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Generalization of the spherical isoperimetric inequality to uniformly convex Banach spaces

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0. In this paper we generalize the classical isoperimetric inequality on S^n to non-invariant measures and prove as a corollary the concentration of measure on spheres $S(X)$ of uniformely convex Banach spaces X. Our argument avoids symmetrization and (or) the calculus of variations by a direct appeal to Cavaliery's principle similar to that used in Hadwiger's proof of the Brunn-Minkowski theorem [H]. In fact, we use the localized Brunn's theorem at the final stage of our proof, though a slight rearrangment of our argument would imply this theorem. (In the Appendix, we give, for the completeness sake, a short proof of Brunn's theorem). One of the applications is the lower exponential bound on the dimension of l_∞ admitting a symmetric map $S(X) \to S(l_\infty)$ with a fixed Lipschitz constant.

In order to keep the presentation transparent we did not attempt to state the most general isoperimetric inequality serving all possible applications. This has unavoidably led to repetitions of some arguments at different places in the paper as some readers may notice.

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1. Let μ be some measure on the Euclidean sphere S^n and let A and B be two disjoint subsets in S^n. We seek an upper bound on dist (A, B) in terms of the measures $\mu(A)$ and $\mu(B)$, where "dist" is some metric on S^n. If B is the complement of the ε-neighbourhood of A, then for $\varepsilon \to 0$ our question reduces to the *isoperimetric problem.*

2. To formulate our main results we have to introduce some notions.

2.1. Consider an open arc $\sigma \subset S^n$ between two opposite points t_+ and t_- in S^n and call a subset Σ σ-admissible if it is a union of open arcs between t_+ and t_- and if every point t in σ lies in the interior of Σ. Next divide σ into three subintervals, say $\sigma = (t_+, a) \cup (a, b) \cup (b, t_-)$, called α_1, α_2 and α_3

respectively, and let A_i, $i = 1, 2, 3$ be open subsets in S^n such that $A_i \cap \sigma = \alpha_i$. Finally, take a Borel measure μ on S^n, and define the *relative canonical measure* $\mu_\sigma(\alpha_2/\alpha_i)$ for $i = 1, 3$ by

$$\mu_\sigma(\alpha_2/\alpha_i) \quad = \quad \inf \liminf_{\Sigma \to \sigma} \frac{\mu(A_2 \cap \Sigma)}{\mu(A_i \cap \Sigma)} \tag{2.1}$$

where inf is taken over all above triples (A_1, A_2, A_3) and where we assume $\infty/\infty = \infty$ and $0/0 = 0$, and where the convergence $\Sigma \to \sigma$ is understood for the Hausdorff topology for subsets in the sphere.

2.2. In the case of a "good" measure μ the definition (2.1) simplifies as follows.

Consider the family of all non-negative measures on S^n with continuous density functions. We will call such measures *regular*. So, for every regular measure μ there exists $f_\mu(t) \in C(S^n)$ such that for any Borel set $A \subset S^n$, $\mu(A) = \int_A f_\mu(t)\mathrm{d}t$. Take two opposite points t and $-t$ on S^n, and consider all maximal arcs σ between t and $-t$. This gives a partition $\mathscr{H}_t$ of $S^n \backslash \{t; -t\}$ and hence every regular measure μ induces a measure (defined up to a constant) on every $\sigma \in \mathscr{H}_t$, called μ_0. We call such partitions *canonical partitions*. Clearly, in this case (2.1) may be rewritten as

$$\mu_\sigma(\alpha_2/\alpha_i) \quad = \quad \frac{\mu_\sigma(\alpha_2)}{\mu_\sigma(\alpha_i)}\,.$$

EXAMPLES 2.3: a. If μ is the standard measure on S^n, then, obviously $\mu_\sigma =$ Const. $(\sin\theta)^{n-1}\,\mathrm{d}\sigma$ for θ – the angle from $[0, \pi]$ parametrizing σ.

b. Let $S^n_+ \subset S^n$ be a hemisphere, and let $S^n_+ \simeq \mathbb{R}^n$ be a projective isomorphism. Then, in case t and $-t$ lie on ∂S^n_+, arcs σ are straight lines in $\mathbb{R}^n$, and so the $\mathbb{R}^n$-invariant measure μ on $\mathbb{R}^n \simeq S^n_+$ gives the Lebesgue measure $\mathrm{d}t$ on c's.

2.4. Next, let A_1 and A_3 be closed subsets in S^n, let A_2^+ be the union of all arcs in S^n between A_1 and A_3 (e.g. $A_2^+ = \mathrm{Conv}\, A_1 \cup A_3$ in the case of convex sets A_1 and A_3), and let $A_2 = A_1 * A_3$ be defined as

$$A_2 \quad = \quad A_1 * A_3 \quad \stackrel{\text{def}}{=} \quad A_2^+ \backslash (A_1 \cup A_3). \tag{2.2}$$

Assume the μ-measures of A_1 and A_3 to be in $(0, \infty)$ and let $\lambda = \mu(A_1)/\mu(A_3)$. Call a pair of points $a \in A_1$ and $b \in A_3$ *extremal* if there is a maximal arc σ

in S^n which contains a and b such that the open interval $(a, b) \subset \sigma$ misses A_1 and A_3. The define

$$\lambda\text{-dist}_\mu\,(a, b)m \;=\; \inf_\sigma \max\,(\lambda\;\mu_\sigma(\alpha_2/\alpha_1),\; \mu_\sigma(\alpha_2/\alpha_3))$$

where a maximal arc $\sigma = (t_-, t_+)$ is divided into intervals $\alpha_1 = (t_-, a)$, $\alpha_2 = [a, b]\ \alpha_3 = (b, t_+)$, and

$$\lambda\text{-dist}_\mu\,(A_1, A_3) \;=\; \inf\,(\lambda\text{-dist}_\mu\,(a, b))$$

over all extremal pairs (a, b) and all σ. In what follows we abbreviate: λ-dist $=$ dist.

THEOREM 3. $\mu(A_2) \geqslant \text{dist}_\mu\,(A_1, A_3)\mu(A_3) = \lambda^{-1}\,\text{dist}_\mu\,(A_1, A_3)\mu(A_1)$.

EXAMPLE 3.1: If μ is the $0(n)$-invariant measure, then the explicit formula for the canonical measure (see Example 2.3.a) gives a *sharp* lower bound for the spherical distance between A_1 and A_3 with equality for balls around opposite points in S^n. Thus, we recapture the classical isoperimetric inequality for S^n.

3.2. The proof of the thereom involves a few constructions which we consider to be of independent interest. By an obvious approximation argument and the definition (2.1), we may (and shall) assume the measure μ is *positive regular* which means, in addition to the regularity condition, that $\mu(A) > 0$ for every open subset A in S^n.

