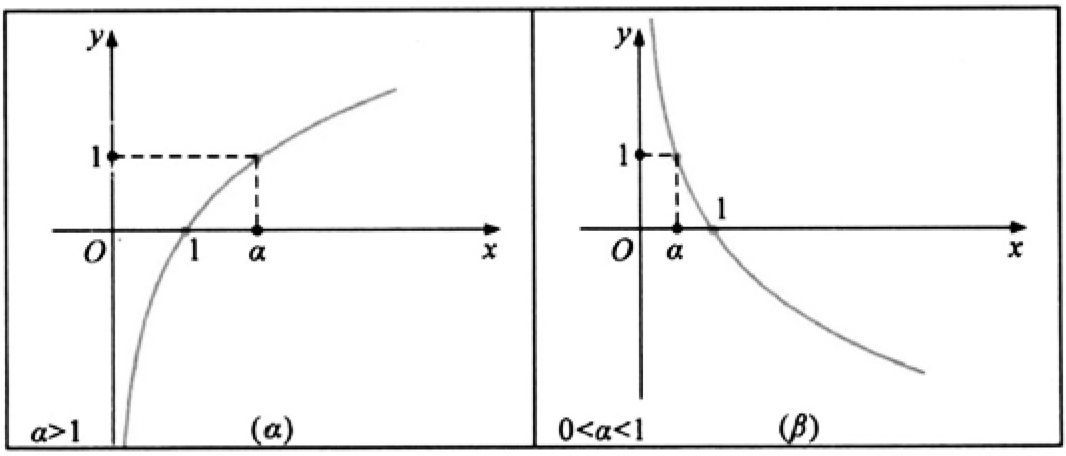
Η λογαριθμική συνάρτηση f(x) = $\log_\alpha x$,



A) $\log_a x = y \Rightarrow a^y = x$

B) $\log_a(x1x2) = \log_a x1 + \log_a x2$

C) $\log_\alpha \frac{\chi 1}{\chi 2} = \log_\alpha \chi 1 - \log_\alpha \chi 2$

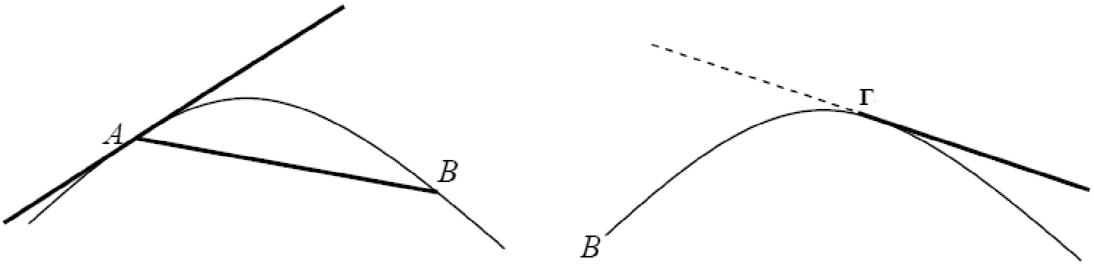
D) $\log_\alpha \chi^\kappa = \kappa \log_\alpha \chi$

Ειδικές περιπτώσεις που χρησιμοποιούνται συχνά στη στατιστική είναι :
$\log_{10} \chi = \log \chi$, $\log_e x = \ln x$

