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INTERACTIONS BETWEEN COMPRESSED SENSING RANDOM MATRICES AND HIGH DIMENSIONAL GEOMETRY

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Abstract. — These notes are an expanded version of short courses given at the occasion of a school held in Université Paris-Est Marne-la-Vallée, 16–20 November 2009, by Djalil Chafaï, Olivier Guédon, Guillaume Lecué, Alain Pajor, and Shahar Mendelson. The central motivation is compressed sensing, involving interactions between empirical processes, high dimensional geometry, and random matrices.

CONTENTS

Introduction 7

1. Empirical methods and high dimensional geometry 11
1.1. Presentation of the Orlicz spaces 11
1.2. Linear combination of centered Psi-alpha random variables 16
1.3. A geometric application: the Johnson-Lindenstrauss lemma 24
1.4. Complexity and covering numbers 25
1.5. Notes and comments 33

2. Compressed sensing and Gelfand widths 35
2.1. Short introduction to compressed sensing 35
2.2. The exact reconstruction problem 36
2.3. The restricted isometry property 42
2.4. The geometry of the null space 43
2.5. Gaussian random matrices satisfy a RIP 47
2.6. RIP for other “simple” subsets: almost sparse vectors 51
2.7. An other complexity measure 57

3. Chaining 59
3.1. The chaining method 59
3.2. An example of a more sophisticated chaining argument 64
3.3. Bibliographical references 70

4. Singular values of random matrices 73
4.1. Singular values of deterministic matrices 73
4.2. Singular values of Gaussian random matrices 81
4.3. Universality of the Gaussian case 90
4.4. Comments 95

5. Empirical methods and selection of characters 97
5.1. Reconstruction of signals with small support by random methods 97

5.2. Empirical processes 104
5.3. Selection of characters 114
5.4. Comments 124

Notations 127

Bibliography 129

INTRODUCTION

Compressed sensing also referred to in the literature as compressive sensing or compressive sampling is framework that enables to get approximate and exact reconstruction of sparse signals from incomplete measurements. The existence of efficient algorithms for the reconstruction, such as the ℓ_1-minimization, and the potential of applications in signal processing and imaging, has led to a fast and wide development of the theory. The ideas and principles underlying the discoveries of these phenomena in high dimensions are related to problems and progresses from Approximation Theory. One significant example of such an interaction is the study of Gelfand and Kolmogorov widths of classical Banach spaces. There is already a large literature on compressed sensing, on both theoretical and numerical aspects. Our aim is not to survey the state of the art of this recent field developing with great speed, but to highlight and to study some interactions with other fields of mathematics, in particular with asymptotic geometric analysis, random matrices and empirical processes.

To introduce the subject, let $T \subset \mathbb{R}^N$ and let A be an $n \times N$ real matrix with rows $Y_1, \dots, Y_n \in \mathbb{R}^N$. Consider the general problem of reconstructing any vector $x \in T$ from the *data* $Ax \in \mathbb{R}^n$, that is from the known *measurements*

$$\langle Y_1, x\rangle, \dots, \langle Y_n, x\rangle.$$

Classical linear algebra suggests that the number n of measurements should be at least as large as its dimension N in order to ensure reconstruction. Compressed sensing provides a way of reconstructing the original signal x from its compression Ax that takes only a small amount of linear measurements, that is with $n \ll N$. Clearly one needs some a priori hypothesis on the subset T of signals that we want to reconstruct from few measurements and of course, the matrix A should be suitably chosen.

The first point concerns T and is a question of *complexity*. Many tools relevant to this matter were developed in Approximation Theory and in Geometry of Banach Spaces. This is one of our objective to introduce these tools.

The second point is concerned with the design of the measurement matrix A. At present the only good matrices so far are random *sampling* matrices. They are obtained in many examples by sampling $Y_1, \dots, Y_n \in \mathbb{R}^N$ in a suitable way. This is where probability enters. These random sampling matrices will be Gaussian or Bernoulli (± 1) type or random sub-matrices of the discrete Fourier $N \times N$ matrix (partial Fourier matrices). There is a huge technical difference in the study of unstructured compressive matrices (with i.i.d entries) and other case such as partial Fourier matrices. This is one of our objective to study the main tools from probability theory that fall within this framework. These are tools from probabilistic inequalities, concentration of measure and empirical processes as well as from random matrix theory.

This is precisely the purpose of Chapter 1 to present the basic tools that will be used within this book. Elementary properties of Orlicz spaces are introduced in relation with tail inequalities of random variables. An important connection between high dimensional geometry and the study of empirical processes comes from the behavior of the sum of independent centered random variables with sub-exponential tails. Discretization is an important step in the study of empirical processes. One approach is given by a net argument. The size of the discrete space may be estimated by the covering numbers. The basic tools to estimate covering numbers from above are presented in the last part of Chapter 1.

Chapter 2 is devoted to compressed sensing. The purpose is to provide some of the key mathematical insights underlying this new sampling method. We present first the exact reconstruction problem as introduced above. The a priori hypothesis on the subset of signals T that we investigate is *sparsity.* A vector is said to be m-sparse if it has at most m non-zero coordinates. An important feature of this subset is its peculiar structure: its intersection with the Euclidean unit sphere is the unions of unit spheres supported on m-dimensional coordinate subspaces. This set is highly compact when the degree of compacity is measured in terms of covering numbers. It makes it a *small* subset of the sphere as far as $m \ll N$, which will be the case. In other words, the set T may be discretized to be reduced to a finite set of reasonably small cardinality.

A fundamental feature of compressive sensing is that practical reconstruction can be performed by using efficient algorithms such as the ℓ_1-minimization method which consists, for a given data $y = Ax$, to perform the "linear programming":

$$\min_{t \in \mathbb{R}^N} \sum_{i=1}^{N} |t_i| \quad \text{subject to} \quad At = y.$$

At this step, the problem comes to find matrices for which this algorithm reconstructs any m-sparse vectors with m relatively large. A study of the cone of constraints to ensure that every m-sparse vector can be reconstructed by the ℓ_1-minimization method leads to a necessary and sufficient condition known as the *null space property* of order m:

$$\forall h \in \ker A,\ h \neq 0,\ \forall I \subset [N],\ |I| \leq m,\ \sum_{i \in I} |h_i| < \sum_{i \in I^c} |h_i|.$$

This property has a nice geometric interpretation on the structure of faces of random polytopes called *neighborliness.* Indeed, if P is the polytope obtained by taking the centrally symmetric convex hull of the columns of A, the *null space property* of order m for A is equivalent to a *neighborliness* property of order m for P. This means that the matrix A which maps the vertices of the cross-polytope

$$B_1^N = \Big\{ t \in \mathbb{R}^N \,:\, \sum_{i=1}^{N} |t_i| \le 1 \Big\}.$$

onto the vertices of P preserves the structure of faces up to the dimension m. A remarkable connection between compressed sensing and high dimensional geometry.

Unfortunately, the null space property is not easy to verify. An ingenious sufficient condition is the so-called Restricted Isometry Property (RIP) of order m that requires that all column sub-matrices of size m of the matrix are well-conditioned. More precisely, we say that A satisfies the RIP of order p with parameter δ if

$$1 - \delta < |Ax|_2^2 < 1 + \delta$$

holds for all p-sparse unit vectors $x \in \mathbb{R}^N$. An important feature of this concept is that if A satisfies the RIP of order $2m$ with parameter δ small enough then every m-sparse vector can be reconstructed by the ℓ_1-minimization method. Even if this RIP condition is difficult to check on a given matrix, it actually holds true with high probability for certain models of random matrices.

Here is the point where probabilistic methods come into play. Among good unstructured sampling matrices we shall study the case of Gaussian and Bernoulli random matrices. The case of partial Fourier matrices, which is more delicate will be studied in Chapter 5. Checking RIP for the first two models may be treated along a simple scheme presented in Chapter 2. More precisely, the method is based on a reduction to a finite number of sparse vectors called a net. For each individual vector x in this net, the Euclidean norm of Ax is concentrated around its mean. This concentration should be strong enough to balance the cardinality of the net in the union bound. The passage from this uniform control over the net to the whole set of sparse vectors leads to the precise choice of the parameters in RIP.

Chapter 3 provides a criterion implying RIP for unstructured models of random matrices, which includes the Bernoulli and Gaussian models. This new degree of generality is allowed by the development of an adequate notion of complexity. The approach, known as the generic chaining, allows to bound the supremum of empirical processes, and is actually quite general.

On the other hand, the RIP can be translated as a control on the largest and smallest singular values of all sub-matrices of a certain size. Chapter 4 aims to provide an accessible introduction to the notion of singular values of matrices, and their behavior when the entries are random, including quite recent striking results from Random Matrix Theory and High Dimensional Geometric Analysis.

Another angle to tackle the problem of reconstruction by the ℓ_1-minimization is to study of the Euclidean diameter of the section of the cross-polytope B_1^N with the kernel of A. This study leads to the notion of Gelfand widths. In this direction, important works were done in the seventies. This viewpoint which comes from Approximation Theory and Asymptotic Geometric Analysis enlighten a new aspect of the problem: if the Euclidean diameter of the section of the cross-polytope B_1^N with the kernel of A is $< D$, then m-sparse vectors can be reconstructed by ℓ_1-minimization with $m = \lfloor 1/D^2 \rfloor$. Then clearly the objective is to estimate this diameter from above.

This approach is pursued in Chapter 5 on the model of partial discrete Fourier matrices. The reconstruction problem is connected to the problem of selecting a large part of a bounded orthonormal system such that on the vectorial span of this family, the L_2 and the L_1 norms are as close as possible. This subject of Harmonic Analysis, goes back to the construction of $\Lambda(p)$ sets which are not $\Lambda(q)$ for $q > p$ where powerful methods of selectors were developed. Again, tools of empirical processes are at the heart of the technics of proof.

CHAPTER 1

EMPIRICAL METHODS AND HIGH DIMENSIONAL GEOMETRY

This chapter is devoted to the presentation of classical tools that will be used within this book. We present some elementary properties of Orlicz spaces and develop the particular case of ψ_α random variables. Several characterization are given in terms of tail estimate, Laplace transform and behavior of its L_p norms. One of the important connection between high dimensional geometry and the study of empirical processes comes from the behavior of the sum of (centered) ψ_α random variables. An important part of these preliminaries concentrate on this subject. We illustrate these connections with the presentation of the Johnson-Lindenstrauss lemma. The last part of this chapter is devoted to the study of covering numbers. We focus our attention on some elementary properties and on the presentation of methods to estimate upper bound of these covering numbers.

1.1. Presentation of the Orlicz spaces

The Orlicz space is a function space which extends naturally the classical L_p spaces when $1 \leq p \leq +\infty$. A function $\psi : [0,\infty) \to [0,\infty)$ is an Orlicz function if it is a convex increasing function such that $\psi(0) = 0$ and $\psi(x) \to \infty$ when $x \to \infty$.

Definition 1.1.1. — *Let ψ be an Orlicz function, for any real random variable X on a measurable space (Ω, σ, μ), we define its L_ψ norm by*

$$\|X\|_\psi = \inf\big\{c > 0 : \mathbb{E}\psi\big(|X|/c\big) \leqslant \psi(1)\big\}.$$

The space $L_\psi(\Omega, \sigma, \mu) = \{X : \|X\|_\psi < \infty\}$ is called an Orlicz space.

It is well known that L_ψ is a Banach space. Classical examples of Orlicz functions are for $p \geqslant 1$ and $\alpha \geqslant 1$

$$\phi_p(x) = x^p/p \quad \text{and} \quad \psi_\alpha(x) = \exp(x^\alpha) - 1.$$

The Orlicz space associated to ϕ_p is the classical L_p space. It is also clear by the theorem of monotone convergence that the infimum in the definition of the L_ψ norm of a random variable X, if finite, is attained at $\|X\|_\psi$.

Let ψ be a convex function. We define its *convex conjugate* ψ^* (also called the *Legendre transform*) by: for every $y > 0$

$$\psi^*(y) := \sup_{x>0}\{xy - \psi(x)\}.$$

The convex conjugate of an Orlicz function is also an Orlicz function.

Proposition 1.1.2. — *Let ψ be an Orlicz function and ψ^* be its convex conjugate. For every real random variables $X \in L_\psi$ and $Y \in L_{\psi^*}$,*

$$\mathbb{E}|XY| \leqslant (\psi(1) + \psi^*(1)) \, \|X\|_\psi \, \|Y\|_{\psi^*} .$$

Proof. — By homogeneity, we can assume $\|X\|_\psi = \|Y\|_{\psi^*} = 1$. By definition of the convex conjugate, we have

$$|XY| \leqslant \psi\left(|X|\right) + \psi^*\left(|Y|\right).$$

Taking the expectation, since $\mathbb{E}\psi(|X|) \leq \psi(1)$ and $\mathbb{E}\psi^*(|Y|) \leqslant \psi^*(1)$, we get that $\mathbb{E}|XY| \leq \psi(1) + \psi^*(1)$. □

It is not difficult to observe that if $\phi_p(t) = t^p/p$ then $\phi_p^* = \phi_q$ where $p^{-1} + q^{-1} = 1$ (it is also known as Young's inequality). In this case, Proposition 1.1.2 corresponds to Hölder inequality.

Any information about the ψ_α norm of a random variable is very useful to describe a tail behavior. This will be explained in Theorem 1.1.5. For instance, we say that X is a *sub-Gaussian* random variable when $\|X\|_{\psi_2} < \infty$, we say that X is a *sub-exponential* random variable when $\|X\|_{\psi_1} < \infty$. In general, we say that X is ψ_α when $\|X\|_{\psi_\alpha} < \infty$. It is important to notice (see Corollary 1.1.6 and Proposition 1.1.7) that for any $1 \leq p < +\infty$, for any $\alpha_2 \geqslant \alpha_1 \geq 1$

$$L_\infty \subset L_{\psi_{\alpha_2}} \subset L_{\psi_{\alpha_1}} \subset L_p.$$

One of the main goal of these preliminaries will be to understand the behavior of the maximum of L_ψ-random variables and of the sum and the product of ψ_α random variables. We start with a general maximal inequality.

Proposition 1.1.3. — *Let ψ be an Orlicz function. Then, for any positive natural integer n and any real valued random variables $X_1, \ldots, X_n$,*

$$\mathbb{E} \max_{1\leq i\leq n} |X_i| \leq \psi^{-1}(n\psi(1)) \max_{1\leq i\leq n} \|X_i\|_\psi,$$

where ψ^{-1} is the inverse function of ψ. Moreover if ψ is such that

$$\exists\, c > 0,\ \forall x,\, y \geq 1/2,\ \psi(x)\psi(y) \leqslant \psi(c\, x\, y) \tag{1.1}$$

then

$$\left\| \max_{1\leqslant i\leqslant n} |X_i| \right\|_\psi \leqslant c\ \max\left\{1/2, \psi^{-1}(2n)\right\} \max_{1\leqslant i\leqslant n} \|X_i\|_\psi .$$

where c is the same as in (1.1).

Remark 1.1.4. — • *Since for any* $x, y \geq 1/2$, $(e^x - 1)(e^y - 1) \leq e^{x+y} \leq e^{4xy} \leq (e^{8xy} - 1)$, *we get that for any* $\alpha \geq 1$, ψ_α *satisfies the assumption* (1.1) *with* $c = 8^{1/\alpha}$.
• *Moreover, the function* ψ_α *is such that* $\psi_\alpha^{-1}(n\psi_\alpha(1)) \leq (1 + \log(n))^{1/\alpha}$ *and* $\psi_\alpha^{-1}(2n) = (\log(1 + 2n))^{1/\alpha}$.
• *The assumption* (1.1) *may be weakened by* $\limsup_{x,y\to\infty} \psi(x)\psi(y)/\psi(cxy) < \infty$.
• *By monotony of* ψ, *for* $n \geq \psi(1/2)/2$, $\max\left\{1/2, \psi^{-1}(2n)\right\} = \psi^{-1}(2n)$.

Proof. — By homogeneity of the statements, we can assume that for any $i = 1, \ldots, n$, $\|X_i\|_\psi \leq 1$.

The first inequality is a simple consequence of Jensen inequality. Indeed,

$$\psi(\mathbb{E} \max_{1\leq i\leq n} |X_i|) \leq \mathbb{E}\psi(\max_{1\leq i\leq n} |X_i|) \leq \sum_{i=1}^{n} \mathbb{E}\psi(|X_i|) \leq n\psi(1).$$

To prove the second assertion, we define $y = \max\{1/2, \psi^{-1}(2n)\}$. For any $i = 1, \ldots, n$, let $x_i = |X_i|/cy$. We observe that if $x_i \geq 1/2$ then we get by (1.1)

$$\psi\left(|X_i|/cy\right) \leq \frac{\psi(|X_i|)}{\psi(y)}.$$

Moreover,

$$\max_{1\leq i\leq n} x_i \leq \max_{1\leq i\leq n} x_i \mathbb{1}_{\left\{\max\limits_{1\leq i\leq n} x_i \leq 1/2\right\}} + \sum_{i=1}^{n} x_i \mathbb{1}_{\{x_i \geq 1/2\}}$$

therefore, we have by monotony of ψ,

$$\begin{aligned}\mathbb{E}\psi\left(\max_{1\leqslant i\leqslant n} |X_i|/cy\right) &\leqslant \psi(1/2) + \sum_{i=1}^{n} \mathbb{E}\psi\left(|X_i|)/cy\right) \mathbb{1}_{\{(|X_i|)/cy) \geq 1/2\}} \\ &\leq \psi(1/2) + \frac{1}{\psi(y)} \sum_{i=1}^{n} \mathbb{E}\psi(|X_i|) \leq \psi(1/2) + \frac{n\psi(1)}{\psi(y)}.\end{aligned}$$

By convexity of ψ and the fact that $\psi(0) = 0$, we have $\psi(1/2) \leq \psi(1)/2$. The proof is finished since by definition of y, $\psi(y) \geq 2n$. □

For every $\alpha \geq 1$, there are very precise connections between the ψ_α norm of a random variable, the behavior of its L_p norms, the tail estimates and the Laplace transform. We sum up these connections in the following Theorem.

Theorem 1.1.5. — *Let* X *be a real valued random variable and* $\alpha \geqslant 1$. *The following assertions are equivalent:*
(1) There exists $K_1 > 0$ *such that* $\|X\|_{\psi_\alpha} \leqslant K_1$.
(2) There exists $K_2 > 0$ *such that for every* $p \geq \alpha$,

$$\left(\mathbb{E}|X|^p\right)^{1/p} \leqslant K_2\, p^{1/\alpha}.$$

(3) There exist $K_3, K_3' > 0$ *such that for every* $t \geq K_3'$,

$$\mathbb{P}\left(|X| \geqslant t\right) \leqslant \exp\left(-t^\alpha/K_3^\alpha\right).$$

Moreover, we have

$$K_2 \le 2eK_1, K_3 \le eK_2, K_3' \le e^2 K_2 \text{ and } K_1 \le 2\max(K_3, K_3').$$

In the case $\alpha > 1$, let β be such that $1/\alpha + 1/\beta = 1$. The preceding assertions are also equivalent to the following:
(4) There exist $K_4, K_4' > 0$ such that for every $\lambda \ge 1/K_4'$,

$$\mathbb{E}\exp\big(\lambda|X|\big) \leqslant \exp\left(\lambda K_4\right)^{\beta}.$$

Moreover, $K_4 \le 2K_1/$, $K_4' \le K_1$, $K_3 \le 2K_4$ and $K_3' \le 2K_4^{\beta}/(K_4')^{\alpha-1}$.