NON IMPORTANT REMARK. One could eliminate λ from the story by multiplying the measure by λ on A_3, and thus reducing the problem to the case $\lambda = 1$. However we prefer to keep λ.

Take an open hemisphere S^n_+, and fix a projective isomorphism $S^n_+ \leftrightarrow \mathbb{R}^n$ sending every straight line of $\mathbb{R}^n$ to a maximal arc on S^n_+ (and conversely). This provides a one-to-one correspondence between positive regular measures on $\mathbb{R}^n$ and S^n_+, thus identifying measures on S^n_+ and $\mathbb{R}^n$ which we denote by the same μ.

4. *Convex restrictions of measures.* Take an affinite (i.e., a translate of a linear) subspace $E \subset \mathbb{R}^n$, fix a projection $p\colon \mathbb{R}^n \to E$, and consider decreasing sequences of convex subsets K_i, $i \in \mathbb{N}$ in $\mathbb{R}^n$, such that $M = \cap K_i \subset E$. By restricting a given measure μ to each K_i, and then by projecting to E, we obtain a sequence of measures μ_i on E. Call a non identically zero measure

ν on E a *convex restriction of μ to M* if for some sequence of above K_i and some sequence of real numbers λ_i,

$$\nu = \lim \lambda_i \mu_i,$$

for the weak limit of measures.

If a measure ν' on a convex subset M' in an affinit subspace $E' \subset E$ is a convex restriction of ν, then obviously, ν' also is a convex restriction of the original μ.

5. *First step. Use of Brunn's Theorem.* Take a k-dimensional subspace $E \subset \mathbb{R}^n$ and a convex body $K \subset \mathbb{R}^n$. Observe that the $(n-k)$-dimensional symmetrization (see [H] or [B.2] $S_E K$ is convex by Brunn's Theorem (see Appendix).

LEMMA 5.1: *Let a decreasing family of convex sets $\{k_i\}$ define a convex restriction measure $\mu_M(M = \cap K_i \subset E)$* of μ. *Then the family $\{S_E K_i\}$ defines the same measure μ_M.*

Proof. This follows from the definition of $S_E K_i$ and the uniqueness of the Radon–Nicodym derivative f_μ which is a continuous function in our case and therefore well defined on M.

REMARK 5.2: If μ is absolutely continuous (rather than regular) with respect to Lebesque measure, then the Lemma only holds true for *almost every* k-dimensional subspace $E \subset \mathbb{R}^n$.

6. *Convex partitions of S^n_+ and $\mathbb{R}^n$.* A set $A \subset S^n_+$ is called *convex* if it contains the arc (in S^n_+) between a and b for all a and b in A. Clearly A is convex iff the set corresponding to it by the projective isomorphism in $\mathbb{R}^n$ is a convex set in $\mathbb{R}^n$.

DEFINITION 6.1: We say that $\mathscr{a}$ is a *k-dimensional convex partition of* $\mathbb{R}^n$ if

i) every $A \in \mathscr{a}$ is convex and k-dimensional, i.e., there exists a k-dimensional affine subspace E such that $A \subset E$ and the interior $\mathring{A}$ of A in E is not emply.

ii) there exists a family of convex open neighbourhood K_i of $\mathring{A}$ such that $\cap K_i = \mathring{A}$ and every $K_i = \cup \mathring{A}_\alpha$ for some $A_\alpha \in \mathscr{a}$.

The image of $\mathscr{a}$ on S^n_+ by a projective isomorphism is called the *k-dimensional convex partition* of S^n_+.

6.2. Consider a 1-dimensional convex partition α of $\mathbb{R}^n$. By Rohlin's measure decomposition theory (see [R]), there is a unique induced measure μ_I on almost every (for the quotient measure on $\mathbb{R}^n/\alpha)I$. In fact, this μ_I equals the convex restriction of μ defined by some sequence of convex bodies $\{K_i\}$ such that $\cap K_i = \mathring{I}$ and every $K_i = \cup \mathring{I}_\alpha$ for some $I_\alpha \in \alpha$ (such family exists by 6.1, ii)). Therefore, (use 5.1) μ_I is the convex restriction of μ defined by the family $\{S_I K_i\}$ obtained from $\{K_i\}$ by the symmetrization around I.

7. *Second step. Construction of 1-dimensional partitions of S^n use of the Borsuk-Ulam Theorem.* We consider a regular positive measure μ on S^n. Let A_1, A_3 and $A_2 = A_1 * A_3 \subset S^n$ be subsets from 2.4. Let

$$\mu(A_1) = \lambda\mu(A_3).$$

Define $H_x^+ = \{y \in S^n : (y, x) \geqslant 0\}$ and $H_x^- = -H_x^+$. Note that $\mathring{H}_x^+ = S_+^n$. We consider a map $\varphi \colon S^n \to \mathbb{R}^2$ such that

$$\varphi(x) = (\mu(A_1 \cap H_x^+); \mu(A_3 \cap H_x^+)).$$

By the Borsuk–Ulam Theorem there exists x_0 such that $\varphi(x_0) = \varphi(-x_0)$ which means that for $i = 1$ and 3

$$\mu(A_i) = 2\mu(A_i \cap H_{x_0}^+)$$

and as a consequence

$$\lambda = \mu(A_1 \cap H_{x_0}^+)/\mu(A_3 \cap H_{x_0}^+).$$

First, we fix one such x_0 and let $S_+^n = \mathring{H}_{x_0}^+$ and $A_i^+ = S_+^n \cap A_i$. Next, we define a convex partition of S_+^n by induction as follows. If $M \subset S_+^n$ is one of the convex sets from the preceding inductive step, then, by assumption

$$\mu(A_1^+ \cap M)/\mu(A_3^+ \cap M) = \lambda;$$

next we construct a map $\varphi \colon S^n \to \mathbb{R}^2$ using $A_i^+ \cap M$ as above (instead of A_i). The map φ determines how to divide M into two convex pieces M^+ and M^- by a hyperplane in such a way that $\mu(A_1^+ \cap M^+)/\mu(A_3^+ \cap M^+) = \lambda$ again. The same holds for the intersections $A_i^+ \cap M^-$. We continue to refine our partitions, and obtain in the limit a partition α_{n-1} whose elements have a strictly smaller dimension than n (using that μ is positive). By Rohlin's theory [R] (compare 6.2) our measure μ defines (up to factor) a (convex restriction) measure μ_α on almost every $M_\alpha \in \alpha_{n-1}$. The construction implies that $\mu_\alpha(A_1 \cap M_\alpha)/\mu_\alpha(A_3 \cap M_\alpha) = \lambda$.

Then we may continue the same procedure with every M_α if $\dim M_\alpha \geqslant 2$. The last condition is important when the Borsuk–Ulam Theorem is used.