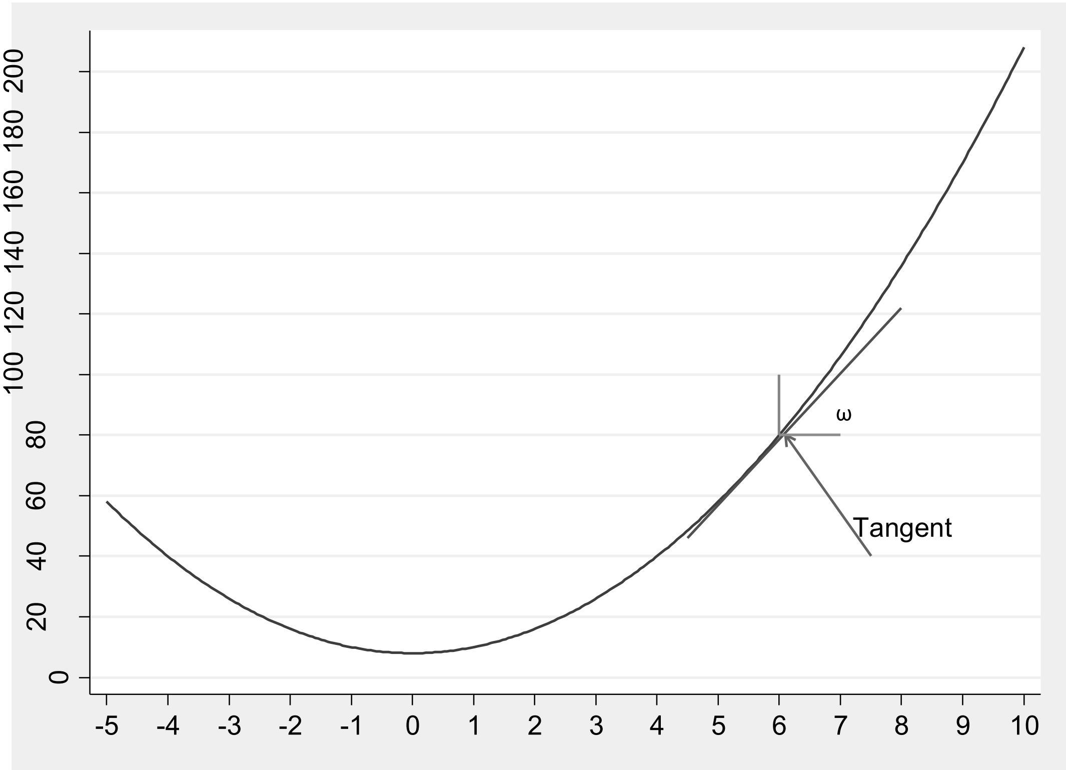
Παράγωγοι

ΟΡΙΣΜΟΣ

Αν δύο μεταβλητά μεγέθη x , y συνδέονται με τη σχέση y = f(x) , τότε ονομάζουμε ρυθμό μεταβολής του y ως προς το x στο σημείο x_0 την παράγωγο $f'(x_0)$.



Εικόνα: Εφαπτομένη σε διάγραμμα



Παράδειγμα Εκτιμώμενα CD4 μετά από την θεραπεία ανάλογα με χώρα προέλευσης

ΚΑΝΟΝΕΣ ΠΑΡΑΓΩΓΙΣΗΣ

Παράγωγος μερικών βασικών συναρτήσεων

(c) ′ = 0

(x) ′ = 1

$(x^{v})' = vx^{v-1}$

$$\left(\sqrt{x}\right)' = \frac{1}{2\sqrt{x}}$$

[ημ(x)] ′ = συν(x)

[συν(x)] ′ = − ημ(x)

(e^x) ′ = e^x

$[\ln(x)]' = \frac{1}{x}$

$(f + g)'(x0) = f'(x0) + g'(x0)$

$(f \bullet g)'(x0) = f'(x0)g(x0) + f(x0)g'(x0)$

$(f/g)'(x0) = [f'(x0)g(x0) - f(x0)g'(x0)]/[g(x0)]^2$

Να βρείτε την παράγωγο της συνάρτησης f στο σημείο x_0 όταν :

i) $f(x) = x^4$, $x_0 = -1$ ii) $f(x) = \sqrt{x}$, $x_0 = 9$

iii) $f(x) = \sigma\upsilon\nu x$, $x_0 = \frac{\pi}{6}$ iv) $f(x) = \ln x$, $x_0 = e$

v) $f(x) = e^x$, $x_0 = \ln 2$.