Proof. — We start by proving that (1) implies (2). By definition of the L_{ψ_α} norm, we get that

$$\mathbb{E}\exp\left(\frac{|X|}{K_1}\right)^{\alpha} \le e.$$

Moreover, for every positive natural integer q and every $x \ge 0$, $\exp x \ge x^q/q!$ hence

$$\mathbb{E}|X|^{\alpha q} \le e\, q! K_1^{\alpha q} \le e q^q K_1^{\alpha q}.$$

For any $p \ge \alpha$, let q be the positive integer such that $q\alpha \le p < (q+1)\alpha$ then

$$\begin{aligned}(\mathbb{E}|X|^p)^{1/p} \le \left(\mathbb{E}|X|^{(q+1)\alpha}\right)^{1/(q+1)\alpha} &\le e^{1/(q+1)\alpha} K_1 (q+1)^{1/\alpha} \\ &\le e^{1/p} K_1 \left(\frac{2p}{\alpha}\right)^{1/\alpha} \le 2eK_1 p^{1/\alpha}\end{aligned}$$

which means that (2) holds true with $K_2 = 2eK_1$.

We now prove that (2) implies (3). We apply Markov inequality and the estimate of (2) to deduce that for every $t > 0$,

$$\mathbb{P}(|X| \ge t) \le \inf_{p>0} \frac{\mathbb{E}|X|^p}{t^p} \le \inf_{p\ge\alpha} \left(\frac{K_2}{t}\right)^p p^{p/\alpha} = \inf_{p\ge\alpha} \exp\left(p \log\left(\frac{K_2 p^{1/\alpha}}{t}\right)\right).$$

We get that for $t \ge e\, K_2 \alpha^{1/\alpha}$, we can choose $p = (t/eK_2)^\alpha \ge \alpha$ and conclude that

$$\mathbb{P}(|X| \ge t) \le \exp\left(-t^\alpha/(K_2 e)^\alpha\right).$$

Since $\alpha \ge 1$, $\alpha^{1/\alpha} \le e$ and we conclude that (3) holds true with $K_3' = e^2 K_2$ and $K_3 = eK_2$.

We conclude by proving that (3) implies (1). Assume that (3) holds true and let $c = 2\max(K_3, K_3')$. Then by integration by parts,

$$\begin{aligned}\mathbb{E}\exp\left(\frac{|X|}{c}\right)^{\alpha} - 1 &= \int_0^{+\infty} \alpha u^{\alpha-1} e^{u^\alpha} \mathbb{P}(|X| \ge uc)du \\ &\le \int_0^{K_3'/c} \alpha u^{\alpha-1} e^{u^\alpha} du + \int_{K_3'/c}^{+\infty} \alpha u^{\alpha-1} \exp\left(u^\alpha\left(1 - \frac{c^\alpha}{K_3^\alpha}\right)\right) du \\ &= \exp\left(\frac{K_3'}{c}\right)^{\alpha} - 1 + \frac{1}{\frac{c^\alpha}{K_3^\alpha} - 1} \exp\left(-\left(\frac{c^\alpha}{K_3^\alpha} - 1\right)\left(\frac{K_3'}{c}\right)^{\alpha}\right) \\ &\le 2\cosh(K_3'/c)^\alpha - 1 \le 2\cosh(1/2) - 1 \le e - 1\end{aligned}$$

by the definition of c and the fact that $\alpha \geq 1$. This proves that (1) holds true with $K_1 = 2\max(K_3, K_3')$.

We now assume that $\alpha > 1$ and prove that (4) implies (3). We apply Markov inequality and the estimate of (4) to get that for every $t > 0$,

$$\begin{aligned}\mathbb{P}(|X| > t) &\leqslant \inf_{\lambda>0} \exp(-\lambda t)\mathbb{E}\exp\big(\lambda|X|\big)\\ &\leqslant \inf_{\lambda\geq 1/K_4'} \exp\big((\lambda K_4)^\beta - \lambda t\big).\end{aligned}$$

Therefore if $t \geq 2K_4^\beta/(K_4')^{\alpha-1}$, we can choose $\lambda t = 2(\lambda K_4)^\beta$ with $\lambda \geq 1/K_4'$ and conclude that

$$\mathbb{P}(|X| > t) \leq \exp\left(-t^\alpha/(2K_4)^\alpha\right).$$

This proves that (3) holds true with $K_3 = 2K_4$ and $K_3' = 2K_4^\beta/(K_4')^{\alpha-1}$.

It remains to prove that (1) implies (4). We have already observed that the convex conjugate of the function $\phi_\alpha(t) = t^\alpha/\alpha$ is ϕ_β which implies that for any $x, y > 0$,

$$xy \leq \frac{x^\alpha}{\alpha} + \frac{y^\beta}{\beta} \leq x^\alpha + y^\beta.$$

Hence for every $\lambda > 0$,

$$\exp(\lambda|X|) \leq \exp\left(\frac{|X|}{K_1}\right)^\alpha \exp\left(\lambda K_1\right)^\beta$$

and taking the expectation, we get by definition of the L_{ψ_α} norm that

$$\mathbb{E}\exp(\lambda|X|) \leq e\exp\left(\lambda K_1\right)^\beta.$$

We conclude that if $\lambda \geq 1/K_1$ then

$$\mathbb{E}\exp(\lambda|X|) \leq \exp\left(2\lambda K_1\right)^\beta$$

which proves that (4) holds true with $K_4 = 2K_1$ and $K_4' = K_1$. □

A simple corollary of Theorem 1.1.5 is the following connection between the L_p norms of a random variable and its ψ_α norm.

Corollary 1.1.6. — *For every $\alpha \geq 1$, for every real random variable X,*

$$\frac{1}{2e^2}\left\|X\right\|_{\psi_\alpha} \leqslant \sup_{p\geqslant\alpha} \frac{(\mathbb{E}|X|^p)^{1/p}}{p^{1/\alpha}} \leqslant 2e\left\|X\right\|_{\psi_\alpha}.$$

Proof. — This follows from the implications (1) ⇒ (2) ⇒ (3) ⇒ (1) in Theorem 1.1.5 and from the computations of the constants K_2, K_3, K_3' and K_1. □

We conclude this part with a kind of Hölder inequality for ψ_α random variables.

Proposition 1.1.7. — *Given p, $q \in [1, +\infty]$ be such that $1/p + 1/q = 1$ and two random variables $X \in L_{\psi_p}$, $Y \in L_{\psi_q}$, we have*

$$\left\|XY\right\|_{\psi_1} \leq \left\|X\right\|_{\psi_p}\left\|Y\right\|_{\psi_q}. \tag{1.2}$$

Moreover, if $1 \leq \alpha \leq \beta$ then for every random variable X

$$\left\|X\right\|_{\psi_1} \leq \left\|X\right\|_{\psi_\alpha} \leq \left\|X\right\|_{\psi_\beta}.$$

Proof. — By homogeneity, we assume that $\|X\|_{\psi_p} = \|Y\|_{\psi_q} = 1$. Since p and q are conjugate, we know by Young inequality that for every $x, y \in \mathbb{R}$, $|xy| \leq \frac{|x|^p}{p} + \frac{|x|^q}{q}$. By convexity of the exponential, we deduce that

$$\mathbb{E}\exp(|XY|) \leq \frac{1}{p}\mathbb{E}\exp|X|^p + \frac{1}{q}\mathbb{E}\exp|X|^q \leq e$$

which proves that $\|XY\|_{\psi_1} \leq 1$.

The moreover part is a consequence of this result. Indeed, by definition of the ψ_q-norm, the random variable $Y = 1$ satisifes $\|Y\|_{\psi_q} = 1$. Hence applying (1.2) with $p = \alpha$ and q being the conjugate of p, we get that for every $\alpha \geq 1$, $\|X\|_{\psi_1} \leq \|X\|_{\psi_\alpha}$. We also observe that for any $\beta \geq \alpha$, if $\delta \geq 1$ is such that $\beta = \alpha\delta$ then we have

$$\|X\|_{\psi_\alpha}^\alpha = \||X|^\alpha\|_{\psi_1} \leq \||X|^\alpha\|_{\psi_\delta} = \|X\|_{\psi_{\alpha\delta}}^\alpha$$

which proves that $\|X\|_{\psi_\alpha} \leq \|X\|_{\psi_\beta}$. □

1.2. Linear combination of centered Psi-alpha random variables

In this part we will focus on the case of centered ψ_α random variables when $\alpha \geq 1$. We will present several results concerning the linear combination of such random variable. The cases $\alpha = 2$ and $\alpha \neq 2$ are different. We will start by looking at the case $\alpha = 2$. Even if we may prove a sharp estimate for their linear combination, we will also consider the simple and well known example of linear combination of Rademacher. This example will show the limitation of the study of the ψ_α norm of certain random variable. However in the case $\alpha \neq 2$, different regime will appear in the tail estimates of such sum. This will be of importance in several chapters of this book.

1.2.1. The sub-Gaussian case.— We start by taking a look to sums of ψ_2 random variables. The following proposition can be seen as a generalization of the classical Hoeffding inequality [**Hoe63**] since $L_\infty \subset L_{\psi_2}$.

Theorem 1.2.1. — *Let $X_1, \ldots, X_n$ be independent real valued random variable such that for any $i = 1, \ldots, n$, $\mathbb{E}X_i = 0$. Then*

$$\left\|\sum_{i=1}^n X_i\right\|_{\psi_2} \leqslant c\left(\sum_{i=1}^n \|X_i\|_{\psi_2}^2\right)^{1/2}$$

where $c \leq 28$.

Before proving the theorem, we start with the following lemma concerning the Laplace transform of a ψ_2 random variable which is centered. The fact that $\mathbb{E}X = 0$ is crucial to improve the assertion (4) of Theorem 1.1.5.

Lemma 1.2.2. — *Let X be a ψ_2 centered random variable. Then, for any $\lambda > 0$, the Laplace transform of X satisfies*

$$\mathbb{E}\exp(\lambda X) \leqslant \exp\left(6\lambda^2\|X\|_{\psi_2}^2\right).$$

Proof. — By homogeneity of the statement, we can assume that $\|X\|_{\psi_2} = 1$. By the definition of the L_{ψ_2} norm of X, we know that

$$\mathbb{E}\exp{(X)^2} \leq e$$

hence, by Markov inequality, we get that for every $t > 0$,

$$\mathbb{P}\big(|X| \geqslant t\big) \leqslant e\exp(-t^2).$$

Thus, for any integer $k \geqslant 2$,

$$\begin{aligned}\mathbb{E}|X|^k &= \int_0^{+\infty} k\,t^{k-1}\mathbb{P}\left(|X| \geqslant t\right)dt \\ &\leq e\int_0^{+\infty} k\,t^{k-1}\exp(-t^2)dt \\ &= e\Gamma\big(k/2+1\big),\end{aligned}$$

where Γ is defined for any $u \geqslant 1$ by $\Gamma(u) = \int_0^\infty t^{u-1}\exp(-t)dt$. Since $\mathbb{E}X = 0$ and for any positive integer k, $\Gamma(k+1) = k\Gamma(k) = k!$ we deduce that for every $\lambda > 0$,

$$\mathbb{E}\exp(\lambda X) = 1 + \sum_{k\geqslant 2}\frac{\lambda^k\mathbb{E}X^k}{k!} \leqslant 1 + e\sum_{k\geqslant 2}\frac{\Gamma(k/2+1)}{\Gamma(k+1)}\lambda^k.$$

By Cauchy-Schwartz, $\Gamma(k/2+1) \leqslant \Gamma(k+1)^{1/2}$ and we obtain

$$\mathbb{E}\exp(\lambda X) \leqslant 1 + e\sum_{k\geqslant 2}\frac{\lambda^k}{\Gamma(k/2+1)} = 1 + e\sum_{k\geqslant 1}\frac{\lambda^{2k}}{\Gamma(k+1)} + \frac{\lambda^{2k+1}}{\Gamma(k+3/2)}.$$

Since Γ is non-decreasing, we get

$$\mathbb{E}\exp(\lambda X) \leqslant 1 + e\sum_{k\geqslant 1}\frac{\lambda^{2k}\,(1+\lambda)}{\Gamma(k+1)} = 1 + e(1+\lambda)(e^{\lambda^2}-1).$$

Using the Taylor expansion, it is easy to see that $1+\lambda \leq 1+e^{\lambda^2}$ and that $e(e^{2\lambda^2}-1) \leq 3(e^{2\lambda^2}-1) \leq e^{6\lambda^2}-1$ and the lemma is proven. □

Remark 1.2.3. — *We could have used assertion* (2) *of Theorem 1.1.5 to get directly an estimate of the k-th moment of the random variable X. This would have led to a slightly worse constant than* 6 *in the estimate.*

Proof of Theorem 1.2.1. — It is enough to get an upper bound of the Laplace transform of the random variable $\big|\sum_{i=1}^n X_i\big|$. Let $Z = \sum_{i=1}^n X_i$ then by independence of the X_i's, we get from Lemma 1.2.2 that for every $\lambda > 0$,

$$\mathbb{E}\exp(\lambda Z) = \prod_{i=1}^n \mathbb{E}\exp(\lambda X_i) \leqslant \exp\big(6\lambda^2\sum_{i=1}^n \|X_i\|_{\psi_2}^2\big).$$

For the same reason, $\mathbb{E}\exp(-\lambda Z) \leqslant \exp\big(6\lambda^2\sum_{i=1}^n \|X_i\|_{\psi_2}^2\big)$. Thus,

$$\mathbb{E}\exp(\lambda|Z|) \leqslant 2\exp\big(6\lambda^2\sum_{i=1}^n \|X_i\|_{\psi_2}^2\big).$$

We conclude that for any $t \geq 1/\left(\sum_{i=1}^n \|X_i\|_{\psi_2}^2\right)^{1/2}$,

$$\mathbb{E}\exp(\lambda|Z|) \leqslant \exp\big(7\lambda^2 \sum_{i=1}^n \|X_i\|_{\psi_2}^2\big)$$

and using the implication ((4) $\Rightarrow$ (1)) in Theorem 1.1.3 with $\alpha = \beta = 2$ (and following the definition of the constants), we get that $\|Z\|_{\psi_2} \leq c\sum_{i=1}^n \|X_i\|_{\psi_2}^2$ with $c \leq 28$. □

Now, we take a particular look to Rademacher processes. Indeed, Rademacher variables are the simplest example of bounded (hence ψ_2) random variables. We denote by $\varepsilon_1, \dots, \varepsilon_n$ independent random variables taking values ± 1 with probability $1/2$. Since $L_\infty \subset L_{\psi_2}$, for any $(a_1, \dots, a_n) \in \mathbb{R}^n$, the random variable $a_i\varepsilon_i$ is centered and has ψ_2 norm equal to $|a_i|$. We apply Theorem 1.2.1 to deduce that

$$\left\|\sum_{i=1}^n a_i\varepsilon_i\right\|_{\psi_2} \leq c|a|_2 = c\left(\mathbb{E}\left|\sum_{i=1}^n a_i\varepsilon_i\right|^2\right)^{1/2}.$$

Therefore we get from Theorem 1.1.3 that for any $p \geq 2$,

$$\left(\mathbb{E}\left|\sum_{i=1}^n a_i\varepsilon_i\right|^2\right)^{1/2} \leq \left(\mathbb{E}\left|\sum_{i=1}^n a_i\varepsilon_i\right|^p\right)^{1/p} \leq 2c\sqrt{p}\left(\mathbb{E}\left|\sum_{i=1}^n a_i\varepsilon_i\right|^2\right)^{1/2}. \qquad (1.3)$$

This is the Khinchine's inequality. It is not difficult to extend it to the case $0 < q \leq 2$ by using Hölder inequality: for any random variable Z, if $0 < q \leq 2$ and $\lambda = q/(4-q)$ then

$$\left(\mathbb{E}|Z|^2\right)^{1/2} \leq \left(\mathbb{E}|Z|^q\right)^{\lambda/q}\left(\mathbb{E}|Z|^4\right)^{(1-\lambda)/4}.$$

Let $Z = \sum_{i=1}^n a_i\varepsilon_i$, we apply (1.3) to the case $p = 4$ to deduce that for any $0 < q \leq 2$,

$$\left(\mathbb{E}\left|\sum_{i=1}^n a_i\varepsilon_i\right|^q\right)^{1/q} \leq \left(\mathbb{E}\left|\sum_{i=1}^n a_i\varepsilon_i\right|^2\right)^{1/2} \leq (4c)^{2(2-q)/q}\left(\mathbb{E}\left|\sum_{i=1}^n a_i\varepsilon_i\right|^q\right)^{1/q}.$$

Since for any $x \geq 0$, $e^{x^2} - 1 \geq x^2$, we also observe that

$$(e-1)\left\|\sum_{i=1}^n a_i\varepsilon_i\right\|_{\psi_2} \geq \left(\mathbb{E}\left|\sum_{i=1}^n a_i\varepsilon_i\right|^2\right)^{1/2}.$$

However the precise knowledge of the ψ_2 norm of the random variable $\sum_{i=1}^n a_i\varepsilon_i$ is not enough to understand correctly the behavior of its L_p norm and consequently of its tail estimate. Indeed, a more precise statement holds true.

Theorem 1.2.4. — *Let $p \geqslant 2$, let $a_1, \dots, a_n$ be real numbers and let $\varepsilon_1, \dots, \varepsilon_n$ be independent Rademacher variables. We have*

$$\left(\mathbb{E}\left|\sum_{i=1}^n a_i\varepsilon_i\right|^p\right)^{1/p} \leq \sum_{i\leqslant p} a_i^* + 2c\sqrt{p}\left(\sum_{i>p} a_i^{*2}\right)^{1/2},$$

where $(a_1^*, \dots, a_n^*)$ is the non-increasing rearrangement of $(|a_1|, \dots, |a_n|)$. Moreover, the estimate is sharp, up to a multiplicative factor.

Remark 1.2.5. — *Even if it is the difficult part of the Theorem, we will not present the proof of the lower bound. It is beyond the scope of this chapter.*

Proof. — Since Rademacher random variables are bounded by 1, we also have the trivial upper bound:

$$\left(\mathbb{E}\left|\sum_{i=1}^n a_i\varepsilon_i\right|^p\right)^{1/p} \le \sum_{i=1}^n |a_i|. \tag{1.4}$$

By independence and by symmetry of the Rademacher we have

$$\left(\mathbb{E}\left|\sum_{i=1}^n a_i\varepsilon_i\right|^p\right)^{1/p} = \left(\mathbb{E}\left|\sum_{i=1}^n a_i^*\varepsilon_i\right|^p\right)^{1/p}.$$

Splitting the sum into two parts, we get that

$$\left(\mathbb{E}\left|\sum_{i=1}^n a_i^*\varepsilon_i\right|^p\right)^{1/p} \le \left(\mathbb{E}\left|\sum_{i=1}^p a_i^*\varepsilon_i\right|^p\right)^{1/p} + \left(\mathbb{E}\left|\sum_{i>p} a_i^*\varepsilon_i\right|^p\right)^{1/p}.$$

We conclude by applying (1.4) to the first term and (1.3) to the second one. □

This provides a good example of one of the main drawback of this strategy. Indeed, being a ψ_α random variable allows only one type of tail estimate. In the sense that, if $Z \in L_{\psi_\alpha}$ then the tail decay of Z behaves like $\exp(-Kt^\alpha)$ for t large enough, but this result is sometimes too weak for a precise study of the L_p norm of Z.

1.2.2. Bernstein's type inequalities, the case $\alpha = 1$. — We start this section with the well known Bernstein's inequalities which hold for an empirical mean of bounded random variables.

Theorem 1.2.6. — *Let $X_1, \dots, X_n$ be n independent random variables and M be a positive number such that for any $i = 1, \dots, n$, $\mathbb{E}X_i = 0$ and $|X_i| \leqslant M$ almost surely. Set $\sigma^2 = n^{-1}\sum_{i=1}^n \mathbb{E}X_i^2$. For any $t > 0$, we have*

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n X_i \geqslant t\right) \leqslant \exp\left(-\frac{n\sigma^2}{M^2} h\left(\frac{Mt}{\sigma^2}\right)\right),$$

where $h(u) = (1+u)\log(1+u) - u$ for all $u > 0$.

