunity to learn within *combining* suggests how use of a modeling task or applied problem that requires additional or novel properties and parameters can draw on students' existing conceptions to introduce new mathematical entities.

Learning during *analyzing* may lead to procedural understanding in several ways. Attempts to use a known procedure can lead to alternative understanding of the procedure. While working on the Hospital Problem, Summer constructed the medians of the triangle with vertices representing Helena, Boise, and Salt Lake City. She knew this procedure would produce a common point of intersection. However, she expected this intersection point to be equidistant from the vertices. She measured to check this equidistance property because the property matched her key condition – the hospital should be the same distance from any of the three cities. Summer's understanding of the construction procedure changed as she learned the median-construction procedure results in a point that is not equidistant from the vertices, that is, as she corrected a piece of knowledge.

Attempts to use a known procedure in an unfamiliar context can lead to generation of a new or modified procedure, which may or may not have conventional mathematical validity. After rejecting the point resulting from her median-construction procedure, Summer altered that construction procedure to create the intersection of segments that connected two vertices to what we call her "third point" (see Figure 5). She created the "third point" as the point of intersection of the segment that connected SL to the point that was one-third of the distance from H to B and the segment that connected H to the point that was one-third of the distance from SL to B. Though this procedure lacks mathematical validity in terms of producing a common point of intersection when the "third-point lines" are constructed from all three vertices, the procedure does show Summer's emerging understanding of proportional division of segments in the generation of a new piece of knowledge.

Analyzing also permits conceptual learning opportunities. Consideration of an entity can reveal new characteristics of the known entity, and

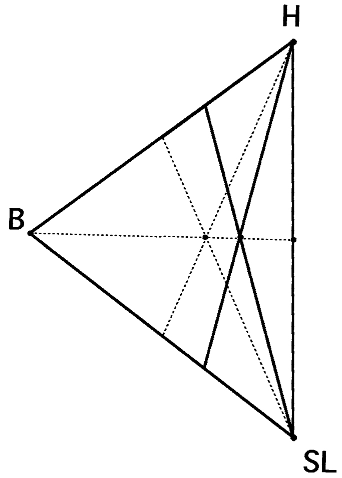

Figure 5. Summer's construction of the intersection of her "third-point line".

thus deepen understanding of the known entity. New understanding of this type emerged in students who did not know that the circumcenter would fall along the median to the base of an isosceles triangle. Their recognition that the centroid must be along the line from Boise to the Helena-Salt Lake City segment led some students to this new understanding. The coincidence that Boise-Helena and Boise-Salt Lake City distances were nearly equal contributed to the learning opportunity. However, we note that while learning did happen in a modeling context, the real-world situation was not needed. Similar learning may have arisen if students simply worked with an isosceles triangle in the absence of the hospital context. Solution of subproblems or lemma problems similarly can deepen understanding of related entities and their properties through the underscoring of characteristics or through the generation of a generalization.

6.2.3. *Changes while working with others*

A student may learn when working with the mathematical entity that another person introduces into a conversation. In the Hospital Quiz, students saw a construction technique of using n-1 points to divide a segment into n segments of equal length. Figure 2 shows some of the screens the students saw as the presenter described the solution steps (but not the reasons). In *aligning*, the students made sense of why this process would work as they determined the merit of the presenter's solution.

Reflection while *communicating* can necessitate the need to reconcile, modify, or justify particulars of mathematical processes and products. Socially based opportunities for learning arise when two or more students are engaged within other subprocesses. In *associating*, students can learn from another's sharing of the association and exploring the aspects of the mathematical idea that are paralleled by aspects of the real world thing as well as the aspects of either the real-world thing or the mathematical idea that are not paralleled in the other venue. When the association is a metaphor, the matches and mismatches are grounds and tensions, respectively (Presmeg, 1998). Students may learn from the extension of the association by looking at things that are paralleled and things that are not paralleled. For example, Carl commented that the centroid of an abstract triangle is like the physical balance point of a metal triangle. Students who try this balancing task see that the rigidity of the triangle and the bigger-than-a-point size of the pencil eraser allows one to balance the triangle when the balance point is moved around a considerably large portion of the interior of the metal triangle. This tension between the uniqueness of the centroid and the many locations on which one might balance the metal triangle would support Carl's contention that the centroid point for the location of hospital could be anywhere in a general region. However, had he used the metal triangle and added weights on the vertices to reflect the populations, Carl could have seen the centroid would not be an adequate balance point if the populations were important. Given this tension, Carl could have generated a new understanding of the centroid as the balance point for the triangle only when its vertices are equally weighted.

7. Conclusion

Mathematical modeling provides a venue in which students can learn curricular mathematics in various ways. We elaborated on the subprocesses of mathematical modeling to provide a way to think about how learning takes place during mathematical modeling. This learning potentially involves both deeper understanding of known curricular mathematics and the motivation to learn new curricular mathematics. Within different mathematical modeling subprocesses lie opportunities for conceptual and procedural development for all modelers.

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