Finally, we construct a partition α of S^n_+ such that for almost every $I \in \alpha$

i) $\dim I = 1$

ii) $\mu_I(A_1 \cap I)/\mu_I(A_3 \cap I) = \lambda$

where

iii) μ_I is a convex restriction measure of μ induced by the partition α.

(The last property follows from 4).

iv) If $\mu_I(A_2 \cap I) \geqslant \alpha_i \mu_I(A_i \cap I)$ for almost every $I \in \alpha$ then $\mu(A_2) \geqslant \alpha_i \mu(A_i)$ (for $i = 1, 3$).

8. *Conclusion of the proof.* We regard α constructed in section 7 as a partition of $\mathbb{R}^n \sim S^n_+$ into straight intervals $I \subset \mathbb{R}^n$. Then, by the property iii) of μ_I (see 7) and Remark 6.2, the measure μ_I on I is a convex restriction of μ defined by a family $\{K_i\}_{i \in \mathbb{N}}$ where K_i are convex sets having the $(n-1)$-dimensional symmetry in the direction perpendicular to I centered at I. Therefore, μ_I is defined by some family of shrinking convex sets $\{T_i\}_{i \in \mathbb{N}}$ in $\mathbb{R}^2$. Let us first consider a case when A_1 and A_3 are convex sets. Then $A_2 \cap I$ is a single interval for every $I \in \alpha$. In this case (see Fig. 1 with $A_2 \cap I$ for $[a, b]$) we may replace T_i by a cone of rotation C_i around I (or in the degenerate case by a cylinder) such that the convex restriction measure μ_C defined on I by μ and the family $C = \{C_i\}_{i \in \mathbb{N}}$ satsifies for $i = 1$ and 3

i) $\mu_C(A_2 \cap I)/\mu_C(A_i \cap I) \leqslant \mu_I(A_2 \cap I)/\mu_I(A_i \cap I)$.

Also it is clear that a family of symmetric cones centered on the straight line containing I is defined by a canonical partition (see 2.2). Therefore,

ii) $\mu_C = \mu_\sigma$ up to a constant factor for some σ containing I.

Let now $\varrho = \mathrm{dist}_\mu (A_1, A_3)$. Then, for extremal points a and b on I and any σ, $I \subset \sigma$, we have either

$$\varrho \leqslant \lambda \mu_\sigma(\alpha_2)/\mu_\sigma(\alpha_1)$$

or

$$\varrho \leqslant \mu_\sigma(\alpha_2)/\mu_\sigma(\alpha_3),$$

where $\alpha_1 = (-t, a]$, $\alpha_3 = [b, t)$ and $\alpha_2 = A_2 \cap I$ and the points $\pm t$ are joined by the arc σ.
So, 8 i) and 7 iv) imply Theorem 3 in this special case (we use again that $\mu_I(A_1 \cap I)/\lambda = \mu_I(A_3 \cap I)$).
In the general case we have again to prove that

$$\mu_I(A_2 \cap I) \geqslant \frac{\varrho}{\lambda} \min \{\mu_I(A_1 \cap I); \lambda \mu_I(A_3 \cap I)\} \tag{8.1}$$

$$\left(= \frac{\varrho}{\lambda} \mu_I(A_1 \cap I) = \varrho \mu_I(A_3 \cap I)\right).$$

By an approximation argument we may assume that $A_2 \cap I$ contains a finite number of intervals. Let A_2'' be one of such intervals and A_2' be the union of all intervals from $A_2 \cap I$ on the one side (say on the left) of A_2''. Call also A_i' the part of $A_i \cap I$ on the same left part of A_2'' for $i = 1$ and 3. Let, for example, A_2'' be joined from the right by an interval from $A_1 \cap I$, called A_1''. We will assume that

$$\mu_I(A_2') \geqslant \frac{\varrho}{\lambda} \min \{\mu_I(A_1'); \lambda \mu_I(A_3')\} \tag{8.2}$$

(i.e. (8.1) is satisifed for the sets A_i'), and we prove (8.1) for the sets $A_i' \cup A_i''$. (We leave for the reader to check the starting point of such induction which will be finished after a finite number of steps and will prove (8.1)). Let $\alpha_2 = A_2'' \subset I$. We choose a maximal arc $\sigma \supset I$ (= a straight line under the projective isomorphism $S_+^n \sim \mathbb{R}^n$) in the same way (see Fig. 1 with α_2 for $[a, b]$) as we did earlier for the convex case where $\alpha_2 = A_2''$ played the role of the entire A_2. Let α_1 be the right hand (corresponding to the right hand

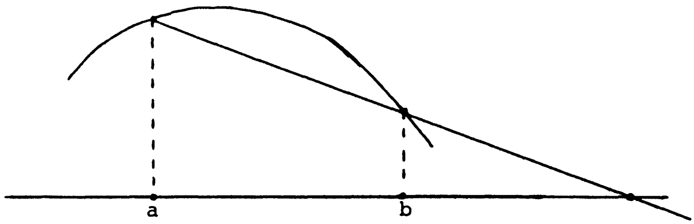

Fig. 1.

side position of A_1'' with respect to A_2'') component of the complement $\sigma\backslash\alpha_2$ and let α_3 be the left hand component. By the definition of the λ-dist. ϱ

$$\mu_\sigma(A_2'') \geqslant \frac{\varrho}{\lambda} \min \{\mu_\sigma(\alpha_1); \lambda\mu_\sigma(\alpha_3)\}.$$

By the construction of σ (see again Fig. 1 and the explanation), we have

$$\mu_I(A_2'') \geqslant \mu_\sigma(A_2'') \quad \text{and} \quad \mu_I(A_1'') \leqslant \mu_\sigma(A_1'') \leqslant \mu_\sigma(\alpha_1),$$

$$\mu_I(A_3') \leqslant \mu_\sigma(A_3') \leqslant \mu_\sigma(\alpha_3).$$

Therefore

$$\mu_I(A_2'') \geqslant \frac{\varrho}{\lambda} \{\min \mu_I(A_1''); \lambda\mu_I(A_3')\} \tag{8.3}$$

Adding (8.2) and (8.3), we have

$$\mu_I(A_2' \cup A_2'') \geqslant \frac{\varrho}{\lambda} \min \{\mu_I(A_1' \cup A_1''); \mu_I(\mathrm{A}_1'') + \lambda\mu_I(\mathrm{A}_3');$$

$$\lambda\mu_I(A_3') + \mu_I(A_1'); 2\lambda\mu_I(A_3')\} \geqslant \frac{\varrho}{\lambda} \{\min \mu_I(A_1' \cup A_1''); \lambda\mu_I(A_3')\}.$$

So, as we wanted, (8.1) is proved for the sets $A_i' \cup A_i''$ (note that A_3'' is empty in our case before). □

REMARK 9: The 1-dimensional partition constructed in Section 7 is not necessarily a convex partition (we passed through intermediate dimensions). We indicate below how to modify this construction to obtain a 1-dimensional convex partition satisfying property ii) from 7.