Proof. — Let $t > 0$, by Markov inequality and by independence we have

$$\begin{aligned}\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n X_i \geqslant t\right) &\leqslant \inf_{\lambda>0} \exp(-\lambda t)\mathbb{E}\exp\left(\frac{\lambda}{n}\sum_{i=1}^n X_i\right)\\ &= \inf_{\lambda>0} \exp(-\lambda t)\prod_{i=1}^n \mathbb{E}\exp\left(\frac{\lambda X_i}{n}\right). \end{aligned}\tag{1.5}$$

Since for any $i = 1, \ldots, n$, $\mathbb{E}X_i = 0$ and $|X_i| \leq M$,

$$\mathbb{E}\exp\left(\frac{\lambda X_i}{n}\right) = 1 + \sum_{k\geqslant 2} \frac{\lambda^k \mathbb{E}X_i^k}{n^k k!} \leq 1 + \mathbb{E}X_i^2 \sum_{k\geqslant 2} \frac{\lambda^k M^{k-2}}{n^k k!}$$
$$= 1 + \frac{\mathbb{E}X_i^2}{M^2}\left(\exp\left(\frac{\lambda M}{n}\right) - \left(\frac{\lambda M}{n}\right) - 1\right).$$

Using the fact that $1 + u \leqslant \exp(u)$ for all $u \in \mathbb{R}$, we get that

$$\prod_{i=1}^n \mathbb{E}\exp\left(\frac{\lambda X_i}{n}\right) \leq \exp\left(\frac{\sum_{i=1}^n \mathbb{E}X_i^2}{M^2}\left(\exp\left(\frac{\lambda M}{n}\right) - \left(\frac{\lambda M}{n}\right) - 1\right)\right).$$

By definition of σ and by (1.5), we conclude that for any $t > 0$,

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n X_i \geqslant t\right) \leq \inf_{\lambda>0} \exp\left(\frac{n\sigma^2}{M^2}\left(\exp\left(\frac{\lambda M}{n}\right) - \left(\frac{\lambda M}{n}\right) - 1\right) - \lambda t\right).$$

The claim follows by choosing λ such that $(1 + tM/\sigma^2) = \exp(\lambda M/n)$. □

Using Taylor expansion, it is not difficult to see that for every $u > 0$ we have $h(u) \geqslant u^2/(2 + 2u/3)$. This proves that if $u \geq 1$, $h(u) \geq 3u/8$ and if $u \leq 1$, $h(u) \geq 3u^2/8$. Therefore the classical Bernstein's inequality for bounded random variables is an immediate corollary of this result.

Theorem 1.2.7. — *Let $X_1, \ldots, X_n$ be n independent random variables such that for all $i = 1, \ldots, n$, $\mathbb{E}X_i = 0$ and $|X_i| \leqslant M$ almost surely. Then, for every $t > 0$,*

$$\mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n X_i \geqslant t\right) \leqslant \exp\left(-\frac{3n}{8}\min\left(\frac{t^2}{\sigma^2}, \frac{t}{M}\right)\right),$$

where $\sigma^2 = \frac{1}{n}\sum_{i=1}^n \mathbb{E}X_i^2$.

From Bernstein's inequality, we can deduce that the tail behavior of a sum of centered, bounded random variables has two regimes. There is a sub-exponential regime with respect to M for large values of t ($t \geqslant \sigma^2/M$) and a sub-Gaussian behavior with respect to σ^2 for small values of t ($t \leqslant \sigma^2/M$). Moreover, this inequality is always stronger than the tail estimate that we could deduce from Theorem 1.2.1 (which is only sub-Gaussian with respect to M^2).

Now, we turn to the important case of sum of sub-exponential centered random variables.

Theorem 1.2.8. — *Let $X_1, \ldots, X_n$ be n independent centered ψ_1 random variables. Then, for every $t > 0$,*

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^n X_i\right| \geqslant t\right) \leqslant 2\exp\left(-c\,n\min\left(\frac{t^2}{\sigma_1^2}, \frac{t}{M_1}\right)\right),$$

where $M_1 = \max_{1\leqslant i\leqslant n} \|X_i\|_{\psi_1}$, $\sigma_1^2 = \dfrac{1}{n}\sum_{i=1}^n \|X_i\|_{\psi_1}^2$ *and* c *is a number that can be taken equal to* $(e-1)/2e(2e-1)$.

Proof. — Since for every $x \geq 0$ and any positive natural integer k, $e^x \geq x^k/k!$, we get by definition of the ψ_1 norm that for any integer $k \geq 1$ and any $i = 1, \dots, n$,

$$\mathbb{E}|X_i|^k \leqslant ek!\, \|X_i\|_{\psi_1}^k .$$

Moreover $\mathbb{E}X_i = 0$ for any $i = 1, \dots, n$ and using Taylor expansion of the exponential, we deduce that for every $\lambda > 0$ such that $\lambda \|X_i\|_{\psi_1} \leq \lambda M_1 < n$,

$$\mathbb{E}\exp(\frac{\lambda}{n}X_i) \leqslant 1 + \sum_{k\geqslant 2} \frac{\lambda^k \mathbb{E}|X_i|^k}{n^k k!} \leqslant 1 + \frac{e\lambda^2 \|X_i\|_{\psi_1}^2}{n^2\left(1 - \frac{\lambda}{n}\|X_i\|_{\psi_1}\right)} \leq 1 + \frac{e\lambda^2 \|X_i\|_{\psi_1}^2}{n^2\left(1 - \frac{\lambda M_1}{n}\right)}.$$

Let $Z = n^{-1}\sum_{i=1}^n X_i$. Since for any real number x, $1+x \leq e^x$, we get by independence of the X_i's that for every $\lambda > 0$ such that $\lambda M_1 < n$

$$\mathbb{E}\exp(\lambda Z) \leq \exp\left(\frac{e\lambda^2}{n^2\left(1 - \frac{\lambda M_1}{n}\right)} \sum_{i=1}^n \|X_i\|_{\psi_1}^2\right) = \exp\left(\frac{e\lambda^2\sigma_1^2}{n - \lambda M_1}\right).$$

We conclude by Markov inequality that for every $t > 0$,

$$\mathbb{P}(Z \geq t) \leq \inf_{0<\lambda<n/M_1} \exp\left(-\lambda\, t + \frac{e\lambda^2\sigma_1^2}{n - \lambda M_1}\right).$$

We consider two cases. If $t \leq \sigma_1^2/M_1$, we choose $\lambda = nt/2e\sigma_1^2 \leq n/2eM_1$. A simple computation gives that

$$\mathbb{P}(Z \geq t) \leq \exp\left(-\frac{e-1}{2e(2e-1)}\,\frac{n\,t^2}{\sigma_1^2}\right).$$

If $t > \sigma_1^2/M_1$, we choose $\lambda = n/2eM_1$. This time, we get

$$\mathbb{P}(Z \geq t) \leq \exp\left(-\frac{e-1}{2e(2e-1)}\,\frac{n\,t}{M_1}\right).$$

We can do the same argument for $-Z$ and this concludes the proof of the announced result. □

1.2.3. The ψ_α case: $\alpha > 1$. — In this part we will focus on the case $\alpha \neq 2$ and $\alpha > 1$. Our goal is to explain the behavior of the tail estimate of a sum of independent ψ_α centered random variables. As in Bernstein inequalities, we will see that they share two different regimes depending on the level of deviation t.

Theorem 1.2.9. — *Let* $\alpha > 1$ *and* β *be such that* $\alpha^{-1} + \beta^{-1} = 1$. *Given* $X_1, \dots, X_n$ *be independent mean zero* ψ_α *real-valued random variables, set*

$$A_1 = \left(\sum_{i=1}^n \|X_i\|_{\psi_1}^2\right)^{1/2} \quad \text{and} \quad B_\alpha = \left(\sum_{i=1}^n \|X_i\|_{\psi_\alpha}^\beta\right)^{1/\beta}.$$

Then, for every $t > 0$,

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \geqslant t\right) \leq \begin{cases} 2\exp\left(-c_\alpha \min\left(\dfrac{t^2}{A_1^2}, \dfrac{t^\alpha}{B_\alpha^\alpha}\right)\right) & \text{if } \alpha < 2, \\ 2\exp\left(-c_\alpha \max\left(\dfrac{t^2}{A_1^2}, \dfrac{t^\alpha}{B_\alpha^\alpha}\right)\right) & \text{if } \alpha > 2 \end{cases}$$

where c_α is a number depending only on α.

Remark 1.2.10. — *We can stated the result with the same normalization as in Bernstein inequalities. Let $\sigma_1^2 = \frac{1}{n}\sum_{i=1}^n \|X_i\|_{\psi_1}^1$ and $M_\alpha^\beta = \frac{1}{n}\sum_{i=1}^n \|X_i\|_{\psi_\alpha}^\beta$, then we have*

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n} X_i\right| \geqslant t\right) \leq \begin{cases} 2\exp\left(-c_\alpha n \min\left(\dfrac{t^2}{\sigma_1^2}, \dfrac{t^\alpha}{M_\alpha^\alpha}\right)\right) & \text{if } \alpha < 2, \\ 2\exp\left(-c_\alpha n \max\left(\dfrac{t^2}{\sigma_1^2}, \dfrac{t^\alpha}{M_\alpha^\alpha}\right)\right) & \text{if } \alpha > 2. \end{cases}$$

Before proving the Theorem, we start by exhibiting a sub-Gaussian behavior of the Laplace transform of any ψ_1 centered random variable.

Lemma 1.2.11. — *Given X a ψ_1 mean-zero random variable, for every λ such that $0 \leq \lambda \leqslant \left(2\|X\|_{\psi_1}\right)^{-1}$ we have*

$$\mathbb{E}\exp(\lambda X) \leqslant \exp\left(4(e-1)\lambda^2\|X\|_{\psi_1}^2\right).$$

Proof. — Let X' be an independent copie of X and denote $Y = X - X'$. Since X is centered, by Jensen inequality,

$$\mathbb{E}\exp\lambda X = \mathbb{E}\exp(\lambda(X - \mathbb{E}X')) \leqslant \mathbb{E}\exp\lambda(X - X') = \mathbb{E}\exp\lambda Y.$$

The random variable Y is symmetric thus, for every λ, $\mathbb{E}\exp\lambda Y = \mathbb{E}\cosh\lambda Y$ and using the Taylor expansion,

$$\mathbb{E}\exp\lambda Y = 1 + \sum_{k\geq 1}\frac{\lambda^{2k}}{(2k)!}\mathbb{E}Y^{2k} = 1 + \lambda^2\sum_{k\geq 1}\frac{\lambda^{2(k-1)}}{(2k)!}\mathbb{E}Y^{2k}.$$

By definition of Y, for every $k \geq 1$, $\mathbb{E}Y^{2k} \leq 2^{2k}\mathbb{E}X^{2k}$. Hence, for every $0 \leq \lambda \leqslant \left(2\|X\|_{\psi_1}\right)^{-1}$, we get that

$$\mathbb{E}\exp\lambda Y \leq 1 + 4\lambda^2\|X\|_{\psi_1}^2\sum_{k\geq 1}\frac{\mathbb{E}X^{2k}}{(2k)!\|X\|_{\psi_1}^{2k}} \leq 1 + 4\lambda^2\|X\|_{\psi_1}^2\left(\mathbb{E}\exp\left(\frac{|X|}{\|X\|_{\psi_1}}\right) - 1\right).$$

By definition of the ψ_1 norm, we conclude that for every $0 \leq \lambda \leqslant \left(2\|X\|_{\psi_1}\right)^{-1}$

$$\mathbb{E}\exp\lambda X \leq 1 + 4(e-1)\lambda^2\|X\|_{\psi_1}^2 \leq \exp\left(4(e-1)\lambda^2\|X\|_{\psi_1}^2\right).$$

□

Proof of Theorem 1.2.9. — We start with the case $1 < \alpha < 2$.
For every $1 = 1, \dots, n$, X_i is a ψ_α random variable with $\alpha > 1$. Then it is a ψ_1 random variable (see Corollary 1.1.6) and from Lemma 1.2.11, we get that

$$\forall\, 0 \le \lambda \le 1/2\left\|X_i\right\|_{\psi_1},\ \mathbb{E}\exp\lambda X_i \le \exp\left(4(e-1)\lambda^2\left\|X_i\right\|_{\psi_1}^2\right).$$

Moreover, from Theorem 1.1.5, we get also that

$$\forall\,\lambda \ge 1/\|X_i\|_{\psi_\alpha},\ \mathbb{E}\exp\lambda X_i \le \exp\left(2\lambda\left\|X_i\right\|_{\psi_\alpha}\right)^\beta.$$

Since $1 < \alpha < 2$ then $\beta > 2$ and it is easy to conclude that there is a number $c_\beta \ge 1$ such that

$$\forall\,\lambda > 0,\ \mathbb{E}\exp\lambda X_i \le \exp\left(c_\beta\left(\lambda^2\|X_i\|_{\psi_1}^2 + \lambda^\beta\|X_i\|_{\psi_\alpha}^\beta\right)\right). \tag{1.6}$$

Indeed when $\|X_i\|_{\psi_\alpha} > 2\|X_i\|_{\psi_1}$, we just have to glue the two estimates. Otherwise, we have $\|X_i\|_{\psi_\alpha} \le 2\|X_i\|_{\psi_1}$ and for every $\lambda \in \left(1/2\left\|X_i\right\|_{\psi_1}, 1/\|X_i\|_{\psi_\alpha}\right)$, we get by Hölder inequality,

$$\mathbb{E}\exp\lambda X_i \le \left(\mathbb{E}\exp\left(\frac{X}{\|X\|_{\psi_\alpha}}\right)\right)^{\lambda\|X_i\|_{\psi_\alpha}} \le \exp\left(2^\beta\lambda\|X\|_{\psi_\alpha}\right) \le \exp\left(2^\beta\lambda^2 4\|X_i\|_{\psi_1}^2\right).$$

Let $Z = \sum_{i=1}^n X_i$, we deduce from (1.6) that for every $\lambda > 0$,

$$\mathbb{E}\exp\lambda Z \le \exp\left(c_\beta\left(A_1^2\lambda^2 + B_\alpha^\beta\lambda^\beta\right)\right)$$

where $c_\beta \ge 1$. From Markov inequality, we have

$$\mathbb{P}(Z \ge t) \le \inf_{\lambda>0} e^{-\lambda t}\mathbb{E}\exp\lambda Z \le \inf_{\lambda>0}\left(c_\beta\left(A_1^2\lambda^2 + B_\alpha^\beta\lambda^\beta\right) - \lambda t\right). \tag{1.7}$$

If $(t/A_1)^2 \ge (t/B_\alpha)^\alpha$ then we have $t^{2-\alpha} \ge A_1^2/B_\alpha^\alpha$ and we choose $\lambda = \frac{t^{\alpha-1}}{4c_\beta B_\alpha^\alpha}$. Therefore,

$$\lambda t = \frac{t^\alpha}{4c_\beta B_\alpha^\alpha}, \quad B_\alpha^\beta\lambda^\beta = \frac{t^\alpha}{(4c_\beta)^\beta B_\alpha^\alpha} \le \frac{t^\alpha}{(4c_\beta)^2 B_\alpha^\alpha}, \quad \text{and}$$
$$A_1^2\lambda^2 = \frac{t^\alpha}{(4c_\beta)^2 B_\alpha^\alpha}\frac{t^{\alpha-2}A_1^2}{B_\alpha^\alpha} \le \frac{t^\alpha}{(4c_\beta)^2 B_\alpha^\alpha}.$$

We conclude from (1.7) that

$$\mathbb{P}(Z \ge t) \le \exp\left(-\frac{1}{4c_\beta}\frac{t^\alpha}{B_\alpha^\alpha}\right).$$

If $(t/A_1)^2 \le (t/B_\alpha)^\alpha$ then we have $t^{2-\alpha} \le A_1^2/B_\alpha^\alpha$ and since $(2-\alpha)\beta/\alpha = (\beta - 2)$ we also have $t^{\beta-2} \le B_\alpha^\beta/A_1^{2(\beta-1)}$. We choose $\lambda = \frac{t}{4c_\beta A_1^2}$ therefore,

$$\lambda t = \frac{t^2}{4c_\beta A_1^2}, \quad A_1^2\lambda^2 = \frac{t^2}{(4c_\beta)^2 A_1^2} \quad \text{and} \quad B_\alpha^\beta\lambda^\beta = \frac{t^2}{(4c_\beta)^\beta A_1^2}\frac{t^{\beta-2}B_\alpha^\beta}{A_1^{2(\beta-1)}} \le \frac{t^2}{(4c_\beta)^2 A_1^2}.$$

We conclude from (1.7) that

$$\mathbb{P}(Z \ge t) \le \exp\left(-\frac{1}{4c_\beta}\frac{t^2}{A_1^2}\right).$$

The proof is complete with $c_\alpha = 1/4c_\beta$.

In the case $\alpha > 2$, we have $1 < \beta < 2$ and the estimate (1.6) for the Laplace transform of each random variable X_i has to be replaced by

$$\forall\, \lambda > 0,\ \mathbb{E}\exp\lambda X_i \le \exp\left(c_\beta \lambda^2 \|X_i\|_{\psi_1}^2\right) \text{ and } \mathbb{E}\exp\lambda X_i \le \exp\left(c_\beta \lambda^\beta \|X_i\|_{\psi_\alpha}^\beta\right).$$

Indeed, when $\lambda \|X\|_{\psi_1} \le 1/2$ then $(\lambda \|X\|_{\psi_1})^2 \le (\lambda \|X\|_{\psi_1})^\beta$ and when $\lambda \|X\|_{\psi_\alpha} \ge 1$ then $(\lambda \|X\|_{\psi_\alpha})^\beta \le (\lambda \|X\|_{\psi_\alpha})^2$. Therefore both inequalities hold true.
We conclude that for $Z = \sum_{i=1}^n X_i$, for every $\lambda > 0$,

$$\mathbb{E}\exp\lambda Z \le \exp\left(c_\beta \min\left(A_1^2\lambda^2, B_\alpha^\beta \lambda^\beta\right)\right).$$

The rest is identical to the preceding proof. □

1.3. A geometric application: the Johnson-Lindenstrauss lemma

The Johnson-Lindenstrauss lemma [**JL84**] is a result concerning low-distortion embeddings of points from high-dimensional into low-dimensional Euclidean space. The lemma states that a small set of points in a high-dimensional space can be embedded into a space of much lower dimension in such a way that distances between the points are nearly preserved. The map used for the embedding is a linear map and can even be taken to be an orthogonal projection. We present here an approach using random Gaussian matrices.

Let $G_1, \ldots, G_k$ be k independent Gaussian vectors in $\mathbb{R}^n$ distributed according to the normal law $\mathcal{N}(0, \mathrm{I}d)$. Let $A : \mathbb{R}^n \to \mathbb{R}^k$ be the random operator defined for every $x \in \mathbb{R}^n$ by

$$Ax = \begin{pmatrix} \langle G_1, x\rangle \\ \vdots \\ \langle G_k, x\rangle \end{pmatrix} \in \mathbb{R}^k. \tag{1.8}$$

We will prove that with high probability, this Gaussian random matrix satisfies the desired property in the Johnson-Lindenstrauss lemma.