Παράδειγμα μέγιστης πιθανοφάνειας 1

$$f(x;p) = p^x(1-p)^{1-x} \quad x = 0,1 \quad 0 \le p \le 1$$

$$P(X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i} = p^{\sum x_i}(1-p)^{n-\sum x_i}$$

Τη συνάρτηση $p^{\sum x_i}(1-p)^{n-\sum x_i}$ την ονομάζουμε συνάρτηση πιθανοφάνειας και τη συμβολίζουμε με L(p). Η L(p) μεγιστοποιείται για την τιμή του p για την οποία $\frac{dL(p)}{dp} = 0$

$$\frac{dL(p)}{dp} = (\sum x_i)p^{\sum x_i - 1}(1-p)^{n-\sum x_i} - (n - \sum x_i)p^{\sum x_i}(1-p)^{n-\sum x_i - 1} = 0$$

$$\frac{\sum x_i}{p} - \frac{n - \sum x_i}{1-p} = 0$$

$$\sum x_i - np = 0$$

$$p = \frac{\sum x_i}{n} = \bar{x}$$

Η στατιστική συνάρτηση $\frac{\sum x_i}{n}$ λέγεται εκτιμητής μέγιστης πιθανοφάνειας και σημειώνεται με $\hat{p}$.

Παράδειγμα μέγιστης πιθανοφάνειας 2

Έστω τυχαίο δείγμα $X_1, X_2, \ldots, X_n$ από εκθετική κατανομή με συνάρτηση πυκνότητας πιθανότητας

$$f(x;\theta) = \frac{1}{\theta} e^{-\frac{x}{\theta}} \quad 0 < x < \infty \quad \theta \in \Omega = \{\theta : 0 < \theta < \infty\}$$

$$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = (\frac{1}{\theta} e^{-\frac{x_1}{\theta}})(\frac{1}{\theta} e^{-\frac{x_2}{\theta}})\ldots(\frac{1}{\theta} e^{-\frac{x_n}{\theta}}) = \frac{1}{\theta^n} e^{-\frac{\sum x_i}{\theta}}$$

$$\ln(L(\theta)) = -n\ln(\theta) - \frac{1}{\theta}\sum_{i=1}^{n} x_i \quad 0 < \theta < \infty$$

Η παραπάνω συνάρτηση μεγιστοποιείται στα σημεία που μηδενίζεται η παράγωγός της ως προς την παράμετρο θ. Έτσι:

$$\frac{dL(\theta)}{d\theta} = \frac{-n}{\theta} + \frac{1}{\theta^2}\sum x_i = 0$$

$$\hat{\theta} = \frac{\sum x_i}{n} = \bar{x}$$

Derivatives

Basic Properties/Formulas/Rules

$\frac{d}{dx}\left(cf(x)\right) = cf'(x)$, c is any constant.

$\frac{d}{dx}\left(f(x) \pm g(x)\right) = f'(x) \pm g'(x)$

$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$, n is any number.

$\frac{d}{dx}\left(c\right) = 0$, c is any constant.

$\left(f(x)\,g(x)\right)' = f'(x)\,g(x) + f(x)\,g'(x)$ – **Product Rule**

$\frac{d}{dx}\left(\mathbf{e}^{g(x)}\right) = g'(x)\mathbf{e}^{g(x)}$

$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{\left(g(x)\right)^2}$ – **Quotient Rule**

$\frac{d}{dx}\left[\ln(g(x))\right] = \frac{g'(x)}{g(x)}$

$\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right)g'(x)$ – **Chain Rule**

Common Derivatives

Polynomials

$\frac{d}{dx}\left(c\right) = 0$ $\quad \frac{d}{dx}(x) = 1$ $\quad \frac{d}{dx}\left(cx\right) = c$ $\quad \frac{d}{dx}\left(x^n\right) = nx^{n-1}$ $\quad \frac{d}{dx}\left(cx^n\right) = ncx^{n-1}$