Let $M \subset S^n_+$ is a convex n-dimensional body from the intermediate step of construction. Then $\mu(A_1^+ \cap M)/\mu(A_3^+ \cap M) = \lambda$. Take a triple of points $x_1, x_2, x_3 \subset M$ which maximize the determinant $|(x_i, x_j)_{i=1,2,3}|$. Define the map φ: $S^2 \to \mathbb{R}^2$ using $A_i^+ \cap M$ as above for $S^2 \subset \mathbb{R}^3 = \text{span}\,\{x_1, x_2, x_3\}$. Using this map, subdivide M into two pieces M^+ and M^- by a hyperplane such that the number λ coincide with the above. Thus, one can obtain a 1-dimensional convex partition $\mathscr{a}$ satisfying ii) from 7.

10. The preceeding construction can be adjusted to various results related to the isoperimetric inequalities.

THEOREM 10.1. *Let A and B be closed subsets of S^n, μ a regular positive measure on S^n and $f(x, y)$ any continuous function on $(S^n \times S^n) - \{(t, -t), t \in S^n\}$. There exists a maximal open arc σ and disjoined sets $\alpha_i \subset \sigma$, $i = 1$, 2, 3 (where $\alpha_2 = \alpha_1 * \alpha_3$) such that*

$$\inf \{f(a, b)\colon a \in \alpha_1, b \in \alpha_3\} \geqslant \inf \{f(x, y)\colon x \in A, y \in \beta, x \neq y\} \tag{10.1}$$

and for $C = A * B$

$$\frac{\mu_\sigma(\alpha_2)}{\mu_\sigma(\alpha_1)} \leqslant \frac{\mu(C)}{\mu(A)}\,;\ \frac{\mu_\sigma(\alpha_2)}{\mu_\sigma(\alpha_3)} \leqslant \frac{\mu(C)}{\mu(B)}\,. \tag{10.2}$$

Proof of the theorem follows from Section 8.

REMARK 10.2. An important case is when $f(x, y)$ is a *distance function* on S^n.

REMARK 10.3. We say that $f(x, y)$ is *monotone* if for any maximal arc σ and for every $x, y, z \subset \sigma$, $y \in (x, z) \subset \sigma$,

$$\max \{f(x, y), f(y, z)\} \leqslant f(x, z)$$

Now, if in the theorem a function $f(x, y)$ is monotone, then there exists a maximal open arc σ (joined some points $\pm\, t \in S^n$) and a partition σ on three intervals: $\alpha_1 = (-t, a]$, $\alpha_2 = (a, b)$, $\alpha_3 = [b, t)$ such that $f(a, b) \geqslant \inf \{f(x, y)\colon x \in A, y \in B, x \neq -y\}$ and (10.2) is satisifed for the above σ and $\alpha_i \subset \sigma$. □

The theorem below follows from the section 7.

THEOREM 10.4. *Let A and B be closed subsets of S^n, μ a regular positive measure on S^n and $C = A * B$.*

i) *There exist maximal open arcs σ_i, intervals $I_i \subset \sigma_i$ and convex restriction measures ν_i on I_i $(i = 1, 2)$ such that*

$$\nu_1(C \cap I_1)/\nu_1(A \cap I_1) \geqslant \mu(C)/\mu(A),\ \nu_1(C \cap I_1)/\nu_1(B \cap I_1) \geqslant \mu(C)/\mu(B)$$

and

$$\nu_2(C \cap I_2)/\nu_2(B \cap I_2) \leqslant \mu(C)/\mu(B),\ \nu_2(C \cap I_2)/\nu_2(B \cap I_2) \vdash \mu(C)/\mu(B).$$

ii) *If, in addition, μ is a probability measure then there exists a maximal open arc σ, an interval $I \subset \sigma$ and a probability convex restriction measure ν on I such that*

$$\nu(A \cap I) = \mu(A) \quad \text{and} \quad \nu(B \cap I) = \mu(B).$$

11. *Concentration of measure on the unit sphere of a uniformly convex normed spaced.* Let a normed space $X = (\mathbb{R}^{n+1}, \|.\|)$ have for fixed $\varepsilon > 0$ *the modulus of convexity* at least $\delta(\varepsilon) > 0$. It means that for every two points x, y in X, $\|x\| = \|y\| = 1$ and $\|x - y\| \geqslant \varepsilon$,

$$1 - \frac{\|x + y\|}{2} \geqslant \delta(\varepsilon).$$

Also let $\delta(\varepsilon)$ be a monotone (increasing) function

11.1. *Linear functionals.* Take a vector $f \in X^*$, $\|f\| = 1$. Define $K = \{x \in X, \|x\| \leqslant 1\}$, $S(X) = \partial K = \{x \in X, \|x\| = 1\}$ and $K_\lambda = K \cap \{x\colon f(x) = \lambda\}$. Clearly, $\mathrm{Vol}_n K_\lambda = \mathrm{Vol}_n K_{-\lambda}$ and $(K_\lambda + K_{-\lambda})/2 = A \subset K_0$. By the Brunn–Minkowski inequality,

$$\mathrm{Vol}_n (K_\lambda + K_{-\lambda})^{1/n} \geqslant (\mathrm{Vol}_n K_\lambda)^{1/n} + (\mathrm{Vol}_n K_{-\lambda})^{1/n},$$

and therefore $\mathrm{Vol}_n K_\lambda \leqslant \mathrm{Vol}_n A$. Note also that for any $x \in K_\lambda$ and $y \in K_{-\lambda}$ we have $\|x - y\| \geqslant 2\lambda$ and, consequently, $\|x + y|/2 \leqslant 1 - \delta(2\lambda)$. Therefore (see [GM2])

LEMMA. $\mathrm{Vol}_n K_\lambda \leqslant (1 - \delta(2\lambda))^n \mathrm{Vol}_n K_0$.

So, we see that the volume of the levels of a linear functional are exponentially *concentrated* at the zero level. We continued this direction in [GM2] to show that (see [GM2], Theorem 3.2).

THEOREM. *For every* $f \in X^*$, $\|f\|^* = 1$,

$$\mathrm{Vol}_{n+1} \{x \in K, |f(x)| \geqslant \varepsilon\} \leqslant (n+1)e^{-n\delta(2\varepsilon)/2} \mathrm{Vol}_{n+1} K.$$

We will extend the results from 11.1 to an arbitrary 1-Lipschitz function on $S(X)$.