Lemma 1.3.1. — *There exists a numerical constant C such that, given $0 < \varepsilon < 1$, a set T of N distinct points in $\mathbb{R}^n$ and an integer $k > k_0 = C\log(N)/\varepsilon^2$ then there exists a linear operator $A : \mathbb{R}^n \to \mathbb{R}^k$ such that for every $x, y \in T$,*

$$\sqrt{1-\varepsilon}\ |x-y|_2 \le |A(x-y)|_2 \le \sqrt{1+\varepsilon}\ |x-y|_2.$$

Proof. — Let Γ be defined by (1.8). For any vector $z \in \mathbb{R}^n$ and every $i = 1, \ldots, k$, we have $\mathbb{E}\langle G_i, z\rangle^2 = |z|_2^2$. Therefore, for every $x, y \in T$,

$$\left|\frac{\Gamma(x-y)}{\sqrt{k}}\right|_2^2 - |x-y|_2^2 = \frac{1}{k}\sum_{i=1}^k \langle G_i, x-y\rangle^2 - \mathbb{E}\langle G_i, x-y\rangle^2.$$

For every $i = 1, \ldots, k$, we define the random variable X_i by $X_i = \langle G_i, x-y\rangle^2 - \mathbb{E}\langle G_i, x-y\rangle^2$. It is a centered random variable. Since $e^u \ge 1+u$, we know that

$\mathbb{E}\langle G_i, x-y\rangle^2 \leq (e-1)\left\|\langle G_i, x-y\rangle^2\right\|_{\psi_1}$. Hence by definition of the ψ_2 norm,

$$\|X_i\|_{\psi_1} \leq 2(e-1)\left\|\langle G_i, x-y\rangle^2\right\|_{\psi_1} = 2(e-1)\left\|\langle G_i, x-y\rangle\right\|_{\psi_2}^2.$$

By definition of the Gaussian law, $\langle G_i, x-y\rangle$ is distributed like $|x-y|_2\, g$ where g is a standard real Gaussian variable. It is not difficult to check that with our definition of the ψ_2 norm, $\|g\|_{\psi_2}^2 = 2(e-1)^2/e(e-2)$. We will call c_0^2 this number and $c_1^2 = 2(e-1)c_0^2$. We conclude that $\|\langle G_i, x-y\rangle\|_{\psi_2}^2 = c_0^2\,|x-y|_2^2$ and that $\|X_i\|_{\psi_1} \leq c_1^2\,|x-y|_2^2$. We apply Theorem 1.2.8. In this case, $M_1 = \sigma_1 = c_1^2\,|x-y|_2^2$ and we get that for $t = \varepsilon|x-y|_2^2$ with $0<\varepsilon<1$,

$$\mathbb{P}\left(\left|\frac{1}{k}\sum_{i=1}^{k}\langle G_i, x-y\rangle^2 - \mathbb{E}\langle G_i, x-y\rangle^2\right| > \varepsilon|x-y|_2^2\right) \leq 2\exp(-c'\,k\,\varepsilon^2)$$

since $t \leq |x-y|_2^2 \leq c_1^2\,|x-y|_2^2 \leq \sigma_1^2/M_1$. The constant c' is defined by $c' = c/c_1^4$ where c comes from Theorem 1.2.8. Since the cardinality of the set $\{(x,y) : x\in T, y\in T\}$ is less than N^2, we get by the union bound that

$$\mathbb{P}\left(\exists\, x,\, y\in T : \left|\left|\frac{\Gamma(x-y)}{\sqrt{k}}\right|_2^2 - |x-y|_2^2\right| > \varepsilon|x-y|_2^2\right) \leq N^2\exp(-c'\,k\,\varepsilon^2)$$

and if $k > k_0 = \log(N^2)/c'\varepsilon^2$ then the probability of this event is strictly less than one. This means that there exists a realization of the matrix $\Gamma/\sqrt{k}$ that defines A and that satisfies the contrary i.e.

$$\forall\, x,\, y\in T,\ \sqrt{1-\varepsilon}\ |x-y|_2 \leq |A(x-y)|_2 \leq \sqrt{1+\varepsilon}\ |x-y|_2.$$

□

Remark 1.3.2. — • *The value of C is less than 1600.*
• *In fact, the proof uses only the ψ_2 behavior of $\langle G_i, x\rangle$. We could replace the Gaussian vectors by any sub-Gaussian (isotropic) vectors, like e.g. by random vectors with independent Rademacher coordinates. In this case, the value of C is less than 170.*

1.4. Complexity and covering numbers

The study of covering and packing numbers is a wide subject. We will only present some basic estimates needed for the purpose of this book. In approximation theory, in compressed sensing, in statistics, it is of importance to measure the complexity of a set. An important notion is the entropy number which measures the compactness of a set. Given U and V two sets of $\mathbb{R}^n$, we define the covering number $N(U,V)$ to be the minimum of translates of V needed to cover U. The formal definition is

$$N(U,V) = \inf\left\{N : \exists x_1,\ldots,x_N \in \mathbb{R}^n, U \subset \bigcup_{i=1}^{N}(x_i+V)\right\}.$$

If moreover V is a symmetric convex set, the packing number $M(U,V)$ is the maximal number of points in U that are 1-separated for the norm induced by the convex set

V. Formally,

$$M(U,V) = \sup\left\{N : \exists x_1,\dots,x_N \in U, \forall i \neq j, x_i - x_j \notin V\right\}.$$

Since V is a symmetric convex set, we can define the norm associated to V: for every $x \in \mathbb{R}^n$

$$\|x\|_V = \inf\{t > 0, x \in tV\}.$$

Hence $x_i - x_j \notin V$ is equivalent to say that $\|x_i - x_j\|_V > 1$. For any positive number ε, we will also use the notation

$$N(U, \varepsilon, \|\cdot\|_V)$$

for $N(U, \varepsilon V)$. Moreover, the family $x_1, \dots, x_N$ is said to be an ε-net if it is such that $U \subset \bigcup_{i=1}^N (x_i + \varepsilon V)$. Also if we define the polar of V by

$$V^o = \{y \in \mathbb{R}^n : \forall x \in V, \langle x, y\rangle \leq 1\}$$

then the dual of the vectorial normed space $(\mathbb{R}^n, \|\cdot\|_V)$ is isometric to $(\mathbb{R}^n, \|\cdot\|_{V^o})$.

In the case V being a symmetric convex set, the notions of packing and covering numbers are closely related.

Proposition 1.4.1. — *If U and V are convex bodies then $N(U,V) \leq M(U,V)$. If U is a convex set and V is a symmetric convex set then $M(U,V) \leq N(U,V/2)$.*

Proof. — Let $N = M(U,V)$ be the maximal number of points $x_1,\dots,x_N$ in U such that for every $i \neq j$, $x_i - x_j \notin V$. Let $u \in U$ then $\{x_1,\dots,x_N,u\}$ is not 1-separated in V and this means that there exists $i \in \{1,\dots,N\}$ such that $u - x_i \in V$. Consequently $U \subset \bigcup_{i=1}^N (x_i + V)$ and $N(U,V) \leq M(U,V)$.

Let $x_1,\dots,x_M$ be a family of vectors of U that are 1-separated. Let $z_1,\dots,z_N$ be a family of vectors such that $U \subset \bigcup_{i=1}^N (z_i + V/2)$. Since for every $i = 1,\dots,M$, $x_i \in U$, we can define an application $j : \{1,\dots,M\} \to \{1,\dots,N\}$ where $j(i)$ is such that $x_i \in z_{j(i)} + V/2$. If $j(i_1) = j(i_2)$ then $x_{i_1} - x_{i_2} \in V/2 - V/2$. By convexity and symmetry of V, $V/2 - V/2 = V$ hence $x_{i_1} - x_{i_2} \in V$. But the family $x_1,\dots,x_M$ is 1-separated in V hence necessarily $i_1 = i_2$. This proves that the map j is injective and this implies that $M(U,V) \leq N(U,V/2)$. □

Moreover, it is not difficult to check that for any U, V, W convex bodies $N(U,W) \leq N(U,V)N(V,W)$. We have the following simple and important volumetric estimate.

Lemma 1.4.2. — *Given V a symmetric convex set in $\mathbb{R}^n$, for every $\varepsilon > 0$,*

$$N(V, \varepsilon V) \leq \left(1 + \frac{2}{\varepsilon}\right)^n.$$

Proof. — By Proposition 1.4.1, $N(V,\varepsilon V) \leq M(V, \varepsilon V)$. Let $M = M(V,\varepsilon V)$ be the maximal number of points $x_1,\dots,x_M$ in V such that for every $i \neq j$, $x_i - x_j \notin \varepsilon V$. Since V is a symmetric convex set, the sets $x_i + \varepsilon V/2$ are disjoint and

$$\bigcup_{i=1}^M (x_i + \varepsilon V/2) \subset V + \varepsilon V/2 = \left(1 + \frac{\varepsilon}{2}\right) V.$$

By taking the volume, we get that

$$M\left(\frac{\varepsilon}{2}\right)^n \leq \left(1+\frac{\varepsilon}{2}\right)^n$$

which proves the desired estimate. □

Another important parameter that we will use to measure the size of a subset T of $\mathbb{R}^n$ is $\ell_*(T)$, defined by

$$\ell_*(T) = \mathbb{E}\sup_{t\in T}\langle G, t\rangle$$

where G is a Gaussian vector in $\mathbb{R}^n$ distributed according to the normal law $\mathcal{N}(0, \mathrm{I}d)$. By definition, $\ell_*(T) = \ell_*(\operatorname{conv} T)$ where $\operatorname{conv} T$ denotes the convex hull of T.

We will present some classical tools to estimate the covering numbers of the unit ball of ℓ_1^n by parallelepipeds and some classical estimates relating covering numbers of T by a multiple of the Euclidean ball and $\ell_*(T)$ or $\ell_*(T^o)$.

1.4.1. The empirical method. — We will introduce this method through a concrete example. Let d be a positive integer and Φ be an $d\times d$ matrix. We assume that the entries of Φ are such that for all $i, j \in \{1, \ldots, d\}$,

$$|\Phi_{ij}| \leqslant \frac{K}{\sqrt{d}} \tag{1.9}$$

where $K > 0$ is an absolute constant.

We denote by $\Phi_1, \ldots, \Phi_d$ the row vectors of Φ and we define for all $p \in \{1, \ldots, d\}$ the semi-norm $\|\cdot\|_{\infty,p}$ for all $x \in \mathbb{R}^d$ by

$$\|x\|_{\infty,p} = \max_{1\leqslant j\leqslant p} |\langle \Phi_j, x\rangle|.$$

Its unit ball is denoted by $B_{\infty,p}$. If $E = \operatorname{span}\{\Phi_1, \ldots, \Phi_p\}$ and P_E is the orthogonal projection on E then we have $B_{\infty,p} = P_E B_{\infty,p} + E^\perp$. Moreover, $P_E B_{\infty,p}$ is a parallelepiped in E. In the next theorem, we obtain an upper bound of the logarithm of the covering numbers of the unit ball of ℓ_1^d, denoted by B_1^d, by a multiple of $B_{\infty,p}$. Observe that from the hypothesis (1.9) on the entries of the matrix Φ, we get that for any $x \in B_1^d$ and any $j = 1, \ldots, p$, $|\langle \Phi_j, x\rangle| \leq |\Phi_j|_\infty |x|_1 \leq K/\sqrt{d}$. Therefore

$$B_1^d \subset \frac{K}{\sqrt{d}} B_{\infty,p} \tag{1.10}$$

and for any $\varepsilon \geq K/\sqrt{d}$, $N(B_1^d, \varepsilon B_{\infty,p}) = 1$.

Theorem 1.4.3. — *With the preceding notations, we have for any* $0 < t < 1$,

$$\log N\left(B_1^d, \frac{tK}{\sqrt{d}} B_{\infty,p}\right) \leqslant \min\left\{c_0 \frac{\log(p)\log(2d+1)}{t^2},\ p\log\left(1+\frac{2}{t}\right)\right\}$$

where c_0 *is an absolute constant.*

The first estimate is proven using an empirical method, while the second one is based on the volumetric estimate.

Proof. — Let x be in B_1^d and define the random variable Z by

$$\mathbb{P}\big(Z = \mathrm{Sign}(x_i)e_i\big) = |x_i| \text{ for all } i = 1, \dots, d \text{ and } \mathbb{P}(Z = 0) = 1 - |x|_1$$

where $(e_1, \dots, e_d)$ is the canonical basis of $\mathbb{R}^d$. Observe that we have $\mathbb{E}Z = x$.

We use a well known symmetrization argument (see Chapter 5 section 5.2.1 for more details). Take m to be chosen later and $Z_1, \dots, Z_m, Z_1', \dots, Z_m'$ be i.i.d. copies of Z. We have by Jensen inequality

$$\mathbb{E}\left\| x - \frac{1}{m}\sum_{i=1}^m Z_i \right\|_{\infty,p} = \left\| \frac{1}{m}\sum_{i=1}^m \mathbb{E}' Z_i' - Z_i \right\|_{\infty,p} \le \mathbb{E}\mathbb{E}' \left\| \frac{1}{m}\sum_{i=1}^m Z_i' - Z_i \right\|_{\infty,p}.$$

The random variable $Z_i' - Z_i$ is symmetric hence it has the same law than $\varepsilon_i(Z_i' - Z_i)$ where $\varepsilon_1, \dots, \varepsilon_m$ are i.i.d. Rademacher random variables. Therefore, by the triangle inequality

$$\mathbb{E}\mathbb{E}' \left\| \frac{1}{m}\sum_{i=1}^m Z_i' - Z_i \right\|_{\infty,p} = \frac{1}{m}\mathbb{E}\mathbb{E}'\mathbb{E}_\varepsilon \left\| \sum_{i=1}^m \varepsilon_i(Z_i' - Z_i) \right\|_{\infty,p} \le \frac{2}{m}\mathbb{E}\mathbb{E}_\varepsilon \left\| \sum_{i=1}^m \varepsilon_i Z_i \right\|_{\infty,p}.$$

We conclude that

$$\begin{aligned} \mathbb{E}\left\| x - \frac{1}{m}\sum_{i=1}^m Z_i \right\|_{\infty,p} &\le \frac{2}{m}\mathbb{E}\mathbb{E}_\varepsilon \left\| \sum_{i=1}^m \varepsilon_i Z_i \right\|_{\infty,p} \\ &= \frac{2}{m}\mathbb{E}\mathbb{E}_\varepsilon \max_{1\le j\le p} \left| \sum_{i=1}^m \varepsilon_i \langle Z_i, \Phi_j\rangle \right|. \end{aligned} \tag{1.11}$$

By definition of Z and by (1.9), we know that $|\langle Z_i, \Phi_j\rangle| \le K/\sqrt{d}$. Let a_{ij} be a sequence of real number such that $|a_{ij}| \le K/\sqrt{d}$. For any j, let X_j be the random variable $X_j = \sum_{i=1}^m a_{ij}\varepsilon_i$. From Theorem 1.2.1, we deduce that

$$\forall j = 1, \dots, p,\ \|X_j\|_{\psi_2} \le c\left(\sum_{i=1}^m a_{ij}^2\right)^{1/2} \le c\frac{K\sqrt{m}}{\sqrt{d}}.$$

Therefore, by Proposition 1.1.3 (and the remark after it), we get

$$\mathbb{E}\max_{1\le j\le p} |X_j| \le c\sqrt{(1+\log p)}\,\frac{K\sqrt{m}}{\sqrt{d}}.$$

From (1.11) and the preceding argument, we conclude that

$$\mathbb{E}\left\| x - \frac{1}{m}\sum_{i=1}^m Z_i \right\|_{\infty,p} \le \frac{2\,c\,K\sqrt{(1+\log p)}}{\sqrt{md}}$$

Let m be the integer such that

$$\frac{4c^2(1+\log p)}{t^2} \le m \le \frac{4c^2(1+\log p)}{t^2} + 1$$

For this choice of m we have
$$\mathbb{E}\left\|x-\frac{1}{m}\sum_{i=1}^{m}Z_i\right\|_{\infty,p}\leqslant\frac{tK}{\sqrt{d}}.$$
In particular, there exists $\omega\in\Omega$ such that
$$\left\|x-\frac{1}{m}\sum_{i=1}^{m}Z_i(\omega)\right\|_{\infty,p}\leqslant\frac{tK}{\sqrt{d}}$$
and so the set
$$\left\{\frac{1}{m}\sum_{i=1}^{m}z_i : z_1,\dots,z_m\in\{\pm e_1,\dots,\pm e_d\}\cup\{0\}\right\}$$
is a $tK/\sqrt{d}$-net of B_1^d with respect to $\|\cdot\|_{\infty,p}$. Since its cardinality is less than $(2d+1)^m$, we get the first estimate:
$$\log N\left(B_1^d,\frac{tK}{\sqrt{d}}B_{\infty,p}\right)\leqslant\frac{c_0(1+\log p)\log(2d+1)}{t^2}$$
where c_0 is an absolute constant.

To prove the second estimate, we recall by (1.10) that $B_1^d\subset K/\sqrt{d}B_{\infty,p}$. Hence
$$N\left(B_1^d,\frac{tK}{\sqrt{d}}B_{\infty,p}\right)\leq N\left(\frac{K}{\sqrt{d}}B_{\infty,p},\frac{tK}{\sqrt{d}}B_{\infty,p}\right)=N\left(B_{\infty,p},tB_{\infty,p}\right).$$
Moreover, we have already observed that $B_{\infty,p}=P_EB_{\infty,p}+E^{\perp}$ which means that
$$N\left(B_{\infty,p},tB_{\infty,p}\right)=N(V,tV)$$
where V is the symmetric convex set $P_EB_{\infty,p}$. Since $\dim E\leq p$, we apply Lemma 1.4.2 to conclude that
$$N\left(B_1^d,\frac{tK}{\sqrt{d}}B_{\infty,p}\right)\leq\left(1+\frac{2}{t}\right)^p.$$
□

1.4.2. Sudakov's inequality and dual Sudakov's inequality. — Classical tools for the computation of the covering numbers of a set by Euclidean balls or in a dual situation covering numbers of Euclidean ball by a symmetric convex set are Sudakov and dual Sudakov inequalities. They relate these covering numbers with the complexity ℓ_* of the sets.

Theorem 1.4.4. — *Let T be a subset of $\mathbb{R}^N$ and V be a symmetric convex set in $\mathbb{R}^N$. Then, the following inequalities hold:*
$$\sup_{\varepsilon>0}\varepsilon\sqrt{\log N(T,\varepsilon B_2^N)}\leqslant c\,\ell_*(T)\tag{1.12}$$
and
$$\sup_{\varepsilon>0}\varepsilon\sqrt{\log N(B_2^N,\varepsilon V)}\leqslant c\,\ell_*(V^o)\tag{1.13}$$

where for a normal Gaussian vector $G \in \mathbb{R}^N$, $\ell_*(T) = \mathbb{E} \sup_{t \in T} \langle G, t \rangle$ *and* $\ell_*(V^o) = \mathbb{E} \sup_{t \in V^o} \langle G, t \rangle = \mathbb{E} \|G\|_V$.

The proof of the Sudakov inequality (1.12) is based on comparison properties between Gaussian processes. We recall the Slepian comparison lemma without proving it.