Trig Functions

$\frac{d}{dx}\left[\sin(x)\right] = \cos(x)$ $\quad \frac{d}{dx}\left[\cos(x)\right] = -\sin(x)$ $\quad \frac{d}{dx}\left[\tan(x)\right] = \sec^2(x)$

$\frac{d}{dx}\left[\csc(x)\right] = -\csc(x)\cot(x)$ $\quad \frac{d}{dx}\left[\sec(x)\right] = \sec(x)\tan(x)$ $\quad \frac{d}{dx}\left[\cot(x)\right] = -\csc^2(x)$

Inverse Trig Functions

$\frac{d}{dx}\left[\sin^{-1}(x)\right] = \frac{1}{\sqrt{1-x^2}}$ $\quad \frac{d}{dx}\left[\cos^{-1}(x)\right] = -\frac{1}{\sqrt{1-x^2}}$ $\quad \frac{d}{dx}\left[\tan^{-1}(x)\right] = \frac{1}{1+x^2}$

$\frac{d}{dx}\left[\csc^{-1}(x)\right] = -\frac{1}{|x|\sqrt{x^2-1}}$ $\quad \frac{d}{dx}\left[\sec^{-1}(x)\right] = \frac{1}{|x|\sqrt{x^2-1}}$ $\quad \frac{d}{dx}\left[\cot^{-1}(x)\right] = -\frac{1}{1+x^2}$

Exponential & Logarithm Functions

$\frac{d}{dx}\left[a^x\right] = a^x\ln(a)$ $\quad \frac{d}{dx}\left[\mathbf{e}^x\right] = \mathbf{e}^x$

$\frac{d}{dx}\left[\ln(x)\right] = \frac{1}{x},\ x > 0$ $\quad \frac{d}{dx}\left[\ln|x|\right] = \frac{1}{x},\ x \neq 0$ $\quad \frac{d}{dx}\left[\log_a(x)\right] = \frac{1}{x\ln(a)},\ x > 0$

Hyperbolic Functions

$\frac{d}{dx}\left[\sinh(x)\right] = \cosh(x)$ $\quad \frac{d}{dx}\left[\cosh(x)\right] = \sinh(x)$ $\quad \frac{d}{dx}\left[\tanh(x)\right] = \operatorname{sech}^2(x)$

$\frac{d}{dx}\left[\operatorname{csch}(x)\right] = -\operatorname{csch}(x)\coth(x)$ $\quad \frac{d}{dx}\left[\operatorname{sech}(x)\right] = -\operatorname{sech}(x)\tanh(x)$ $\quad \frac{d}{dx}\left[\coth(x)\right] = -\operatorname{csch}^2(x)$

Integrals

The integral of f(x) is another function whose derivative is equal to g(x).

Basic Properties/Formulas/Rules

$\int cf(x)\,dx = c\int f(x)\,dx$, c is a constant. $\int f(x) \pm g(x)\,dx = \int f(x)dx \pm \int g(x)\,dx$

$\int_a^b f(x)\,dx = f(x)\Big|_a^b = F(b) - F(a)$ where $f(x) = \int f(x)\,dx$

$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$, c is a constant. $\int_a^b f(x) \pm g(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

$\int_a^a f(x)\,dx = 0$ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ $\int_a^b c\,dx = c(b-a)$, c is a constant.