11.2. *A measure on* $S(X)$. A standard $(n+1)$-dimensional volume on $\mathbb{R}^{n+1}$ induces the probability measure μ on $S(X)$: for any Borel set $A \subset S(X)$,

$$\mu(A) \stackrel{\text{def}}{=} \mathrm{Vol}_{n+1} \{\cup tA, 0 \leqslant t \leqslant 1)/\mathrm{Vol}\, K. \tag{11.1}$$

To apply Theorem 3 to this measure μ on $S(X)$ we have to estimate μ_σ for any maximal arc σ. However our estimate will also work for any convex restriction measure, and the application of Theorem 10.4 will be easier.

Note that Theorem 3 and Theorem 10.4 concern measures on S^n in the projective sense and are applicable for μ on $S(X)$ (the reader who feels uncomfortable at this point, may choose *any* euclidean sphere S^n and, using the radial projection of $S(X)$ to S^n, transport all constructions and results from $S(X)$ to S^n and vice versa).

Fix $z \in S(X)$. Let $f \in X^*$, $\|f\| = 1$, be the support functional at z, i.e. $f(z) = 1$. Consider $\mathrm{Ker} f \cap S(X) = S_0$. Take any $x \in S_0$. We study σ which joins $\pm z$ and pass through x (i.e. a "half" of the two-dimensional sphere $S(X) \cap \mathrm{span}\,\{x; z\}$). Choose a parametrization of the arc σ. Take $x_0 \in \theta$, such that $\theta = \varrho(x_0, -z) = \int_{-z}^{x_0} \|dx_t\|$, i.e., θ is the length of the arc $(-z, x_0)$. It is known [S] that $a = \varrho(z, -z)$ changes between $3 \leqslant a \leqslant 4$ (a is the "π" of a normed space $E = \mathrm{span}\,\{z; x\}$) and for $C < t$; $\theta < a$

$$\|x_t - x_0\| \leqslant \varrho(x_t, x_0) \leqslant 2\|x_t - x_0\| \tag{11.2}$$

So for any $t \in (0, a)$ we have the unique $x_t \in \sigma$.

PROPOSITION 11.3. *Let $\delta(\varepsilon)$ be as in 11. If ν is a convex restriction probability measure on σ then there exists $t_0 \in [0, a]$ such that for any $\theta > 0$*

$$\begin{cases} \nu[-z, x_{t_0-\theta}] \leqslant \dfrac{[1-\delta(\theta/4)]^{n-1}}{1-[1-\delta(\theta/4)]^{n-1}}\, \nu[x_{t_0-\theta}, x_{t_0}] \quad (\text{if } t_0 > \theta) \\ \text{and} \\ \nu[x_{t_0+\theta}, z] \leqslant \dfrac{[1-\delta(\theta/4)]^{n-1}}{1-[1-\delta(\theta/4)]^{n-1}}\, \nu[x_{t_0}, x_{t_0+\theta}] \quad (\text{if } t_0+\theta < a) \end{cases} \tag{11.3}$$

where one may choose t_0 as the (unique) maximum point of the density $f_\nu(t)$ of the measure ν. The function $\psi(t) = [f_\nu(t)]^{1/(n-1)}$ satisfies the following "weak concavity" condition: there exists a number α, $0 < \alpha < 1/2$, such that for any $0 < t_1 < t_2 < a$ and $\theta = t_2 - t_1$

$$\frac{\psi(t_1)+\psi(t_2)}{2} \leqslant \max_{\tau\in(t_1+\alpha\theta,\, t_2-\alpha\theta)} \psi(\tau)$$

It follows from (11.3) that (if $\theta < t_0 < a - \theta$)

$$\nu\{(-z, x_{t_0-\theta}) \cup (x_{t_0+\theta}, z)\} \leqslant (1-\delta(\theta/4))^{n-1} \simeq \mathrm{e}^{-\delta(\theta/4)(n-1)} \tag{11.4}$$

Proof. We use an argument similar to that of 11.1 where a concentration property of linear functionals was proved. Define $\Delta\nu(x_t)$ to be an infinitesimal $(n-1)$-dimensional volume of infinitesimal convex neighborhood Δ_t of x_t in $S(X) \cap \{x : f(x) = f(x_t)\}$ which induces the density $f_\nu(t)$ of the probability convex restriction measure ν on σ at the point x_t. Then, by the Brunn–Minkowski inequality, for any $0 < t_1 < t_2 < a$

$$\operatorname{Vol}\left(\frac{\Delta_{t_1}+\Delta_{t_2}}{2}\right)^{1/(n-1)} \geqslant \frac{[\Delta\nu(x_{t_1})]^{1/(n-1)} + [\Delta\nu(x_{t_2})]^{1/(n-1)}}{2}$$
$$\geqslant \min_{i=1,2} \{\Delta\nu(x_{t_i})\}^{1/(n-1)}.$$

Also, for any $y_t \in \Delta_{t_1}$ and $y_{t_2} \in \Delta_{t_2}$, we have

$$\frac{y_{t_1}+y_{t_2}}{2} = \lambda y_{t_3} \tag{11.5}$$

for some $0 < \lambda < 1 - \delta(\|x_{t_2} - x_{t_1}\|)$ where $y_{t_3} \in S(X)$ and y_{t_3} belongs to the arc joining y_{t_1} and y_{t_2}, i.e., y_{t_3} belongs to any convex neighborhood of the arc $\{x_{t_1}, x_{t_2}]$ which is contained Δ_{t_1} and Δ_{t_2}. Therefore,

$$\mathrm{Vol}\left(\frac{\Delta_{t_1} + \Delta_{t_2}}{2}\right) \leqslant [1 - \delta(\|x_{t_2} - x_{t_1}\|)]^{n-1} \max_{t_1 \leqslant \tau \leqslant t_2} \mathrm{Vol}\,\Delta_\tau \tag{11.6}$$

$$\leqslant \left[1 - \delta\left(\frac{t_2 - t_1}{2}\right)\right]^{n-1} \max_{t_1 < \tau < t_2} \mathrm{Vol}\,\Delta_\tau$$

(we use 11.2: $\|x_{t_2} - x_{t_1}\| \geqslant (t_2 - t_1)/2$). So we have the following inequalities for the density function $f_v(t)$:

$$\frac{[f_v(t_2)^{1/(n-1)} + [f_v(t_1)]^{1/(n-1)}}{2} \leqslant \left(1 - \delta\left(\frac{t_2 - t_1}{2}\right)\right) \max_{t_1 < \tau < t_2} f_v(\tau)^{1(n-1)} \tag{11.7$'$}$$

and

$$\min_{i=1,2} f_v(t_i) < \left(1 - \delta\left(\frac{t_2 - t_1}{2}\right)\right)^{n-1} \max_{t_1 < \tau < t_2} f_v(\tau). \tag{11.7$''$}$$

It follows from (11.7″) that $f_v(t)$ has no local minima.