Lemma 1.4.5. — *Let* $X_1, \ldots, X_M$ *and* $Y_1, \ldots, Y_M$ *be Gaussian random variables such that for all* $i, j = 1, \ldots, M$

$$\mathbb{E}|Y_i - Y_j|^2 \le \mathbb{E}|X_i - X_j|^2$$

then

$$\mathbb{E} \max_{1 \le k \le M} Y_k \le 2\, \mathbb{E} \max_{1 \le k \le M} X_k.$$

Proof of Theorem 1.4.4. — We start by proving (1.12). Let $x_1, \ldots, x_M$ be M points of T that are ε-separated with respect to the Euclidean norm $|\cdot|_2$ and define for every $i = 1, \ldots, M$, the Gaussian variables $X_i = \langle x_i, G \rangle$ where G is a standard Gaussian vector in $\mathbb{R}^N$. We have

$$\mathbb{E}|X_i - X_j|^2 = |x_i - x_j|_2^2 \ge \varepsilon^2 \quad \text{for all } i \ne j.$$

Let $g_1, \ldots, g_M$ be M standard independent Gaussian random variables and for every $i = 1, \ldots, M$ let Y_i be defined by $Y_i = \frac{\varepsilon}{\sqrt{2}} g_i$. We have for all $i \ne j$

$$\mathbb{E}|Y_i - Y_j|^2 = \varepsilon^2$$

and we conclude from Lemma 1.4.5 that

$$\frac{\varepsilon}{\sqrt{2}} \mathbb{E} \max_{1 \le k \le M} g_k \le 2\, \mathbb{E} \max_{1 \le k \le M} \langle x_k, G \rangle \le 2\ell(T).$$

Moreover there exists a constant $c > 0$ such that for every positive integer M

$$\mathbb{E} \max_{1 \le k \le M} g_k \ge \sqrt{\log M}/c \tag{1.14}$$

and this proves that $\varepsilon\sqrt{\log M} \le 2c\sqrt{2}\ell(T)$. By Proposition 1.4.1, the proof of inequality (1.12) is complete. The lower bound (1.14) is a classical exercise about Gaussian random variables. First, we observe that $\mathbb{E} \max(g_1, g_2)$ is computable, it is equal to $1/\sqrt{\pi}$. Hence we can assume that M is large enough (say greater than 10^4). In this case, we observe that

$$2\, \mathbb{E} \max_{1 \le k \le M} g_k \ge \mathbb{E} \max_{1 \le k \le M} |g_k| - \mathbb{E}|g_1|.$$

Indeed,

$$\mathbb{E} \max_{1 \le k \le M} g_k = \mathbb{E} \max_{1 \le k \le M} (g_k - g_1) = \mathbb{E} \max_{1 \le k \le M} \max((g_k - g_1), 0)$$

and by symmetry of the g_i's,

$$\begin{aligned} \mathbb{E} \max_{1 \le k \le M} |g_k - g_1| &\le \mathbb{E} \max_{1 \le k \le M} \max((g_k - g_1), 0) + \mathbb{E} \max_{1 \le k \le M} \max((g_1 - g_k), 0) \\ &= 2\, \mathbb{E} \max_{1 \le k \le M} (g_k - g_1) = 2\mathbb{E} \max_{1 \le k \le M} g_k. \end{aligned}$$

But, by independence of the g_i's

$$\mathbb{E} \max_{1\le k\le M} |g_k| = \int_0^{+\infty} \mathbb{P}\left(\max_{1\le k\le M} |g_k| > t\right) dt = \int_0^{+\infty} \left(1 - \mathbb{P}\left(\max_{1\le k\le M} |g_k| \le t\right)\right) dt$$
$$= \int_0^{+\infty} \left(1 - \left(1 - \sqrt{\frac{2}{\pi}} \int_t^{+\infty} e^{-u^2/2} du\right)^M\right) dt$$

and it is easy to see that for every $t > 0$,

$$\int_t^{+\infty} e^{-u^2/2} du \ge e^{-(t+1)^2/2}.$$

Let $t_0 + 1 = \sqrt{2\log M}$ then

$$\mathbb{E} \max_{1\le k\le M} |g_k| \ge \int_0^{t_0} \left(1 - \left(1 - \sqrt{\frac{2}{\pi}} \int_t^{+\infty} e^{-u^2/2} du\right)^M\right) dt$$
$$\ge t_0 \left(1 - \left(1 - \frac{\sqrt{2}}{M\sqrt{\pi}}\right)^M\right) \ge t_0(1 - e^{-\sqrt{2/\pi}})$$

which concludes the proof of (1.14).

We will now prove the dual Sudakov inequality (1.13). The argument is very similar to the volumetric argument introduced in Lemma 1.4.2, replacing the Lebesgue measure by the Gaussian measure. Let $r > 0$ to be chosen later. Observe that $N(B_2^N, \varepsilon V) = N(rB_2^N, r\varepsilon V)$ and let $x_1, \ldots, x_M$ be M points in rB_2^N that are $r\varepsilon$ separated for the norm induced by the symmetric convex set V. By Proposition 1.4.1, it is enough to prove that

$$\varepsilon \sqrt{\log M} \le c\, \ell_*(V^o).$$

The balls centered at the points x_i and of radius $r\varepsilon/2$ are disjoints and by taking the Gaussian measure of the union of these sets, we get that

$$\gamma_N\left(\bigcup_{i=1}^M (x_i + r\varepsilon/2\, V)\right) = \sum_{i=1}^M \int_{\|z-x_i\|_V \le r\varepsilon/2} e^{-|z|_2^2/2} \frac{dz}{(2\pi)^{N/2}} \le 1.$$

However, by the change of variable $z - x_i = u_i$, we have

$$\int_{\|z-x_i\|_V \le r\varepsilon/2} e^{-|z|_2^2/2} \frac{dz}{(2\pi)^{N/2}} = e^{-|x_i|_2^2/2} \int_{\|u_i\|_V \le r\varepsilon/2} e^{-|u_i|_2^2/2} e^{-\langle u_i, x_i\rangle} \frac{du_i}{(2\pi)^{N/2}}$$

and from Jensen inequality and the fact that V has barycenter at the origin,

$$\int_{\|z-x_i\|_V \le r\varepsilon/2} e^{-|z|_2^2/2} \frac{dz}{(2\pi)^{N/2}} \ge e^{-|x_i|_2^2/2}\, \gamma_N\left(\frac{r\varepsilon}{2} V\right).$$

Since $x_i \in rB_2^N$, we have proved that

$$M\, e^{-r^2/2}\, \gamma_N\left(\frac{r\varepsilon}{2} V\right) \le 1.$$

To conclude, we choose r such that $r\varepsilon/2 = 2\ell_*(V^o)$. Hence by Markov inequality, $\gamma_N\left(\frac{r\varepsilon}{2}V\right) \geq 1/2$ and we have proved that $M \leq 2e^{r^2/2}$ which means that for a constant c,

$$\varepsilon\sqrt{\log M} \leq c\,\ell_*(V^o).$$

□

1.4.3. The metric entropy of the Schatten balls. — To finish this chapter, we show how to apply Sudakov and dual Sudakov inequalities to compute the metric entropy of Schatten balls with respect to Schatten norms. We denote by $B_p^{m,n}$ the unit ball of the Banach spaces of matrices in $\mathcal{M}_{m,n}$ endowed with the Schatten norm $\|\cdot\|_{S_p}$ defined for any $A \in \mathcal{M}_{m,n}$ by

$$\|A\|_{S_p} = \left(\mathrm{tr}(A^*A)^{p/2}\right)^{1/p}.$$

It is also the ℓ_p-norm of the singular values of A and we refer to Chapter 4 for more informations about the singular values of a matrix.

Proposition 1.4.6. — *For every $m \geq n \geqslant 1$, $p, q \in [1, +\infty]$ and $\varepsilon > 0$,*

$$\varepsilon\sqrt{\log N(B_p^{m,n}, \varepsilon B_2^{m,n})} \leqslant c_1\sqrt{m}\; n^{(1-1/p)} \tag{1.15}$$

and

$$\varepsilon\sqrt{\log N(B_2^{m,n}, \varepsilon B_q^{m,n})} \leqslant c_2\sqrt{m}\; n^{1/q} \tag{1.16}$$

where c_1 and c_2 are numerical constants. Moreover, for $n \geq m \geq 1$ the same result holds by exchanging m and n.

Proof. — We start by proving a rough upper bound of the operator norm of a Gaussian random matrix $\Gamma \in \mathcal{M}_{m,n}$ i.e. a matrix with independent standard Gaussian entries:

$$\mathbb{E}\,\|\Gamma\|_{S_\infty} \leq 2\sqrt{2}(\sqrt{n} + \sqrt{m}). \tag{1.17}$$

Let $X_{u,v}$ be the Gaussian process defined for any $u \in B_2^m, v \in B_2^n$ by

$$X_{u,v} = \langle \Gamma v, u\rangle.$$

It is defined such that

$$\mathbb{E}\,\|\Gamma\|_{S_\infty} = \mathbb{E}\sup_{u\in B_2^m, v\in B_2^n} X_{u,v}.$$

We have for any $u, u' \in B_2^m, v, v' \in B_2^n$

$$\begin{aligned}\mathbb{E}|X_{u,v} - X_{u',v'}|^2 &= \mathbb{E}|\langle \Gamma v, u-u'\rangle + \langle \Gamma(v-v'), u'\rangle|^2 \\ &\leq 2(\mathbb{E}|\langle \Gamma v, u-u'\rangle|^2 + \mathbb{E}\langle \Gamma(v-v'), u'\rangle|^2) \leq 2(|u-u'|^2 + |v-v'|^2).\end{aligned}$$

Let $Y_{u,v}$ be the Gaussian process defined for any $u \in B_2^m, v \in B_2^n$ by

$$Y_{u,v} = \sqrt{2}\,(\langle G_1, u\rangle + \langle G_2, v\rangle)$$

where G_1 is a random standard Gaussian vector in $\mathbb{R}^m$ and G_2 is a random standard Gaussian vector in $\mathbb{R}^n$ independent with G_1. Then for any $u, u' \in B_2^m, v, v' \in B_2^n$

$$\mathbb{E}|Y_{u,v} - Y_{u',v'}|^2 = 2(|u-u'|^2 + |v-v'|^2) \geq \mathbb{E}|X_{u,v} - X_{u',v'}|^2$$

and by Lemma 1.4.5, we deduce that
$$\mathbb{E} \sup_{u\in B_2^m, v\in B_2^n} X_{u,v} \le 2\,\mathbb{E} \sup_{u\in B_2^m, v\in B_2^n} Y_{u,v}.$$
Since
$$\mathbb{E} \sup_{u\in B_2^m, v\in B_2^n} Y_{u,v} = \sqrt{2}(\mathbb{E}|G_1|_2 + \mathbb{E}|G_2|) \le \sqrt{2}(\sqrt{m}+\sqrt{n})$$
we conclude that (1.17) holds true.

We first prove (1.15) in the case $m \ge n \ge 1$. Using Sudakov inequality (1.12), we have for all $\varepsilon > 0$,
$$\varepsilon\sqrt{\log N(B_p^{m,n}, \varepsilon B_2^{m,n})} \leqslant c\ell_*(B_p^{m,n}).$$
However
$$\ell_*(B_p^{m,n}) = \mathbb{E} \sup_{A\in B_p^{m,n}} \langle \Gamma, A\rangle$$
where $\langle \Gamma, A\rangle = \mathrm{Tr}(\Gamma\, A^*)$. If p' is such that $1/p + 1/p' = 1$ then we have by the trace duality
$$\langle \Gamma, A\rangle \le \|\Gamma\|_{S_{p'}} \|A\|_{S_p} \leqslant n^{1/p'} \|\Gamma\|_{S_\infty} \|A\|_{S_p}.$$
By taking the supremum over $A \in B_p^{m,n}$, the expectation and using (1.17), we deduce that
$$\ell_*(B_p^{m,n}) \le n^{1/p'}\mathbb{E}\, \|\Gamma\|_{S_\infty} \le c\sqrt{m}\; n^{1/p'}$$
which ends the proof of (1.15)

We prove (1.16) in the case $m \ge n \ge 1$. Using the dual Sudakov inequality (1.13) and (1.17) we get that for any $q \in [1, +\infty]$:
$$\varepsilon\sqrt{\log N(B_2^{m,n}, \varepsilon B_q^{m,n})} \leqslant c\,\mathbb{E}\, \|\Gamma\|_{S_q} \leqslant c\, n^{1/q}\mathbb{E}\, \|\Gamma\|_{S_\infty} \leqslant c' n^{1/q}\sqrt{m}.$$
The proof of the case $n \ge m$ is completely similar.

□

1.5. Notes and comments

We focused in this chapter on the study of some very particular concentration inequalities. Of course, there exist different and powerful other type of concentration inequalities. Several books and surveys are devoted to this subject and we refer for example to [**LT91, vdVW96, Led01, BBL04, Mas07**] for the interested reader. The classical references for the study of Orlicz spaces are [**KR61, LT77, LT79, RR91, RR02**].

Tail and moment estimates for Rademacher averages are well understood. Theorem 1.2.4 is due to Montgomery-Smith [**MS90**] and several extensions to the vector valued case are known [**DMS93, MS95**]. The case of sum of independent random variables with logarithmically concave tails has been studied by Gluskin and Kwapien [**GK95**]. For the proof of Theorem 1.2.9, we could have followed a classical probabilistic trick which reduces the proof of the result to the case of Weibull random variables. These variables are defined such that the tails are equals to e^{-t^α}. Hence, the tails are logarithmically concave and the result is a corollary of the results of Gluskin and

Kwapien [**GK95**]. We have presented here an approach which follows the line of [**Tal94**]. Even if the results of Talagrand [**Tal94**] are only written for random variables with densities $c_\alpha e^{-t^\alpha}$, the proofs work in the general context of ψ_α random variables.

Originally, Lemma 1.3.1 is proved in [**JL84**] and the operator is chosen at random in the set of orthogonal projections onto a random k-dimensional subspace of ℓ_2, uniformly according to the Haar measure on the Grassman manifold $\mathcal{G}_{n,k}$.

The classical references for the study of entropy numbers are [**Pie72, Pie80, Pis89, CS90**]. The method of proof of Theorem 1.4.3 has been introduced by Maurey, in particular for studying entropy numbers of operators from ℓ_1^d into a Banach space of type p. This was published in [**Pis81**]. The method was extended and developed by Carl in [**Car85**]. Sudakov inequality 1.12 is due to Sudakov [**Sud71**] while the dual Sudakov inequality 1.13 is due to Pajor and Tomczak-Jaegermann [**PTJ86**]. The proof that we presented follows the lines of Ledoux-Talagrand [**LT91**]. We have made the choice to speak only about Slepian inequality, Lemma 1.4.5. The result of Slepian [**Sle62**] is more general, it tells about distribution inequality. In the context of Lemma 1.4.5, Fernique [**Fer74**] proved that the constant 2 can be replaced by 1. Gordon [**Gor85, Gor87**] extended these results and he proved results not only for the max but also for the min-max of some Gaussian processes. We would like to note that inequality (1.17) is due to Chevet [**Che78**]. It is known that the constant $2\sqrt{2}$ can be replaced by 1 and this is a result of Gordon [**Gor85**]. About the covering numbers of the Schatten balls, Proposition 1.4.6 is due to Pajor [**Paj99**] and more general estimates are proved in that paper.

CHAPTER 2

COMPRESSED SENSING AND GELFAND WIDTHS

2.1. Short introduction to compressed sensing

Compressed Sensing is a quite new framework that enables to get approximate and exact reconstruction of sparse signals from incomplete measurements. The ideas and principles are strongly related to other problems coming from different fields such as approximation theory, in particular to the study of Gelfand and Kolmogorov width of classical Banach spaces (diameter of sections). Since the seventies an important work was done in that direction, in Approximation Theory and in Asymptotic Geometric Analysis (called Geometry of Banach spaces at that time).

It is not in our aim to give here an introduction to compressed sensing, there are many good references for that, but mainly to emphasize some interactions with other fields of mathematics, in particular with asymptotic geometric analysis, random matrices and empirical processes. The possibility of reconstructing any vector from a given subset is highly related to the *complexity* of this subset and in the field of Geometry of Banach spaces, many tools were developed to analyze various concepts of complexity.

In this introduction to compressive sensing, for simplicity, we will consider the real case, real vectors and real matrices. Let $1 \leq n \leq N$ be integers. We are given a rectangular $n \times N$ real matrix A. One should think of $N \gg n$; we have in mind to compress some vectors from $\mathbb{R}^N$ for large N into vectors in $\mathbb{R}^n$. Let $X_1, \ldots, X_N \in \mathbb{R}^n$ be the columns of A and let $Y_1, .., Y_n \in \mathbb{R}^N$ its rows. One has

$$A = \begin{pmatrix} X_1 \cdots\cdots\cdots X_N \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}.$$

We are also given a subset $T \subset \mathbb{R}^N$ of vectors. Now let $x \in T$ be an *unknown* vector. The data one is given are n linear measurements of x (again, think of $N \gg n$)

$$\langle Y_1, x \rangle, \ldots, \langle Y_n, x \rangle$$

or equivalently

$$y = Ax.$$

We wish to *recover* x or more precisely to *reconstruct* x, exactly or approximately, within a given accuracy and in an efficient way (fast algorithm).

2.2. The exact reconstruction problem

Let us first discuss the exact reconstruction question. Let $x \in T$ be unknown and recall that the given data is $y = Ax$. When $N \gg n$, the problem is *ill-posed* because the system $At = y$, $t \in \mathbb{R}^N$ is highly under-determined. Thus if we want to recover x we need some more information on its *nature*. Moreover if we want to recover any x from T, one should have some a priori information on the set T, on its *complexity* whatever it means at this stage. We shall see various parameters of complexity in these notes. The a priori hypothesis that we investigate now is *sparsity*.

2.2.1. Sparsity. — We first introduce some notation. We equip $\mathbb{R}^n$ and $\mathbb{R}^N$ with the canonical scalar product $\langle \cdot, \cdot \rangle$ and Euclidean norm $|\cdot|_2$. We use the notation $|\cdot|$ to denote the cardinality of a set. By B_2^N we denote the unit Euclidean ball and by S^{n-1} its unit sphere.

Definition 2.2.1. — *Let $0 \le m \le N$ be integers. For any $x = (x_1, \dots, x_N) \in \mathbb{R}^N$, denote by $\operatorname{supp} x = \{k : 1 \le k \le N, x_k \neq 0\}$ the subset of non-zero coordinate of x. The vector x is said m-sparse if $|\operatorname{supp} x| \le m$. The subset of m-sparse vectors of $\mathbb{R}^N$ is denoted by $\Sigma_m = \Sigma_m(\mathbb{R}^N)$ and its unit sphere by*

$$S_2(\Sigma_m) = \{x \in \mathbb{R}^N : |x|_2 = 1 \text{ and } |\operatorname{supp} x| \le m\}.$$

Similarly let

$$B_2(\Sigma_m) = \{x \in \mathbb{R}^N : |x|_2 \le 1 \text{ and } |\operatorname{supp} x| \le m\}.$$

Note that Σ_m is not a linear subspace and that $B_2(\Sigma_m)$ is not convex.

Problem 2.2.2. — **The exact reconstruction problem.** *We wish to reconstruct exactly any m-sparse vector $x \in \Sigma_m$ from the given data $y = Ax$. Thus we are looking for a* decoder Δ *such that*

$$\forall x \in \Sigma_m, \qquad \Delta(A, Ax) = x.$$

Claim 2.2.3. — *Linear algebra tells us that such a decoder Δ exists iff*

$$\ker A \cap \Sigma_{2m} = \{0\}.$$

Example 2.2.4. — *Let $m \ge 1$, $N \ge 2m$ and $0 < a_1 < \cdots < a_N = 1$. Let $n = 2m$ and build the Vandermonde matrix $A = (a_j^{i-1}), 1 \le i \le n, 1 \le j \le N$. Clearly all the $2m \times 2m$ minors of A are non singular Vandermonde matrices. Unfortunately it is known that such matrices are ill-conditioned. Therefore reconstructing $x \in \Sigma_m$ from $y = Ax$ is numerically unstable.*

2.2.2. Metric entropy. — As already said, there are many different approaches to seize and measure complexity of a metric space. The most simple is probably to estimate a degree of compactness via the so-called covering and packing numbers.

Since all the metric spaces we will consider here are subsets of normed spaces, we restrict to this setting.

Definition 2.2.5. — *Let B and C be subsets of a vector space and let $\varepsilon > 0$. An ε-net of B by translates of εC is a subset Λ of B such that for every $x \in B$, there exit $y \in \Lambda$ and $z \in C$ such that $x = y + \varepsilon z$. In other words, one has*

$$B \subset \Lambda + \varepsilon C = \bigcup_{x \in \Lambda} (x + \varepsilon C),$$

where $\Lambda + \varepsilon C := \{a + \varepsilon c : a \in \Lambda, c \in C\}$ is the Minkowski sum of the sets Λ and εC. The covering number of B by εC is the smallest cardinality of such an ε-net and is denoted by $N(B, \varepsilon C)$. The function $\varepsilon \to \log N(B, \varepsilon C)$ is called the metric entropy of B by C.