If $f(x) \geq 0$ on $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq 0$

If $f(x) \geq g(x)$ on $a \leq x \leq b$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$

Common Integrals

Polynomials as the derivative of x+c is equal to 1 as the derivative of kx+c is equal to k

$\int dx = x + c$ $\int k\,dx = kx + c$ $\int x^n dx = \frac{1}{n+1}x^{n+1} + c,\ n \neq -1$

$\int \frac{1}{x}\,dx = \ln|x| + c$ $\int x^{-1}\,dx = \ln|x| + c$ $\int x^{-n}dx = \frac{1}{-n+1}x^{-n+1} + c,\ n \neq 1$

$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$ $\int x^{\frac{p}{q}}dx = \frac{1}{\frac{p}{q}+1}x^{\frac{p}{q}+1} + c = \frac{q}{p+q}x^{\frac{p+q}{q}} + c$

Trig Functions not so useful for statistics

$\int \cos(u)\,du = \sin(u) + c$ $\int \sin(u)\,du = -\cos(u) + c$ $\int \sec^2 u\,du = \tan(u) + c$

$\int \sec(u)\tan(u)\,du = \sec(u) + c$ $\int \csc(u)\cot(u)\,du = -\csc(u) + c$ $\int \csc^2 u\,du = -\cot(u) + c$

$\int \tan(u)\,du = -\ln|\cos(u)| + c = \ln|\sec(u)| + c$ $\int \cot(u)\,du = \ln|\sin(u)| + c = -\ln|\csc(u)| + c$

$\int \sec(u)\,du = \ln|\sec(u) + \tan(u)| + c$ $\int \sec^3(u)\,du = \frac{1}{2}\left(\sec(u)\tan(u) + \ln|\sec(u) + \tan(u)|\right) + c$

$\int \csc(u)\,du = \ln|\csc(u) - \cot(u)| + c$ $\int \csc^3(u)\,du = \frac{1}{2}\left(-\csc(u)\cot(u) + \ln|\csc(u) - \cot(u)|\right) + c$

Exponential & Logarithm Functions

$$\int \mathbf{e}^{u}\,du = \mathbf{e}^{u} + c \qquad \int a^{u}\,du = \frac{a^{u}}{\ln(a)} + c \qquad \int \ln(u)\,du = u\ln(u) - u + c$$

$$\int \mathbf{e}^{au}\sin(bu)\,du = \frac{\mathbf{e}^{au}}{a^2+b^2}\Big(a\sin(bu) - b\cos(bu)\Big) + c \qquad \int u\mathbf{e}^{u}du = (u-1)\mathbf{e}^{u} + c$$

$$\int \mathbf{e}^{au}\cos(bu)\,du = \frac{\mathbf{e}^{au}}{a^2+b^2}\Big(a\cos(bu) + b\sin(bu)\Big) + c \qquad \int \frac{1}{u\ln(u)}\,du = \ln\big|\ln(u)\big| + c$$

Inverse Trig Functions

$$\int \frac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\frac{u}{a}\right) + c \qquad \int \sin^{-1}(u)\,du = u\sin^{-1}(u) + \sqrt{1-u^2} + c$$

$$\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + c \qquad \int \tan^{-1}(u)\,du = u\tan^{-1}(u) - \frac{1}{2}\ln\left(1+u^2\right) + c$$

$$\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\sec^{-1}\left(\frac{u}{a}\right) + c \qquad \int \cos^{-1}(u)\,du = u\cos^{-1}(u) - \sqrt{1-u^2} + c$$

Hyperbolic Functions

$$\int \sinh(u)du = \cosh(u) + c \qquad \int \operatorname{sech}(u)\tanh(u)du = -\operatorname{sech}(u) + c \qquad \int \operatorname{sech}^2(u)du = \tanh(u) + c$$

$$\int \cosh(u)du = \sinh(u) + c \qquad \int \operatorname{csch}(u)\coth(u)du = -\operatorname{csch}(u) + c \qquad \int \operatorname{csch}^2(u)du = -\coth(u) + c$$

$$\int \tanh(u)du = \ln\left(\cosh(u)\right) + c \qquad \int \operatorname{sech}(u)du = \tan^{-1}\big|\sinh(u)\big| + c$$