Note that for a euclidean space the point y_{t_3} in (11.5) is in the middle of the arc joining y_{t_1} and y_{t_2} and $t_3 = (t_2 + t_1)/2$. Also the (Banach–Mazur) distance between an arbitrary two-dimensional normed space and the euclidean two-dimensional space is at most $\sqrt{2}$. Therefore, there exists a numerical constant α, $0 < \alpha < 1/2$, so that $t_3 \in (t_1 + \alpha(t_2 - t_1), t_2 - \alpha(t_2 - t_1))$. By this reason, we may take the maximum in (11.6), (11.7′) and 11.7″) in the interval $(t_1 + \alpha(t_2 - t_1), t_2 - \alpha(t_2 - t_1))$. It follows that $[f_v(t)]^{1/(n-1)}$ satisfies the "weak concavity" condition. Let f_v attain the maximum at $t_0 \in [0, a]$. If $t_0 <$ a, then for every $0 < t_2 < t_0$

$$f_v(t_2) \leqslant \left(1 - \delta\left(\frac{t_2 - t_0}{2}\right)\right)^{n-1} f_v(t_0)$$

(we have a similar inequality for every $0 < t_1 < t_0$ if $t_0 > 0$).

By monotonicity of $f_v(t)$ on the intervals $[t_0, a]$ and $[0, t_0]$, we also have

$$f_v(t + \theta) \leqslant (1 - \delta(\theta/2))^{n-1} f_v(t) \tag{11.8}$$

for $a > t + \theta > t > t_0$ (if, of course, $t_0 < a$) and, similarly, if $t_0 > 0$ and $0 < t - \theta < t < t_0$

$$f_v(t - \theta) < (1 - \delta(\theta/2)^{n-1} f_v(t).$$

Now, integrate (11.8) from $t_0 + \theta$ to $a - \theta$ (assuming $t_0 < a - 2\theta$) and obtain (to simplify notations we write $v[t, \tau]$ instead of $v[x_t, x_\tau]$)

$$X(\theta) \overset{\text{def}}{=} v[t_0 + 2\theta; a] \leqslant (1 - \delta(\theta/2))^{n-1} v[t_0 + \theta; a - \theta]$$

$$\leqslant (1 - \delta(\theta/2))^{n-1}\{v[t_0 + \theta; t_0 + 2\theta] + \chi(\theta)\}$$

Therefore,

$$X(\theta) \leqslant \frac{[1 - \delta(\theta/2)]^{n-1}}{1 - [1 - \delta(\theta/2)]^{n-1}} v[t_0 + \theta; t_0 + 2\theta]$$

for $\theta > 0$. Similarly we deal with the comparison of $v[0; t_0 - 2\theta]$ and $v[t_0 - 2\theta; t_0 - \theta]$. Then the statement of the proposition follows. □

REMARK: For the canonical measure μ_σ (i.e., for a convex restriction of μ corresponding to the canonical partition) the maximum t_0 of f_{μ_σ} is strictly inside the interval $0 < t_0 < a$, i.e. $x_{t0} \in \sigma \backslash \{\pm z\}$.

COROLLARY 11.4: *Let* $I_\varepsilon(x_{t_0}) = \{x \in \sigma: \|x - x_{t_0}\| \leqslant 2\varepsilon\}$
Then

$$v(\sigma - I_\varepsilon) \leqslant \left(1 - \delta\left(\frac{\varepsilon}{4}\right)\right)^{n-1} \simeq e^{-\delta(\varepsilon/4)(n-1)}.$$

COROLLARY 11.5: *Let an arc* $[a, b] \subset \sigma$, *where* σ *is a maximal arc joined points* $\pm z$, *and* $\|a - b\| \geqslant \varepsilon > 0$. *Then there exists a number* $\lambda(\varepsilon) > 0$ *depending only on* $\delta_X(\varepsilon) > 0$ *such that for any convex measure* v *induced by* μ *either*

$$\frac{v(-z, a]}{v(a, b)} \leqslant e^{-\lambda(\varepsilon)n}$$

or

$$\frac{v(b, z)}{v(a, b)} \leqslant e^{-\lambda(\varepsilon)n}.$$

Proof. Use (11.4).

REMARK 11.6: Proposition 11.3 and Corollaries 11.4 and 11.5 remain valid when the following changes are made in these results: consider any closed interval $I = [\alpha, \beta] \subset \sigma$ instead of σ and any convex restriction probability measure ν on I instead of on σ. In this case, the maximum point $x_{t_0} \in I$. We also have to replace $-z$ by α and z by β.

THEOREM 11.7: *Let $\delta_X(\varepsilon)$ be the modulus of convexity of a normed $(n + 1)$-dimensional space X and μ be the probability measure (11.1) on $S(X)$. Let $a(\varepsilon) = \delta((\varepsilon/8) - \theta_n)$ and $\delta(\theta_n/4) = 1 - (1/2)^{1/(n-1)} \simeq \ln 2/(n-1)$. Then, for every Borel set $A \subset S(X)$, $\mu(A) \geqslant 1/2$ and every $\varepsilon > 0$*

$$\mu(A_\varepsilon) \geqslant 1 - e^{-a(\varepsilon)n}$$

where $A_\varepsilon = \{x \in S(X);\ \varrho(x, A_\varepsilon) \leqslant \varepsilon\}$ and $\varrho(x, y) = \|x - y\|$.

Proof. Note that the principal part of the Theorem is the existence of a number $a(\varepsilon) > 0$ depending only on $\delta_X(\varepsilon) > 0$ such that $\mu(A_\varepsilon) \geqslant 1 - e^{-a(\varepsilon)n}$. This follows straightforwardly from Theorem 3 and Corollary 11.5. Indeed, use Theorem 3 with A instead of A_1 and $B = (A_e)^c$ instead of A_3. By Corollary 11.5 we have that one of the numbers $\mu_\sigma(\alpha_2/\alpha_1)$ or $\mu_\sigma(\alpha_2/\alpha_3)$ is at least $e^{\lambda(\varepsilon)n}$ (i.e. exponentially large). Therefore, λ must be at most $2e^{-\lambda(\varepsilon)n}$ (because $\mu(A_2) \leqslant 1$). However, to compute $a(\varepsilon)$ we use Theorem 10.4.ii). By this theorem there exists an interval $I \subset \sigma$ and a probability convex restriction measure ν on I such that