Remark 2.2.6. — *If (B, d) is a metric space, an ε-net of (B, d) is a covering of B by balls of radius ε for the metric d. The covering number is the smallest cardinality of an ε-net and is denoted by $N(B, d, \varepsilon)$. In our setting, the metric d will be defined by a norm with unit ball say C. Then $x + \varepsilon C$ is a ball of radius ε centered at x.*

Let us start by an easy but important fact. Let $C \subset \mathbb{R}^N$ be a symmetric convex body, that is a symmetric convex compact subset of $\mathbb{R}^N$, with non-empty interior (that is, the unit ball of a norm on $\mathbb{R}^N$). Consider a subset $\Lambda \subset C$ of maximal cardinality such that the points of Λ are εC-apart in the sense that:

$$\forall x \neq y, x, y \in \Lambda, \text{ one has } x - y \notin \varepsilon C$$

(recall that $C = -C$). It is clear that Λ is an ε-net of C by εC. Moreover the balls

$$(x + (\varepsilon/2)C)_{x \in \Lambda}$$

of radius $(\varepsilon/2)$ centered at the points of Λ are pairwise disjoint and their union is a subset of $(1 + (\varepsilon/2))\, C$ (this is where convexity is involved). Taking volume of this union, we get that $N(C, \varepsilon C) \leq (1 + (2/\varepsilon))^N$. Let us conclude:

Proposition 2.2.7. — *Let $C \subset \mathbb{R}^N$ be a symmetric convex body (the unit ball of a norm). There exists an ε-net Λ of C by translates of εC such that $|\Lambda| \leq (1 + 2/\varepsilon)^N$. Moreover $\Lambda \subset C \subset (1 - \varepsilon)^{-1} \operatorname{conv}(\Lambda)$.*

Let us prove the moreover part of the Proposition by successive approximation.

Proof. — Since Λ is an ε-net of C by translates of εC, every $z \in C$ can be written as $z = x_0 + \varepsilon z_1$, where $x_0 \in \Lambda$ and $z_1 \in C$. Iterating, it follows that $z = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots$, with $x_i \in \Lambda$, which implies by convexity that $C \subset (1 - \varepsilon)^{-1} \operatorname{conv}(\Lambda)$. □

This gives the first claim:

Claim 2.2.8. — *Covering the unit Euclidean sphere by Euclidean balls of radius ε. One has*

$$\forall\varepsilon \in (0,1), \qquad N(S^{N-1}, \varepsilon B_2^N) \le \left(\frac{3}{\varepsilon}\right)^N.$$

Now, since $S_2(\Sigma_m)$ is the union of spheres of dimension m,

$$N(S_2(\Sigma_m), \varepsilon B_2^N) \le \binom{N}{m} N(S^{m-1}, \varepsilon B_2^m).$$

Using $\binom{N}{m} \le (eN/m)^m$, we get:

Claim 2.2.9. — *Covering the set of sparse unit vectors by Euclidean balls of radius ε: let $1 \le m \le N$ and $\varepsilon \in (0,1)$, then*

$$N(S_2(\Sigma_m), \varepsilon B_2^N) \le \left(\frac{3eN}{m\varepsilon}\right)^m.$$

2.2.3. The ℓ_1-minimization method. — Coming back to the exact reconstruction problem, if we want to solve the system

$$At = y$$

where $y = Ax$ is given and x is m-sparse, it is tempting to test all possible support of the unknown vector x. This is the so-called ℓ_0-method. But there are $\binom{N}{m}$ possible supports, too many to answer the request of a fast algorithm.

A very clever approach was proposed by D. Donoho in [**Don06**] and E. Candes, J. Romberg and T. Tao in [**CRT06**]. This is the convex relaxation of the ℓ_0-method. Let x be the unknown vector. The given data is $y = Ax$. For $t = (t_i) \in \mathbb{R}^N$ denote by

$$|t|_1 = \sum_{i=1}^{N} |t_i|$$

its ℓ_1 norm. The ℓ_1-minimization method (also called *basis-pursuit*) is the following program:

$$(P) \qquad \min_{t\in\mathbb{R}^N} |t|_1 \quad \text{subject to} \quad At = y.$$

This program may be recast as a linear programming by

$$\min \sum_{i=1}^{N} s_i, \text{ subject to } s \ge 0, -s \le t \le s, At = y.$$

Definition 2.2.10. — **The exact reconstruction problem by ℓ_1-minimization.** *We say that the matrix A has the exact reconstruction property of order m by ℓ_1-minimization if for every $x \in \Sigma_m$ the problem*

$$(P) \ \min_{t\in\mathbb{R}^N} |t|_1 \quad \textit{subject to} \quad At = Ax \textit{ has a unique solution equal to } x. \tag{2.1}$$

Note that the above property is not specific to the matrix A but rather a property of its null space. In order to emphasize this point, let us introduce some notation.

For any subset $I \subset [N]$ where $[N] = \{1, \dots, N\}$, let I^c be its complement and for any $x \in \mathbb{R}^N$, let us write x_I for the vector in $\mathbb{R}^N$ with the same coordinates as x for indices in I and 0 for indices in I^c. We are ready for a criterium on the null space.

Proposition 2.2.11. — **The null space property.** *The following are equivalent*

i) *For any $x \in \Sigma_m$, the problem*

$$(P) \qquad \min_{t \in \mathbb{R}^N} |t|_1 \quad \textit{subject to} \quad At = Ax$$

has a unique solution equal to x (that is A has the exact reconstruction property of order m by ℓ_1-minimization)

ii)

$$\forall h \in \ker A, h \neq 0, \forall I \subset [N], |I| \leq m, |h_I|_1 < |h_{I^c}|_1. \tag{2.2}$$

Proof. — On one side, let $h \in \ker A, h \neq 0$ and $I \subset [N], |I| \leq m$. Put $x = -h_I$. Then $x \in \Sigma_m$ and (2.1) implies that $|x + h|_1 > |x|_1$, that is $|h_{I^c}|_1 > |h_I|_1$.

For the reverse implication, suppose that

$$\forall h \in \ker A, h \neq 0, \forall I \subset [N], |I| \leq m, \ |h_I|_1 < |h_{I^c}|_1.$$

Let $x \in \Sigma_m$ and let $I = \mathrm{supp}(x)$. Then $|I| \leq m$ and

$$|x + h|_1 = |x_I + h_I|_1 + |h_{I^c}|_1 > |x_I + h_I|_1 + |h_I|_1 \geq |x|_1.$$

□

Definition 2.2.12. — *We say that A satisfies the null space property of order m if (2.2) is satisfied.*

This property has a nice geometric interpretation. To introduce it, we need some more notation. Let $(e_i)_{1 \leq i \leq N}$ be the canonical basis of $\mathbb{R}^N$. Let ℓ_1^N be the N-dimensional space $\mathbb{R}^N$ equipped with the ℓ_1-norm and B_1^N be its unit ball. Denote also

$$S_1(\Sigma_m) = \{x \in \Sigma_m : |x|_1 = 1\} \quad \text{and} \quad B_1(\Sigma_m) = \{x \in \Sigma_m : |x|_1 \leq 1\}.$$

Let $1 \leq m \leq N$. Any $(m-1)$-dimensional face of B_1^N is of the form $\mathrm{conv}(\{\varepsilon_i e_i : i \in I\})$ with $I \subset [N], |I| = m$ and $(\varepsilon_i) \in \{-1, 1\}^I$, where we denoted by $\mathrm{conv}(\,\cdot\,)$ the convex hull. From the geometric point of view, $S_1(\Sigma_m)$ is the union of all the $(m-1)$-dimensional faces of B_1^N.

Let A be an $n \times N$ matrix and $X_1, \dots, X_N \in \mathbb{R}^n$ be its columns. A polytope $P \subset \mathbb{R}^n$ is said centrally symmetric if $P = -P$. Observe that

$$A(B_1^N) = \mathrm{conv}(\pm X_1, \dots, \pm X_N).$$

Proposition 2.2.11 can be reformulated in the following geometric language:

Proposition 2.2.13. — **The geometry of faces of $A(B_1^N)$.** *Let $1 \le m \le n \le N$. Let A be an $n \times N$ matrix with columns $X_1, \dots, X_N \in \mathbb{R}^n$. Then A satisfies the null space property (2.2) iff one has*

$$\forall I \subset [N],\ 1 \le |I| \le m, \forall (\varepsilon_i) \in \{-1,1\}^I,$$
$$\operatorname{conv}(\{\varepsilon_i X_i \,:\, i \in I\}) \cap \operatorname{conv}(\{\pm X_j \,:\, j \notin I\}) = \emptyset \tag{2.3}$$

Taking advantage of the symmetries, property (2.3) is equivalent to the following:

$$\forall I \subset [N],\ 1 \le |I| \le m, \forall (\varepsilon_i) \in \{-1,1\}^I,$$
$$\operatorname{Aff}(\{\varepsilon_i X_i \,:\, i \in I\}) \cap \operatorname{conv}(\{\pm X_j \,:\, j \notin I\}) = \emptyset \tag{2.4}$$

where $\operatorname{Aff}(\{\varepsilon_i X_i \,:\, i \in I\})$ denotes the affine subspace generated by $\{\varepsilon_i X_i \,:\, i \in I\}$.

The proof of this equivalence is left as exercise.

Definition 2.2.14. — *Let $1 \le m \le n$. A centrally symmetric polytope $P \subset \mathbb{R}^n$ is said to be symmetric m-neighborly if every set of m of its vertices, containing no antipodal pair, is the set of all vertices of some face of P.*

Note that any centrally symmetric polytope is symmetric 1-neighborly. This property becomes non-trivial when $m \ge 2$. In that case, observe that $A(B_1^N) = \operatorname{conv}(\pm X_1, \dots, \pm X_N)$ has $\{\pm X_1, \dots, \pm X_N\}$ as vertices AND is symmetric m-neighborly iff A maps every $(m-1)$-dimensional face of B_1^N onto a $(m-1)$ dimensional face of $\operatorname{conv}(\pm X_1, \dots, \pm X_N)$. From (2.3) and (2.4), we deduce the following criterium.

Proposition 2.2.15. — [**Don05**] *Let $1 \le m \le N$. The matrix A has the null space property of order m iff its columns $\pm X_1, \dots, \pm X_N$ are the $2N$ vertices of $A(B_1^N)$ and $A(B_1^N)$ is m-neighborly.*

Consider the quotient map

$$Q : \ell_1^N \longrightarrow \ell_1^N / \ker A$$

If A has maximum rank n, then $\ell_1^N / \ker A$ is n-dimensional. Denote by $\|\,.\,\|$ the quotient norm on $\ell_1^N / \ker A$ defined by

$$\|Qx\| = \min_{h \in \ker A} |x + h|_1.$$

Property (2.1) implies that Q is norm preserving on Σ_m. Since $\Sigma_{\lfloor m/2 \rfloor} - \Sigma_{\lfloor m/2 \rfloor} \subset \Sigma_m$, Q is an isometry on $\Sigma_{\lfloor m/2 \rfloor}$ equipped with the ℓ_1 metric. In other words,

$$\forall x, y \in \Sigma_{\lfloor m/2 \rfloor} \quad \|Qx - Qy\| = |x - y|_1.$$

As it is classical in approximation theory, we can take benefit of such an isometric embedding to bound the complexity by comparing the metric entropy of the source space $(\Sigma_{\lfloor m/2 \rfloor}, \ell_1^N)$ with the target space, which lives in a much lower dimension.

The following lemma is a well known fact on packing.

Lemma 2.2.16. — *There exists a family Λ of subset of $[N]$ with cardinality m such that for every $I, J \in \Lambda, I \ne J, |I \cap J| \le \lfloor m/2 \rfloor$ and $|\Lambda| \ge \left\lfloor \frac{N}{32em} \right\rfloor^{\lfloor m/2 \rfloor}$.*

Proof. — We use successive enumeration of the subsets of cardinality m and exclusion of wrong items. Without loss of generality, assume that $m/2$ is an integer. Pick any subset I_1 of $\{1,...,N\}$ of cardinality m and throw away all subsets J of $\{1,...,N\}$ of size m such that the Hamming distance $|I_1 \vartriangle J| \le m/2$, where Δ stands for the symmetrical difference. There are at most

$$\sum_{k=m/2}^{m} \binom{m}{k}\binom{N-m}{m-k}$$

such subsets and since $m \le N/2$ we have

$$\sum_{k=m/2}^{m} \binom{m}{k}\binom{N-m}{m-k} \le 2^m \max_{m/2\le k\le m} \binom{N-m}{m-k} \le 2^m \binom{N}{m/2}.$$

Now, select a new subset I_2 of size m from the remaining subsets. Repeating this argument, we obtain a family $\Lambda = \{I_1, I_2, \dots, I_p\}$, $p = |\Lambda|$, of subsets of cardinality m which are $(m/2)$-separated in the Hamming metric and such that

$$|\Lambda| \ge \left\lfloor \binom{N}{m} \Big/ 2^m \binom{N}{m/2} \right\rfloor .$$

Since for $m \le N/2$ we have $\left(\frac{N}{2m}\right)^m \le \binom{N}{m} \le \left(\frac{eN}{m}\right)^m$, we get that

$$|\Lambda| \ge \left\lfloor \frac{(N/2m)^m}{2^m (Ne/(m/2))^{(m/2)}} \right\rfloor \ge \left\lfloor \left(\frac{N}{32em}\right)^{m/2} \right\rfloor \ge \left\lfloor \frac{N}{32em} \right\rfloor^{\lfloor m/2\rfloor}$$

which concludes the proof. □

Let Λ be the family constructed in the previous Lemma. For every $I \in \Lambda$, define $x(I) = \frac{1}{m}\sum_{i\in I} e_i$. Then $x(I) \in S_1(\Sigma_m)$ and for every $I, J \in \Lambda, I \ne J$

$$|x(I) - x(J)|_1 = 2\left(1 - \frac{|I\cap J|}{m}\right) \geqslant 2\left(1 - \frac{[m/2]}{m}\right) \ge 1.$$

If the matrix A has the exact reconstruction property of order m, then

$$\forall I, J \in \Lambda\ I \ne J, \quad \|Q(x(I)) - Q(x(J))\| = \|Q(x(I) - x(J))\| = |x(I) - x(J)|_1 \ge 1.$$

On one side $|\Lambda| \ge \left\lfloor C\frac{N}{[m/2]} \right\rfloor^{[m/2]}$, but on the other side, the cardinality of the set $(Q(x(I)))_{I\in\Lambda}$ cannot be too big. Indeed, it is a subset of the unit ball $Q(B_1^N)$ of the quotient space and we already saw that the maximum cardinality of a set of points of a unit ball which are 1-apart is less than 3^n. It follows that

$$\lfloor N/32em \rfloor^{\lfloor m/2\rfloor} \le 3^n.$$

Proposition 2.2.17. — *If the matrix A has the exact reconstruction property of order m by ℓ_1-minimization, then*

$$m\log(cN/m) \le Cn.$$

where $C, c > 0$ are universal constants.

Whatever is the matrix A, this proposition gives an upper bound on the size m of sparsity of vectors such that any vectors from Σ_m can be exactly reconstructed by ℓ_1-minimization method (see [**FPRU10**] for more details and [**LN06**] where the analogous problem for neighborliness is studied).

2.3. The restricted isometry property

So far, we do not know of any "simple" condition in order to check whether a matrix A satisfies the exact reconstruction property (2.1).

Let us start with the following definition which was introduced in [**CT05**] and plays an important role in compressed sensing.

Definition 2.3.1. — *Let A be a $n \times N$ matrix. For any $0 \le p \le N$, the restricted isometry constant of order p of A is the smallest number $\delta_p = \delta_p(A)$ such that*

$$(1-\delta_p)|x|_2^2 \le |Ax|_2^2 \le (1+\delta_p)|x|_2^2$$

for all p-sparse vectors $x \in \mathbb{R}^N$. Let $\delta \in (0,1)$. We say that the matrix A satisfies the Restricted Isometry Property of order p with parameter δ, shortly $\mathrm{RIP}_p(\delta)$, if $0 \leqslant \delta_p(A) < \delta$.

The relevance of the Restricted Isometry parameter for the reconstruction property was for instance revealed in [**CT06**], [**CT05**], where it was shown that if

$$\delta_m(A) + \delta_{2m}(A) + \delta_{3m}(A) < 1$$

then the encoding matrix A has the exact reconstruction property of order m. This result was improved in [**Can08**] as follows:

Theorem 2.3.2. — *Let $1 \leqslant m \leqslant N/2$. Let A be an $n \times N$ matrix. If*

$$\delta_{2m}(A) < \sqrt{2} - 1.$$

then A satisfies the exact reconstruction property of order m by ℓ_1-minimization.

Remark 2.3.3. — *The constant $\sqrt{2}-1$ was recently improved in* [**FL09**].

For simplicity, we shall discuss an other parameter involving a more general concept which was introduced in [**FL09**]. The aim is to relax the constraint $\delta_{2m}(A) < \sqrt{2}-1$, in Theorem 2.3.2 and still get an exact reconstruction property of a certain order by ℓ_1-minimization.

Definition 2.3.4. — *Let $0 \le p \le n$ be integers and let A be an $n \times N$ matrix. Define $\alpha_p = \alpha_p(A)$ and $\beta_p = \beta_p(A)$ as the best constants such that*

$$\forall x \in \Sigma_p, \quad \alpha_p |x|_2 \le |Ax|_2 \le \beta_p |x|_2.$$

Thus $\beta_p = \max\{|Ax|_2 : x \in \Sigma_p \ |x|_2 = 1\}$ *and* $\alpha_p = \min\{|Ax|_2 : x \in \Sigma_p \ |x|_2 = 1\}$. *Now we define the parameter $\gamma_p = \gamma_p(A)$ by*

$$\gamma_p(A) := \frac{\beta_p(A)}{\alpha_p(A)}\cdot$$

In other words, let $I \subset [N]$ with $|I| = p$. Denote by A^I the $n \times p$ matrix with columns $(X_i)_{i \in I}$ obtained by extracting from A the columns X_i with index $i \in I$. Then α_p is the smallest singular value among all the block matrices A^I with $|I| = p$, and β_p is the largest. In other words, denoting by $B^\top$ the transposed matrix of a matrix B and $\lambda_{min}((A^I)^\top A^I)$, respectively $\lambda_{max}((A^I)^\top A^I)$, the smallest and largest eigenvalues of $(A^I)^\top A^I$, then

$$\alpha_p^2 = \alpha_p^2(A) = \min_{I \subset [N], |I|=p} \lambda_{min}((A^I)^\top A^I)$$

whereas

$$\beta_p^2 = \beta_p^2(A) = \max_{I \subset [N], |I|=p} \lambda_{max}((A^I)^\top A^I).$$

Of course, if A satisfies $\mathrm{RIP}_p(\delta)$, then $\gamma_p(A)^2 \leq \frac{1+\delta}{1-\delta}$. The concept of RIP is not homogenous, in the sense that A may satisfy $\mathrm{RIP}_p(\delta)$ but not a multiple of A. One can "rescale" the matrix to satisfy a Restricted Isometry Property. This does not ensure that the new matrix, say A' will satisfy $\delta_{2m}(A') < \sqrt{2} - 1$ and will not allow us to conclude to an exact reconstruction from Theorem 2.3.2 (compare with Corollary 2.4.4 in the next section). Also note that the Restricted Isometry Property for A can be written

$$\forall x \in S_2(\Sigma_p) \quad \left| |Ax|_2^2 - 1 \right| \leq \delta$$

expressing a form of concentration property of $|Ax|_2$. Such a property may not be satisfied despite the fact that A does satisfy the exact reconstruction property of order p by ℓ_1-minimization (see Example 2.5.7).

2.4. The geometry of the null space

Let $1 \leqslant m \leqslant p \leqslant N$. Let $h \in \mathbb{R}^N$ and let $\varphi = \varphi_h : [N] \to [N]$ be a one-to-one mapping associated to a non-increasing rearrangement of $(|h_i|)$; in others words $|h_{\varphi(1)}| \geq |h_{\varphi(2)}| \geq \cdots \geq |h_{\varphi(N)}|$. Denote by $I_1 = \varphi_h(\{1, \ldots, m\})$ (a subset of indices of the largest m coordinates of $(|h_i|)$) then by $I_2 = \varphi_h(\{m+1, \ldots, m+p\})$ (a subset of indices of the next p largest coordinates of $(|h_i|)$) and iterate $I_{k+1} = \varphi_h(\{m+(k-1)p+1, \ldots, m+kp\})$, for $k \geq 2$, as far as $m + kp \leq N$, in order to partition $[N]$ in subsets of cardinality p, except the first one, I_1 which has cardinality m and the last one, which may have cardinality less than p.