Miscellaneous

$$\int \frac{1}{a^2-u^2}\,du = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + c \qquad \int \sqrt{a^2+u^2}\,du = \frac{u}{2}\sqrt{a^2+u^2} + \frac{a^2}{2}\ln\left|u+\sqrt{a^2+u^2}\right| + c$$

$$\int \frac{1}{u^2-a^2}\,du = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + c \qquad \int \sqrt{u^2-a^2}\,du = \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u+\sqrt{u^2-a^2}\right| + c$$

$$\int \sqrt{a^2-u^2}\,du = \frac{u}{2}\sqrt{a^2-u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right) + c$$

$$\int \sqrt{2au-u^2}\,du = \frac{u-a}{2}\sqrt{2au-u^2} + \frac{a^2}{2}\cos^{-1}\left(\frac{a-u}{a}\right) + c$$

Standard Integration Techniques

u Substitution : $\int_a^b f\big(g(x)\big)\,g'(x)\,dx$ will convert the integral into $\int_a^b f\big(g(x)\big)\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$ using the substitution $u = g(x)$ where $du = g'(x)dx$. For indefinite integrals drop the limits of integration.

Integration by Parts : $\int u\,dv = uv - \int v\,du$ and $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$. Choose u and dv from integral and compute du by differentiating u and compute v using $v = \int dv$.

Trig Substitutions : If the integral contains the following root use the given substitution and formula.

$$\sqrt{a^2 - b^2x^2} \quad \Rightarrow \quad x = \frac{a}{b}\sin(\theta) \quad \text{and} \quad \cos^2(\theta) = 1 - \sin^2(\theta)$$

$$\sqrt{b^2x^2 - a^2} \quad \Rightarrow \quad x = \frac{a}{b}\sec(\theta) \quad \text{and} \quad \tan^2(\theta) = \sec^2(\theta) - 1$$

$$\sqrt{a^2 + b^2x^2} \quad \Rightarrow \quad x = \frac{a}{b}\tan(\theta) \quad \text{and} \quad \sec^2(\theta) = 1 + \tan^2(\theta)$$

Partial Fractions : If integrating a rational expression involving polynomials, $\int \frac{P(x)}{Q(x)}\,dx$, where the degree (largest exponent) of $P(x)$ is smaller than the degree of $Q(x)$ then factor the denominator as completely as possible and find the partial fraction decomposition of the rational expression. Integrate the partial fraction decomposition (P.F.D.). For each factor in the denominator we get term(s) in the decomposition according to the following table.

Factor of $Q(x)$	Term in P.F.D	Factor is $Q(x)$	Term in P.F.D
$ax+b$	$\frac{A}{ax+b}$	$(ax+b)^k$	$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$	$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c} + \cdots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Products and (some) Quotients of Trig Functions :

For $\int \sin^n(x)\cos^m(x)\,dx$ we have the following :

1. n **odd.** Strip 1 sine out and convert rest to cosines using $\sin^2(x) = 1 - \cos^2(x)$, then use the substitution $u = \cos(x)$.
2. m **odd.** Strip 1 cosine out and convert rest to sines using $\cos^2(x) = 1 - \sin^2(x)$, then use the substitution $u = \sin(x)$.
3. n **and** m **both odd.** Use either 1. or 2.
4. n **and** m **both even.** Use double angle and/or half angle formulas to reduce the integral into a form that can be integrated.

For $\int \tan^n(x)\sec^m(x)\,dx$ we have the following :

1. n **odd.** Strip 1 tangent and 1 secant out and convert the rest to secants using $\tan^2(x) = \sec^2(x) - 1$, then use the substitution $u = \sec(x)$.
2. m **even.** Strip 2 secants out and convert rest to tangents using $\sec^2(x) = 1 + \tan^2(x)$, then use the substitution $u = \tan(x)$.
3. n **odd and** m **even.** Use either 1. or 2.
4. n **even and** m **odd**. Each integral will be dealt with differently.

Convert Example : $\cos^6(x) = \left(\cos^2(x)\right)^3 = \left(1 - \sin^2(x)\right)^3$