$$\nu(A \cap I) = \mu(A) \geqslant 1/2, \quad \nu(B \cap I) = \mu(B),$$

where, as above, $B = (A_\varepsilon)^c$. Let x_{t_0} be the point on I where the maximum of the density function of ν is attained. Take θ_0 such that $\nu(I_{\theta_0}(x_{t_0})) = \nu\{x \in I\colon \|x - x_{t_0}\| \leqslant 2\theta_0\} = 1/2$. Then, by Corollary 11.4 and Remark 11.6

$$[1 - \delta(\theta_0)]^{n-1} \geqslant 1/2.$$

It means that

$$\delta(\theta_0) \leqslant 1 - (1/2)^{1/n-1} \simeq \frac{\ln 2}{n-1}.$$

Let θ_n be such that $\delta(\theta_n) = 1 - (1/2)^{1/(n-1)} \simeq \ln 2/(n-1)$. Then $\nu[I_{0_n}(x_{t_0})] \geqslant 1/2$. Therefore, if $\nu(A \cap I) \geqslant 1/2$, then there exists $x_t \in A \cap I$ and $\|x_t - x_{t_0}\| \leqslant 2\theta_n$. Now, take ε-neighborhood of $\{x_t\}$ and let $\varepsilon = 2\theta + 4\theta_n$. Then $\{x_t\}_\varepsilon \supset \{x_{t_0}\}_{2\theta}$ and $B \cap \{x_t\}_\varepsilon = \emptyset$. Therefore, by Remark 11.6 applied to Corollary 11.4 we have

$$\mu(B) = \nu(B \cap I) \leqslant \nu(I - I_0(x_{t_0}) \leqslant \left(1 - \delta\left(\frac{\theta}{4}\right)\right)^{n-1} \simeq e^{-\delta(\theta/4)(n-1)}$$

$$= e^{-(\varepsilon/8 - \theta_n)(n-1)}. \qquad \square$$

Remark 11.8: Note that Theorem 11.7 shows that any family of finite dimensional spaces $\{X_n, \dim X_n \to \infty\}$, such that $\delta_{X_n}(\varepsilon) \geqslant \delta(\varepsilon) > 0$ for $\varepsilon > 0$, is a Levy family (see definition and a number of related examples in [GM1], [AM]).

Let $f(x)$ be a continuous function on $S(X)$, $\dim X = n + 1$ where $S(X)$, $\dim X = n + 1$ where $S(X)$ is the unit sphere of X. We call L_f the *median* of $f(x)$ (or *Levy mean*) if

$$\mu\{x \in S(X): f(x) \geqslant L_f\} \geqslant 1/2 \quad \text{and} \quad \mu\{x \in S(X): f \leqslant L_f\} \geqslant 1/2.$$

Let $\omega_f(\varepsilon)$ be the modulus of continuity of the function $f(x)$.
It follows from Theorem 11.7 that

$$\mu\{x \in S(X): |f(x) - L_f| \leqslant \omega_f(\varepsilon)\} \geqslant 1 - 2e^{-a(\varepsilon)n}. \tag{11.9}$$

12. *Application to a Lipschitz embedding problem.* Let $X = (\mathbb{R}^n, \|.\|)$ be a uniformly convex space with the modulus of convexity $\delta(\varepsilon) > 0$ (for $\varepsilon > 0$). Let $S(X) = \{X \in: \|x\| = 1\}$ and, similarly, $S(l_\infty^N)$ be the unit sphere of the space l_∞^N of dimension N.

Theorem: *Fix* $1 > \varepsilon > 0$. *If* $N < \frac{1}{2} e^{a(\varepsilon)n}$ *where* $a(\varepsilon) > 0$ *was defined in Theorem 11.7, then there exists no 1-Liptschitz antipodal* (i.e., $\varphi(-x) = -\varphi(x)$) *map*

$$\varphi: S(X) \to S(l_\infty^N)$$

Proof. Assume φ exists. Let $f_i(x)$, $i = 1, \dots N$, be the i-th coordinate in l_∞^N of $\varphi(x)$, i.e., $\varphi(x) = (f_i(x))N_{i=1} \in S(l_\infty^N)$. Then,

i) $\max_i |f_i(x)| = 1$ for $x \in S(X)$,

ii) $f_i(-x) = -f_i(x)$ for any $i = 1, \dots, N$ and $x \in S(X)$

iii) $|f_i(x) - f_i(y)| \leqslant \|x - y\|$, $i = 1, \dots, N$; $x, y \in S(X)$

(because $\|\varphi(x) - \varphi(y)\| \leqslant \|x - y\|$ implies $\max_i |f_i(x) - f_i(y)| \leqslant \|x - y\|$). Define $A_i = \{x \in S(X): |f_i(x)| \leqslant \varepsilon\}$. By ii), 0 is the median of f_i for every $i \in \{1, \dots, N\}$ (see 11.8 for the definition). Also, by iii), and (11.9),

$$\mu(A_i) \geqslant 1 - 2e^{-a(\varepsilon)n}.$$

Then,

$$\mu\left(\bigcap_{i=1}^{N} A_i\right) \geqslant 1 - 2Ne^{-a(\varepsilon)n}$$

and, in the case of $N < \frac{1}{2} e^{a(\varepsilon)n}$, there exists

$$x \in \bigcap_{i}^{N} A_i.$$

Hence, $|f_i(x)| \leqslant \varepsilon$ for every $i = 1, \dots, N$ which contradicts i).

Note that in the case of a linear embedding $\varphi: X \to l_\infty^N$, $\dim X = n$, the above Theorem was proved by Pisier [P].

Appendix

Let $P: \mathbb{R}^n \to \mathbb{R}^{n-k}$ be a linear projection, and let K be a convex subset in $\mathbb{R}^n$. We study the k-dimensional volume of the intersections $K \cap P^{-1}(x)$ for $x \in \mathbb{R}^{n-k}$.

BRUNN'S THEOREM: *Let* $\varphi(x) = \mathrm{Vol}_k\, K \cap P^{-1}(x)$. *Then the function* $\varphi^{1/k}$ *is concave on the image* $P(K) \subset \mathbb{R}^{n-k}$.

We will prove a more general statement.

DEFINITION: We say that a function $f: K \to \mathbb{R}$ is *α-concave* ($\alpha > 0$) if

i) K is a convex set in $\mathbb{R}^n$,

ii) $f(x) \geqslant 0$ for $x \in K$

iii) $f^{1/\alpha}$ is concave on K, i.e.

$$f^{1/\alpha}\left(\frac{x_1 + x_2}{2}\right) \geqslant \frac{f^{1/\alpha}(x_1) + f^{1/\alpha}(x_2)}{2}$$

for any $x_1, x_2 \in K$.