Claim 2.4.1. — *Let $h \in \mathbb{R}^N$. Suppose that $1 \leqslant m \leqslant p \leqslant N$ and $N \geq m + p$. With the previous notation, we have*

$$\forall k \geq 2, \quad |h_{I_{k+1}}|_2 \leq \frac{1}{\sqrt{p}} |h_{I_k}|_1$$

and

$$\sum_{k \geq 3} |h_{I_k}|_2 \leq \frac{1}{\sqrt{p}} \; |h_{I_1^c}|_1.$$

Proof. — Let $k \geq 1$. We have

$$|h_{I_{k+1}}|_2 \leq \sqrt{|I_{k+1}|}\ \max\{|h_i| \,:\, i \in I_{k+1}\}$$

and

$$\max\{|h_i| \,:\, i \in I_{k+1}\} \leq \min\{|h_i| \,:\, i \in I_k\} \leq |h_{I_k}|_1/|I_k|.$$

We deduce that

$$\forall k \geq 1 \quad |h_{I_{k+1}}|_2 \leq \frac{\sqrt{|I_{k+1}|}}{|I_k|}\,|h_{I_k}|_1.$$

Adding up these inequalities for all $k \geq 2$, for which $\sqrt{|I_{k+1}|}/|I_k| = 1/\sqrt{p}$, this prove the claim. □

We are ready for the main Theorem of this section

Theorem 2.4.2. — *Let $1 \leqslant m \leqslant p \leqslant N$ and $N \geq m+p$. Let A be an $n \times N$ matrix. Then*

$$\forall h \in \ker A, h \neq 0, \quad \forall I \subset [N], |I| \leq m, \quad |h_I|_1 < \sqrt{\frac{m}{p}}\ \gamma_{2p}(A)\ |h_{I^c}|_1 \tag{2.5}$$

and $\forall h \in \ker A, h \neq 0,\ \forall I \subset [N], |I| \leq m,$

$$|h|_2 \leq \sqrt{\frac{1+\gamma_{2p}^2(A)}{p}}\ |h_{I_1^c}|_1 \leq \sqrt{\frac{1+\gamma_{2p}^2(A)}{p}}\ |h|_1. \tag{2.6}$$

In particular,

$$\operatorname{diam}(\ker A \cap B_1^N) \leq \sqrt{\frac{1+\gamma_{2p}^2(A)}{p}}$$

where $\operatorname{diam}(B) = \sup_{x\in B} |x|_2$.

Proof. — Let $h \in \ker A$, $h \neq 0$ and organize the coordinates of h as above. By definition of α_{2p} (see 2.3.4), one has

$$|h_{I_1} + h_{I_2}|_2 \leq \frac{1}{\alpha_{2p}}|A(h_{I_1} + h_{I_2})|_2.$$

Using that $h \in \ker A$ we obtain

$$|h_{I_1} + h_{I_2}|_2 \leq \frac{1}{\alpha_{2p}}|A(h_{I_1} + h_{I_2} - h)|_2 \leq \frac{1}{\alpha_{2p}}|A(-\sum_{k\geq 3} h_{I_k})|_2.$$

Then from the definition of β_p and γ_p (2.3.4), using Claim 2.4.1, we get

$$|h_{I_1}|_2 < |h_{I_1} + h_{I_2}|_2 \leq \frac{\beta_p}{\alpha_{2p}}\sum_{k\geq 3}|h_{I_k}|_2 \leq \frac{\gamma_{2p}(A)}{\sqrt{p}}|h_{I_1^c}|_1. \tag{2.7}$$

This inequality is strict because $h_{I_2} = 0$ would imply $h_{I_1^c} = 0$ and subsequently $h_{I_1} = 0$. To conclude the proof of (2.5), note that for any subset $I \subset [N], |I| \leq m$, $|h_{I_1^c}|_1 \leq |h_{I^c}|_1$ and $|h_I|_1 \leq |h_{I_1}|_1 \leq \sqrt{m}|h_{I_1}|_2$.

To prove (2.6), we start from

$$|h|_2^2 = |h - h_{I_1} - h_{I_2}|_2^2 + |h_{I_1} + h_{I_2}|_2^2$$

Using Claim (2.4.1), the first term satisfies

$$|h - h_{I_1} - h_{I_2}|_2 \leq \sum_{k \geq 3} |h_{I_k}|_2 \leq \frac{1}{\sqrt{p}} |h_{I_1^c}|_1.$$

From(2.7), $|h_{I_1} + h_{I_2}|_2 \leq \frac{\gamma_{2p}(A)}{\sqrt{p}} |h_{I_1^c}|_1$ and putting things together, we derive that

$$|h|_2 \leq \sqrt{\frac{1 + \gamma_{2p}^2(A)}{p}} \, |h_{I_1^c}|_1 \leq \sqrt{\frac{1 + \gamma_{2p}^2(A)}{p}} \, |h|_1.$$

□

Remark 2.4.3. — *Relation (2.5) was proved in* [**FL09**] *with a better numerical constant.*

From relation (2.5) and the null space property (Proposition 2.2.11) we derive the following corollary.

Corollary 2.4.4. — *Let $1 \leqslant p \leqslant N/2$. Let A be a $n \times N$ matrix. If $\gamma_{2p}(A) \leq \sqrt{p}$, then A satisfies the exact reconstruction property of order m by ℓ_1-minimization with*

$$m = \left\lfloor p / \gamma_{2p}^2(A) \right\rfloor.$$

The objective now is to find p such that γ_{2p} is bounded by some numerical constant. This means that we need a uniform control of the smallest and largest singular values of all block matrices of A with $2p$ columns. By Corollary 2.4.4 this is a sufficient condition for the exact reconstruction of m-sparse vectors by ℓ_1-minimization with $m \sim p$. When $|Ax|_2$ satisfies good concentration properties, the restricted isometry property is more adapted. In this situation, $\gamma_{2p} \sim 1$. When the isometry constant δ_{2p} is sufficiently small, A satisfies the exact reconstruction of m-sparse vectors with $m = p$ (see Theorem 2.3.2).

Conversely, an estimate of diam $(\ker A \cap B_1^N)$ gives an estimate of the size of sparsity of vectors which can be reconstructed by ℓ_1-minimization.

Proposition 2.4.5. — *Let A be an $n \times N$ matrix and let $0 < D \leq 1$. If*

$$\operatorname{diam}(\ker A \cap B_1^N) < D$$

then the matrix A satisfies the exact reconstruction property of order m by ℓ_1-minimization with

$$m = \left\lfloor \frac{1}{D^2} \right\rfloor.$$

Proof. — Let $h \in \ker A$ and $I \subset [N], |I| \leq m$. By our assumption, we have that

$$\forall h \in \ker A, h \neq 0 \quad |h|_2 < D\,|h|_1.$$

Now $|h_I|_1 \leq \sqrt{m}\ |h_I|_2 \leq \sqrt{m}\ |h|_2 < D\sqrt{m}\,|h|_1$. Thus if $m \leqslant \frac{1}{D^2}$ and $h \neq 0$, then $|h_I|_1 < |h_{I^c}|_1$. We conclude using the null space property (Proposition 2.2.11). □

This study leads to the notion of Gelfand widths.

Definition 2.4.6. — *Let T be a bounded subset of a normed space E. Let $k \geq 0$ be an integer. Its k-th Gelfand width is defined as*

$$d^k(T, E) := \inf_G \sup_{x \in G \cap T} \|x\|_E,$$

where $\| \, . \, \|_E$ denotes the norm of E and where the infimum is taken over all linear subspaces G of codimension $\leq k$.

Remark 2.4.7. — *A different notation is used in Banach space and Operator Theory. Let $u : X \longrightarrow Y$ be an operator between two normed spaces X and Y. The k-th Gelfand number is defined by*

$$c_k(u) = \inf_G \sup_{x \in G \cap B_X} \|u(x)\|_Y,$$

where B_X denotes the unit ball of X and the infimum is taken over all subspaces G of X with codimension $< k$. Thus

$$c_{k+1}(u) = d^k(u(B_X), Y).$$

If F is a linear space ($\mathbb{R}^N$ for instance) equipped with two norms defining two normed spaces X and Y and if $I_{X \to Y}$ is the identity mapping of F considered from X to Y, then

$$d^k(B_X, Y) = c_{k+1}(I_{X \to Y}).$$

Let $0 < D \leq 1$, Proposition 2.4.5 shows that if

$$d^n(B_1^N, \ell_2^N) < D$$

then there exists a matrix A satisfying the exact reconstruction property of order $m = \lfloor 1/D^2 \rfloor$. Thus if A is any $n \times N$ matrix so that the n-Gelfand width of $I_{\ell_1^N \to \ell_2^N}$ is reached at $G = \ker A$, then A satisfies the exact reconstruction property of order m by ℓ_1-minimization with $m = \lfloor 1/d^n(B_1^N, \ell_2^N)^2 \rfloor$. A famous result from [**Kaš77**] improved in [**Glu83**] reads as follows:

Theorem 2.4.8. — [**GG84**] *Let $1 \leqslant n \leqslant N$. Then*

$$d^n(B_1^N, \ell_2^N) \leq \frac{c_1}{\sqrt{n}} \sqrt{\log c_2 \frac{N}{n}} \tag{2.8}$$

where $c_1, c_2 > 0$ are universal constants.

We deduce from Theorem 2.4.8 that there exists a matrix A satisfying the exact reconstruction property of order

$$\lfloor c_1 n / \log(c_2 N/n) \rfloor$$

where c_1, c_2 are universal constants. This result was proved in [**CRT06**], [**CT06**] and [**Don06**] using compressive sensing methods. From Proposition 2.2.17 it is the optimal order.

Remark 2.4.9. — *The result of* [**Kaš77**] *was proved using Kolmogorov widths (dual to the Gelfand widths) and with a non-optimal power of the logarithm (power 3/2 instead of 1/2 later improved in* [**GG84**]*). The upper bound of Kolmogorov widths was obtained via random matrices with i.i.d. Bernoulli entries, whereas* [**Glu83**] *and* [**GG84**] *use some properties of random Gaussian matrices. It was also shown in* [**GG84**] *that the estimates of Gelfand numbers given in Theorem 2.4.8 are optimal.*

Remark 2.4.10. — *The method of proofs of results of this section follow* [**CT05**], [**CDD09**],[**FL09**],[**FPRU10**] *and* [**KT07**].

2.5. Gaussian random matrices satisfy a RIP

So far, we did not give yet any example of matrices satisfying the exact reconstruction property of order m with large m. It is known that with high probability Gaussian matrices do satisfy this property.

2.5.1. The subgaussian Ensemble. — We consider a probability $\mathbb{P}$ on the space of real $n \times N$ matrices $M(n,N)$ satisfying the following concentration inequality: there exists an absolute constant c_0 such that for every $x \in \mathbb{R}^N$ we have

$$\mathbb{P}\left(\{A : \left||Ax|_2^2 - |x|^2\right| \geq t|x|_2^2\}\right) \leq 2e^{-c_0 t^2 n} \qquad \text{for all } 0 < t \leq 1. \tag{2.9}$$

Definition 2.5.1. — *For a real random variable Z we define the ψ_2-norm by*

$$\|Z\|_{\psi_2} = \inf\left\{s > 0 \,:\, \mathbb{E}\exp\left(|Z|/s\right)^2 \leqslant e\right\}.$$

We say that a random vector $Y \in \mathbb{R}^N$ is isotropic if

$$\forall y \in \mathbb{R}^N, \quad \mathbb{E}|\langle Y, y\rangle|^2 = |y|_2^2.$$

A random vector $Y \in \mathbb{R}^N$ satisfies a ψ_2 estimate with constant α (shortly Y is ψ_2 with constant α) if

$$\forall y \in \mathbb{R}^N, \quad \|\langle Y, y\rangle\|_{\psi_2} \leqslant \alpha|y|_2.$$

It is well-known that a real random variable Z is ψ_2 (with some constant) if and only if it satisfies a subgaussian tail estimate. In particular if Z is a real random variable with $\|Z\|_{\psi_2} \leq \alpha$, then for every $t \geq 0$,

$$\mathbb{P}(|Z| \geq t) \leq e^{-(t/\alpha)^2+1}$$

This ψ_2 property can also be characterized by the growth of moments. Well known examples are Gaussian random variables and bounded centered random variables (see Chapter 1 for details).

Let us consider $Y_1, \ldots, Y_n \in \mathbb{R}^N$ be i.i.d. isotropic random vectors which are ψ_2 with the same constant α. Let A be the matrix with $Y_1, \ldots, Y_n \in \mathbb{R}^N$ as rows. We consider the probability $\mathbb{P}$ on the space of matrices $M(n,N)$ induced by the mapping $(Y_1, \ldots, Y_n) \to A$.

Let us recall Bernstein's inequality (see Chapter 1). For $y \in S^{N-1}$ consider the average of n independent copies of the random variable $\langle Y_1, y\rangle^2$. Then for every $t > 0$,

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{n}\langle Y_i, y\rangle^2 - 1\right| > t\right) \leqslant 2\exp\left(-cn\min\left\{\frac{t^2}{\alpha^4}, \frac{t}{\alpha^2}\right\}\right),$$

where c is an absolute constant. Note that since $\mathbb{E}\langle Y_1, y\rangle^2 = 1$, one has $\alpha \geqslant 1$ and that $\left|\frac{Ax}{\sqrt{n}}\right|_2^2 = \frac{1}{n}\sum_{i=1}^{n}\langle Y_i, y\rangle^2$. This shows the next claim:

Claim 2.5.2. — *Let $Y_1, \ldots, Y_n \in \mathbb{R}^N$ be i.i.d. isotropic random vectors that are ψ_2 with constant α. Let $\mathbb{P}$ be the probability induced on $M(n, N)$. Then for every $x \in \mathbb{R}^N$ we have*

$$\mathbb{P}\left(\left|\left|\frac{A}{\sqrt{n}}x\right|_2^2 - |x|^2\right| \geq t|x|_2^2\right) \leq 2e^{-\frac{c}{\alpha^4}t^2 n} \qquad \text{for all } 0 < t \leq 1$$

where $c > 0$ is an absolute constant.

The most important examples for us of model of random matrices satisfying (2.9) are matrices with independent subgaussian rows, normalized in the right way.

Example 2.5.3. — *Some classical examples:*

- $Y_1, \ldots, Y_n \in \mathbb{R}^N$ *are independent copies of the Gaussian vector* $Y = (g_1, \ldots, g_N)$ *where the* g_i*'s are independent* $\mathcal{N}(0, 1)$ *Gaussian variables*
- $Y_1, \ldots, Y_n \in \mathbb{R}^N$ *are independent copies of the random sign vector* $Y = (\varepsilon_1, \ldots, \varepsilon_N)$ *where the* ε_i*'s are independent, symmetric* ± 1 *(Bernoulli) random variables*
- $Y_1, \ldots, Y_n \in \mathbb{R}^N$ *are independent copies of a random vector uniformly distributed on the Euclidean sphere of radius* $\sqrt{N}$.

In all these cases the (Y_i) are isotropic with a ψ_2 constant α, for a suitable $\alpha \geqslant 1$. For the last case see e.g. [**LT91**]. For more details on Orlicz norm and probabilistic inequalities used here see Chapter 1.

2.5.2. Sub-Gaussian matrices are almost norm preserving on Σ_m. — An important feature of Σ_m and its subsets $S_2(\Sigma_m)$ and $B_2(\Sigma_m)$ is their peculiar structure: the two last are the unions of the unit spheres, and unit balls, respectively, supported on m-dimensional coordinate subspaces of $\mathbb{R}^N$.

We begin with the following well known lemma (see Chapter 1) which allows to step up from an ε-net to the whole unit sphere.

Lemma 2.5.4. — *Let $m \geq 1$ be an integer, $\|\,.\,\|$ be a semi-norm in $\mathbb{R}^m$ and $\varepsilon \in (0, 1/3)$. Let $\Lambda \subset S^{m-1}$ be an ε-net of S^{m-1} by εB_2^m. If*

$$\forall y \in \Lambda \quad 1 - \varepsilon \leq \|y\| \leq 1 + \varepsilon,$$

then

$$\forall y \in S^{m-1} \quad \frac{1 - 3\varepsilon}{1 - \varepsilon} \leq \|y\| \leq \frac{1 + \varepsilon}{1 - \varepsilon}.$$

Proof. — Proposition 2.2.7 implies that $S^{m-1} \subset (1-\varepsilon)^{-1}\mathrm{conv}\,\Lambda$. Therefore we have

$$\sup_{y\in S^{m-1}} \|y\| \le (1+\varepsilon)(1-\varepsilon)^{-1}.$$

To get a lower estimate, write any $y \in S^{m-1}$ as $y = y_1 + \varepsilon y_2$, with $y_1 \in \Lambda$ and $y_2 \in B_2^m$. Then $\|y\| \ge \|y_1\| - \varepsilon\|y_2\| \ge (1-\varepsilon) - \varepsilon(1+\varepsilon)(1-\varepsilon)^{-1} = (1-3\varepsilon)/(1-\varepsilon)$ which proves the claim. □

We can give now a simple proof that subgaussian matrices satisfy the exact reconstruction property of order m by ℓ_1-minimization with large m (see [**BDDW08**] and [**MPTJ08**]).

Theorem 2.5.5. — *Let $\mathbb{P}$ be a probability on $M(n,N)$ satisfying (2.9). Then there exist positive constants c_1, c_2 and c_3 depending only on c_0 from (2.9), for which the following holds: with probability at least $1-2\exp(-c_3 n)$, A satisfies the exact reconstruction property of order m by ℓ_1-minimization with*

$$m = \left\lfloor \frac{c_1 n}{\log(c_2 N/n)} \right\rfloor.$$

Moreover, A satisfies $RIP_m(\delta)$ for any $\delta \in (0,1)$ with $m \sim c\delta^2 n/\log(CN/\delta^3 n)$ where c and C depend only on c_0.

Proof. — Let $\varepsilon \in (0,1/3)$ to be fixed later. Let $1 \le p \le N/2$. Let y_i, $i = 1,2,\dots,n$, be the rows of A. For every subset I of $[N]$ of cardinality $2p$ let Λ_I be an ε-net of the unit sphere of $\mathbb{R}^I$ by εB_2^I satisfying Claim 2.2.8, that is with $|\Lambda_I| \le \left(\frac{3}{\varepsilon}\right)^{2p}$. Apply Lemma 2.5.4 to the semi-norm

$$||y|| := \left(\frac{1}{n}\sum_{i=1}^n \langle y_i, y\rangle^2\right)^{1/2}$$

on the unit sphere of $\mathbb{R}^I$. Let $\Lambda \subset \mathbb{R}^N$ be the union of all these Λ_I for $|I| = 2p$.

Suppose that

$$\sup_{y\in\Lambda}\Big|\frac{1}{n}\sum_{i=1}^n(\langle y_i, y\rangle^2 - 1)\Big| \le \varepsilon,$$

then

$$\forall y \in S_2(\Sigma_{2p}) \qquad \frac{1-3\varepsilon}{1-\varepsilon} \le \big(\sum_{i=1}^n \langle y_i, y\rangle^2\big)^{1/2} \le \frac{1+\varepsilon}{1-\varepsilon}.$$

Note that there is nothing random in that relation. This is why we change the notation of the rows from (Y_i) to (y_i). Thus checking how well the matrix A defined by the rows (y_i) is acting on Σ_{2p} is reduced to checking that on the finite set Λ. Now recall that $|\Lambda| \le \binom{N}{2p}\left(\frac{3}{\varepsilon}\right)^{2p} \le \exp\left(2p\log\left(\frac{3eN}{2p\varepsilon}\right)\right)$.