LEMMA 1: *Let f be α-concave, g be β-concave and let* $\operatorname{Dom} f = \operatorname{Dom} g = K \subset \mathbb{R}^n$. *Then the product fg is $(\alpha + \beta)$-concave.*

Proof. Let $x_1, x_2 \in K$. Then

$$\frac{[f(x_1)g(x_1)]^{1/(\alpha+\beta)} + f(x_2)g(x_2)]^{1/(\alpha+\beta)}}{2} \leqslant \left[\frac{f(x_1)^{1/\alpha} + f(x_2)^{1/\alpha}}{2}\right]^{\alpha/(\alpha+\beta)}$$

$$\times \left[\frac{g(x_1)^{1/\beta} + g(x_2)^{1/\beta}}{2}\right]^{\beta/(\alpha+\beta)}$$

(by the Hölder inequality for $p = (\alpha + \beta)/\alpha$ and $q = (\alpha + \beta)/\beta$)

$$\leqslant \left[f\left(\frac{x_1 + x_2}{2}\right)\right]^{1/(\alpha+\beta)} \left[g\left(\frac{x_1 + x_2}{2}\right)\right]^{1/(\alpha+\beta)}$$

(by α- and β-concavity of the functions f and g). □

Consider a linear projection $P: \mathbb{R}^m \xrightarrow{\text{onto}} \mathbb{R}^{m-1}$. Then $\mathbb{R}^m = \mathbb{R}^{m-1} + \operatorname{Ker} P$. Let K be a convex set $K \subset \mathbb{R}^m$. We have for every $x \in K$: $x = y + t$, where $y \in PK \subset \mathbb{R}^{m-1}$ and $t \in I_y = \{x \in K: Px = y\}$ is an interval in $y + \operatorname{Ker} P$. Let f be a function, $\operatorname{Dom} f = K$. We will write $f(x) = f(y; t)$ where $x = y + t$, $y \in PK$ and $t \in I_y$. Define a *projection Pf* of a function f as the function with $\operatorname{Dom} Pf = P \operatorname{Dom} f (= PK)$ and

$$(Pf)(y) \stackrel{\text{def}}{\equiv} \int_{t \in I_y} f(y; t)\, \mathrm{d}t.$$

LEMMA 2: *If f is α-concave, then Pf is $(1 + \alpha)$-concave.*

Proof. It is sufficient, by the definition of concavity, to consider the case $m = 2$. Then PK is an interval. Let $x_1, x_2 \in PK$ and $x = (x_1 + x_2)/2$ and let $I_{x_i} = [a_i, b_i]$, $i = 1, 2$. We may assume that

$$I_x = \left[\frac{a_1 + a_2}{2}, \frac{b_1 + b_2}{2}\right].$$

Let $c_i \in I_{x_i}$ $(i = 1, 2)$ satisfy

$$\int_{a_i}^{b_i} f(t, x_i)\mathrm{d}t = 2\int_{a_i}^{c_i} f(t, x_i)\mathrm{d}t = 2\int_{a_i}^{b_i} f(t, x_i)\mathrm{d}t.$$

Let K_{up} be the convex hull of the two intervals (for $i = 1, 2$) $[(c_i; x_i), (b_i; x_i)] \subset K$ and K_{down} be the convex hull of the intervals (again for $i = 1, 2$) $[(a_i; x_i), (c_i; x_i)] \subset K$. Let $f_1 = f|_{K_{\mathrm{up}}}$ and $f_2 = f|_{K_{\mathrm{down}}}$. It is easy to check (we leave it to the reader) that if $\varphi_i = Pf_i$ $(i = 1, 2)$ are $(1 + \alpha)$-concave, then the same is true for the original projection Pf. Therefore, our problem is reduced to the $(1 + \alpha)$-concavity of the functions φ_1 and φ_2. We continue this procedure, and build the partitions of the intervals $[a_i, b_i]$ for $i = 1, 2$: $t_{0,i} = a_i < t_{1,i} < \cdots < t_{n-1,i} < b_i = t_{n,i}$, such that

$$\int_{a_i}^{b_i} f(t, x_i)\mathrm{d}t = n\int_{t_{p-1,i}}^{t_{p,i}} f(t, x_i)\mathrm{d}t$$

for every $p = 1, \ldots, n$ and $i = 1, 2$. Let $K_p \subset K$ be a trapez which is the convex hull of the two intervals $[(t_{p-1,i}; x_i), (t_{p,i}; x_i)]$ $(i = 1, 2)$. Then, by the above remark, one only needs to check that the functions $P(f|_{K_p})$ are $(1 + \alpha)$-concave for every $p = 1, \ldots, n$. By the obvious approximation argument, the problem is reduced now to the following observation:

Let $t_i \in I_{x_i}$ and $\Delta_i > 0$ $(i = 1, 2)$; for $x = \lambda x_1 + (1 - \lambda)x_2$. Set $t(x) = \lambda t_1 + (1 - \lambda)t_2$ and $\Delta(x) = \lambda\Delta_1 + (1 - \lambda)\Delta_2$. Then, by Lemma 1, the function $f(t(x), x)$. $\Delta(x)$ is $(1 + \alpha)$-concave because f is α-concave and the linear function $\Delta(x)$ is 1-concave.

Now, we prove Brunn's theorem as follows. We start from the characteristic function $\chi_K(x)$ of the set K which is α-concave for energy $\alpha > 0$. After k consequent projections we come to the function $\varphi(x)$ on $\tilde{K}$ which is, by Lemma 2, $(k + \alpha)$-concave for every $\alpha > 0$ and, therefore, k-concave.

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References

[A.M] D. Amir and V.D. Milman: A quantitative finite-dimensional Krivine Theorem, *Israel J. Math.* (1985) to appear.

[GM1] M. Gromov and V.D. Milman: A topological application of the isoperimetric inequality. *Amer. J. Math.* 105 (1983) 843–854.

[GM2] M. Gromov and V.D. Milman: Brunn Theorem and a concentration of volume phenomena for symmetric convex bodies. *Geometric Aspects of Functional Analysis*, Seminar Notes, Tel Aviv (1983–1984).

[H] H. Hadwiger: *Vorlesungen Über Inhalt: Oberfläche und Isoperimetrie*, Springer-Verlag (1957).

[P] G. Pisier: Remarques sur un Résultat non publié de B. Maurey, Séminaire d'Analyse Fonctionnelle (1980–1981) Exp. V.

[S] J. J. Schäffer: Inner diameter, perimeter, and girth of spheres. *Math. Ann.* 173 (1967) 59–82.

[R] V.A. Rohlin: On the fundamental ideas of measure theory. *Mat. Sbornic, N.S.* 25 (67), (1949), 107–150 (Russian). (Amer. Math. Soc. Translation 71, 55pp., (1952)).

[B.Z.] Y.D. Burago and V.A. Zalgaler: *Geometrical Inequalities*. Leningrad, "Nauka" (1980).