Given a probability $\mathbb{P}$ on $M(n,N)$ satisfying (2.9), and using a union bound estimate, we get that the inequalities

$$\forall x \in S_2(\Sigma_{2p}) \quad \frac{1-3\varepsilon}{1-\varepsilon} \le |Ax|_2 \le \frac{1+\varepsilon}{1-\varepsilon}$$

hold with probability at least

$$1 - |\Lambda| e^{-c_0 \varepsilon^2 n} \geq 1 - \exp\left(2p \log\left(\frac{3eN}{2p\varepsilon}\right)\right) e^{-c_0\varepsilon^2 n} \geq 1 - e^{-c_0 \varepsilon^2 n/2}$$

whenever

$$2p \log\left(\frac{3eN}{2p\varepsilon}\right) \leq c_0 \varepsilon^2 n/2.$$

Assuming these inequalities, we get

$$\gamma_{2p}(A) \leq (1+\varepsilon)/(1-3\varepsilon)$$

with probability larger than $1 - \exp(-c_0\varepsilon^2 n/2)$. From Corollary 2.4.4, we deduce that A satisfies the exact reconstruction property of order m by ℓ_1-minimization with

$$m = \lfloor p/\gamma_{2p}(A)^2 \rfloor.$$

This gives the announced result by fixing ε (say $\varepsilon = 1/4$) and solving $2p \log\left(\frac{3eN}{2p\varepsilon}\right) \leq c_0 \varepsilon^2 n/2$. □

Remark 2.5.6. — *Concerning $RIP_m(\delta)$ a better estimate $m \sim c\delta^2 n/\log(CN/\delta^2 n)$ will be shown later in Theorem 2.7.4.*

The strategy that we used in the proof of Theorem 2.5.5 is the following:

- *discretization*: discretization of the set Σ_{2p}, it is a net argument
- *concentration*: $|Ax|_2$ concentrates around its mean for each individual x of the net
- *union bound*: concentration should be good enough to balance the cardinality of the net and to conclude to a uniform concentration on the net of $|Ax|_2$ around its mean
- *from the net to the whole set*, that is checking RIP, is obtained by Lemma 2.5.4.

Such a strategy is very classical in Approximation Theory, see [**Kaš77**] and in Banach space theory where it has played an important role in quantitative version of Dvoretsky's theorem on almost spherical sections, see [**FLM77**].

We conclude this section by an example of an $n \times N$ matrix A which is a good compressed sensing matrix but none of the $n \times N$ matrices with the same kernel as A satisfy a restricted isometry property of any order ≥ 1 with good parameter. As we already noticed, if A has parameter γ_p, one can find $t_0 > 0$ and rescale the matrix so that $\delta_p(t_0 A) = \gamma_p^2 - 1/\gamma_p^2 + 1 \in [0, 1)$. In this example, γ_p is large, $\delta_p(t_0 A) \sim 1$ and one cannot deduce any result about exact reconstruction from Theorem 2.3.2.

Example 2.5.7. — *Let $1 \leq n \leq N$. Let $\delta \in (0,1)$. There exists an $n \times N$ matrix A such that for any $p \leq cn/\log(CN/n)$, one has $\gamma_{2p}(A)^2 \leq c'(1-\delta)^{-1}$. Thus, for any $m \leq c"(1-\delta)n/\log(CN/n)$, the matrix A satisfies the exact reconstruction property of m-sparse vectors by ℓ_1-minimization. Nevertheless, for any $n \times n$ matrix U, the restricted isometry constant of order 1 of UA satisfies, $\delta_1(UA) \geq \delta$ (think of $\delta \geq 1/2$). Here, $C, c, c', c" > 0$ are universal constants.*

The proof is left as exercise.

2.6. RIP for other "simple" subsets: almost sparse vectors

As already mentioned, various "random projection" operators may act as "almost norm preserving" on "thin" subsets of the sphere. We analyze a simple structure of the metric entropy of a set $T \subset \mathbb{R}^N$ in order that, with high probability, (a multiple of) Gaussian or subgaussian matrices act almost like an isometry on T. This will apply to a more general case than sparse vectors. We follow the lines of [**MPTJ08**].

Theorem 2.6.1. — *Consider a probability on the space of $n \times N$ matrices satisfying*

$$\forall x \in \mathbb{R}^N \qquad \mathbb{P}\left(\left||Ax|_2^2 - |x|_2^2\right| \geq t|x|_2^2\right) \leq 2e^{-c_0 t^2 n} \qquad \text{for all } 0 < t \leq 1.$$

Let $T \subset S^{N-1}$ and $0 < \varepsilon < 1/15$. Assume the following:

(i) *There exists an ε-net $\Lambda \subset S^{N-1}$ of T satisfying $|\Lambda| \leq \exp(c_0\varepsilon^2 n/2)$*
(ii) *There exists a subset Λ' of εB_2^N such that $(T - T) \cap \varepsilon B_2^N \subset 2\,\mathrm{conv}\,\Lambda'$ and $|\Lambda'| \leq \exp(c_0 n/2)$.*

Then with probability at least $1 - 2\exp(-c_0\varepsilon^2 n/2)$, one has that for all $x \in T$,

$$1 - 15\varepsilon \leq |Ax|_2^2 \leq 1 + 15\varepsilon. \tag{2.10}$$

Proof. — The idea is to show that A acts on Λ in an almost norm preserving way. This is the case because the degree of concentration of each variable $|Ax|_2^2$ around its mean defeats the cardinality of Λ. Then one shows that $A(\mathrm{conv}\,\Lambda')$ is contained in a small ball - thanks to a similar argument.

Consider the set Ω of matrices A such that

$$||Ax_0|_2 - 1| \leq \left||Ax_0|_2^2 - 1\right| \leq \varepsilon \qquad \text{for all } \ x_0 \in \Lambda, \tag{2.11}$$

and

$$|Az|_2 \leq 2\varepsilon \qquad \text{for all } \ z \in \Lambda'. \tag{2.12}$$

From our assumption (2.9), i) and ii)

$$\mathbb{P}(\Omega) \geq 1 - \exp(-c_0\varepsilon^2 n/2) - \exp(-c_0 n/2) \geq 1 - 2\exp(-c_0\varepsilon^2 n/2).$$

Let $x \in T$ and consider $x_0 \in \Lambda$ such that $|x - x_0|_2 \leq \varepsilon$. Then for every $A \in \Omega$

$$|Ax_0|_2 - |A(x - x_0)|_2 \leq |Ax|_2 \leq |Ax_0|_2 + |A(x - x_0)|_2\,.$$

Since $x - x_0 \in (T - T) \cap \varepsilon B_2^N$, property ii) and (2.12) give that

$$|A(x - x_0)|_2 \leq 2 \sup_{z \in \mathrm{conv}\,\Lambda'} |Az|_2 = 2 \sup_{z \in \Lambda'} |Az|_2 \leq 4\varepsilon. \tag{2.13}$$

Combining this with (2.11) implies that $1 - 5\varepsilon \leq |Ax|_2 \leq 1 + 5\varepsilon$. The proof is completed by squaring. □

2.6.1. Approximate reconstruction of almost sparse vectors. — After analyzing the restricted isometry property for thin sets like Σ_m, we consider the ℓ_1-minimization method in order to get approximate reconstruction of vectors which are not far from sparse vectors.

Let A be a $n \times N$ matrix and $x \in \mathbb{R}^N$. Let us recall the ℓ_1-minimization method:

$$(P) \qquad \min_{t \in \mathbb{R}^N} |t|_1 \quad \text{subject to} \quad At = y$$

where $y = Ax$. Let $x^\sharp$ be a minimizer of (P) and let $h = x^\sharp - x \in \ker A$. For any subset $I \subset [N]$, observe that

$$|x|_1 \geq |x+h|_1 = |x_I + h_I|_1 + |x_{I^c} + h_{I^c}|_1 \geq |x_I|_1 - |h_I|_1 + |h_{I^c}|_1 - |x_{I^c}|_1$$

and thus

$$|h_{I^c}|_1 \leq |h_I|_1 + 2|x_{I^c}|_1.$$

Let $1 \leq m \leq p \leq n$ and assume that $|I| \leq m$. Using (2.5) we get

$$\left(1 - \sqrt{\frac{m}{p}}\ \gamma_{2p}(A)\right) |h|_1 \leq 2 \left(1 + \sqrt{\frac{m}{p}}\ \gamma_{2p}(A)\right) |x_{I^c}|_1.$$

Imposing $\sqrt{\frac{m}{p}}\ \gamma_{2p}(A) \leq 1/2$, we get

$$|h|_1 \leq 6\, |x_{I^c}|_1,$$

and using (2.6) we obtain

$$|h|_2 \leq \sqrt{\frac{1+\gamma_{2p}^2(A)}{p}}\, |h|_1 \leq 6 \sqrt{\frac{1+\gamma_{2p}^2(A)}{p}} |x_{I^c}|_1.$$

This yields

$$|h|_2 \leq 6 \sqrt{\frac{1}{p} + \frac{1}{4m}}\ |x_{I^c}|_1 \leq 3\sqrt{5} \sqrt{\frac{1}{m}}\ |x_{I^c}|_1.$$

The minimum of $|x_{I^c}|_1$ over all subsets I, $|I| \leq m$, is obtained when I is the support of the m-largest coordinates of x. The vector x_I is henceforth the best m-sparse approximation of x.

From this analysis, we derive the following proposition that we formulate here in terms of γ_p rather than in terms of constants of isometry (see [**Can08**] and [**CDD09**] for more details).

Proposition 2.6.2. — *Let A be a $n \times N$ matrix and let $1 \leq m \leq p \leq n$. Let $x \in \mathbb{R}^N$ and let $x^\sharp$ be a minimizer of*

$$(P) \qquad \min_{t \in \mathbb{R}^N} |t|_1 \quad \textit{subject to} \quad At = Ax.$$

Assume that $m \leq p/4\gamma_{2p}(A)^2$, then

$$|x - x^\sharp|_1 \leq 6\, |x - x_I|_1$$

and

$$|x - x^\sharp|_2 \leq 3\sqrt{5}\, \frac{1}{\sqrt{m}}\ |x_{I^c}|_1$$

where $I \subset [N], |I| = m$ and x_I is the best m-sparse approximation of x.

Let us give an immediate application of this proposition. Let $0 < p < 1$ and consider

$$T = B_{p,\infty}^N = \left\{x = (x_1, \dots, x_N) \in \mathbb{R}^N \,:\, |\{i \,:\, |x_i| \geq s\}| \leq s^{-p} \text{ for all } \ s > 0\right\}$$

the unit ball of $\ell_{p,\infty}^N$, the so-called weak ℓ_p^N space. Observe that for any $x \in B_{p,\infty}^N$, one has $x_i^* \leq 1/i^{1/p}$, for every $i \geq 1$, where $(x_i^*)_{i=1}^N$ is a non-increasing rearrangement of $(|x_i|)_{i=1}^N$. Let $I \subset [N], |I| = m$ and let x_I be the best m-sparse approximation of x. Note that $\sum_{i>m} i^{-1/p} \leq (1/p-1)^{-1} m^{1-1/p}$. Under the assumption of Proposition 2.6.2 and with the same notation, we get that if $m \leq p/4\gamma_{2p}(A)^2$ and if $x^\sharp$ is a minimizer of (P), then

$$|x - x^\sharp|_2 \leq 3\sqrt{5} \ \left((1/p) - 1\right)^{-1} m^{1/2-1/p}.$$

2.6.2. Reducing the computation of Gelfand widths by truncation. — We begin with a simple principle which reduces the computation of Gelfand widths to the width of a truncated set. This method goes back to [**Glu83**].

Definition 2.6.3. — *We say that a subset $T \subset \mathbb{R}^N$ is star-shaped in 0 or shortly, star-shaped, if $\lambda T \subset T$ for every $0 \leq \lambda \leq 1$. Let $\rho > 0$ and let $T \subset \mathbb{R}^N$ be star-shaped, we denote by T_ρ the subset*

$$T_\rho = T \cap \rho S^{N-1}.$$

Recall that $\operatorname{diam}(S) = \sup_{x \in S} |x|_2$.

Lemma 2.6.4. — *Let $\rho > 0$ and let $T \subset \mathbb{R}^N$ be star-shaped. Then for any linear subspace $E \subset \mathbb{R}^N$ such that $E \cap T_\rho = \emptyset$ we have* $\operatorname{diam}(E \cap T) < \rho$.

Proof. — If $\operatorname{diam}(E \cap T) \geqslant \rho$, there would be $x \in E \cap T$ of norm greater or equal to ρ. Since T is star-shaped, so is $E \cap T$ and thus $x/|x|_2 \in E \cap T_\rho$; a contradiction. □

This easy lemma will be a useful tool in the next sections and in Chapter 5. The subspace E will be the kernel of our matrix A, ρ a parameter that we try to estimate as small as possible such that $\ker A \cap T_\rho = \emptyset$, that is such that $Ax \neq 0$ for all $x \in T$ with $|x| = \rho$. This will be in particular the case if A or a multiple of A acts on T_ρ in an almost norm preserving way.

With Theorem 2.6.1 in mind, we apply this plan to subsets T like Σ_m.

Corollary 2.6.5. — *Let $\mathbb{P}$ be a probability on $M(n, N)$ satisfying (2.9). Consider a star-shaped set $T \subset \mathbb{R}^N$ and let $\rho > 0$. Assume that $\frac{1}{\rho} T_\rho \subset S^{N-1}$ satisfies the hypothesis of Theorem 2.6.1 for some $0 < \varepsilon < 1/15$. Then $\operatorname{diam}(\ker A \cap T) < \rho$, with probability at least $1 - 2\exp(-cn)$ where $c > 0$ is an absolute constant.*

2.6.3. Application to subsets related to ℓ_p unit balls. — To illustrate this method, we consider some examples of set T, for $0 < p < 2$:

- the unit ball of ℓ_1^N, denoted by B_1^N
- the unit ball $B_p^N = \{x \in \mathbb{R}^N : \sum_1^N |x_i|^p \leq 1\}$ of ℓ_p^N, $0 < p < 1$
- the unit ball $B_{p,\infty}^N = \left\{x \in \mathbb{R}^N : |\{i : |x_i| \geq s\}| \leq s^{-p} \text{ for all } \ s > 0\right\}$ of $\ell_{p,\infty}^N$ (weak ℓ_p^N), for $0 < p < 1$.

Note that for $0 < p < 1$, the "unit ball" B_p^N is not convex and that $B_p^N \subset B_{p,\infty}^N$, so that for estimating Gelfand widths, we can restrict to the balls $B_{p,\infty}^N$.

We need two lemmas. The first lemma comes from [**MPTJ08**] and uses the following classical fact:

Claim 2.6.6. — *Let $(a_i), (b_i)$ two sequences of positive numbers such that (a_i) is non-increasing. Then the sum $\sum a_i b_{\pi(i)}$ is maximized over all permutations π of the index set, if $b_{\pi(1)} \geq b_{\pi(2)} \geq \dots$.*

Lemma 2.6.7. — *Let $0 < p < 1$, $1 \leqslant m \leqslant N$ and set $r = (1/p - 1)m^{1/p-1/2}$. Then, for every $x \in \mathbb{R}^N$,*

$$\sup_{z \in rB_{p,\infty}^N \cap B_2^N} \langle x, z\rangle \leqslant 2 \left(\sum_{i=1}^m {x_i^*}^2\right)^{1/2},$$

where $(x_i^*)_{i=1}^N$ is a non-increasing rearrangement of $(|x_i|)_{i=1}^N$. Equivalently,

$$rB_{p,\infty}^N \cap B_2^N \subset 2\,\mathrm{conv}\,(S_2(\Sigma_m)). \tag{2.14}$$

Moreover one has

$$\sqrt{m} B_1^N \cap B_2^N \subset 2\,\mathrm{conv}\,(S_2(\Sigma_m)). \tag{2.15}$$

Proof. — We treat only the case of $B_{p,\infty}^N$, $0 < p < 1$. The case of B_1^N is similar. Note first that if $z \in B_{p,\infty}^N$, then for any $i \geq 1$, $z_i^* \leq 1/i^{1/p}$, where $(z_i^*)_{i=1}^N$ is a non-increasing rearrangement of $(|z_i|)_{i=1}^N$. Using Claim 2.6.6 we get that for any $r > 0, m \geq 1$ and $z \in rB_{p,\infty}^N \cap B_2^N$,

$$\begin{aligned}
\langle x, z\rangle &\leqslant \left(\sum_{i=1}^m {x_i^*}^2\right)^{1/2} + \sum_{i>m} \frac{r x_i^*}{i^{1/p}} \\
&\leqslant \left(\sum_{i=1}^m {x_i^*}^2\right)^{1/2} \left(1 + \frac{r}{\sqrt{m}} \sum_{i>m} \frac{1}{i^{1/p}}\right) \\
&\leqslant \left(\sum_{i=1}^m {x_i^*}^2\right)^{1/2} \left(1 + \left(\frac{1}{p} - 1\right)^{-1} \frac{r}{m^{1/p-1/2}}\right).
\end{aligned}$$

By the definition of r, this completes the proof. □

The second lemma shows that $m^{1/p-1/2} B_{p,\infty}^N \cap S^{N-1}$ is well approximated by vectors on the sphere with short support.

Lemma 2.6.8. — *Let $0 < p < 2$ and $\delta > 0$, and set $\varepsilon = 2(2/p-1)^{-1/2}\delta^{1/p-1/2}$. Let $1 \le m \le N$. Then $S_2(\Sigma_{\lceil m/\delta \rceil})$ is an ε-net of $m^{1/p-1/2}B^N_{p,\infty} \cap S^{N-1}$ with respect to the Euclidean metric.*

Proof. — Let $x \in m^{1/p-1/2}B^N_{p,\infty} \cap S^{N-1}$ and assume without loss of generality that $x_1 \ge x_2 \ge \ldots \ge x_n \ge 0$. Define z' by $z'_i = x_i$ for $1 \le i \le \lceil m/\delta \rceil$ and $z'_i = 0$ otherwise. Then

$$|x - z'|_2^2 = \sum_{i > m/\delta} |x_i|^2 \le m^{2/p-1} \sum_{i>m/\delta} 1/i^{2/p} < (2/p-1)^{-1}\, \delta^{2/p-1}.$$

Thus $1 \ge |z'|_2 \ge 1 - (2/p-1)^{-1/2}\, \delta^{1/p-1/2}$. Put $z = z'/|z'|_2$. Then $z \in S_2(\Sigma_{\lceil m/\delta \rceil})$ and

$$|z - z'|_2 = 1 - |z'|_2 \le (2/p-1)^{-1/2}\, \delta^{1/p-1/2}.$$

By the triangle inequality $|x - z|_2 < \varepsilon$, completing the proof. □

The preceding lemmas are used to show that the hypothesis of Theorem 2.6.1 are satisfied for an appropriate choice of T and ρ. Before that, property ii) of Theorem 2.6.1, brings us to the following definition.

Definition 2.6.9. — *We say that a subset T of $\mathbb{R}^N$ is quasi-convex with constant $a \ge 1$, if T is star-shaped and $T + T \subset 2aT$.*

Let us note the following easy fact.

Claim 2.6.10. — *Let $0 < p < 1$, then $B^N_{p,\infty}$ and B^N_p are quasi-convex with constant $2^{(1/p)-1}$.*

We come up now with the main claim:

Claim 2.6.11. — *Let $0 < p < 1$ and $T = B^N_{p,\infty}$. Then $(1/\rho)T_\rho$ satisfies properties i) and ii) of Theorem 2.6.1 with*

$$\rho = C_p \left(\frac{n}{\log(cN/n)} \right)^{1/p-1/2}$$

where C_p depends only on p and $c > 0$ is an absolute constant. Moreover if $T = B^N_1$, then $(1/\rho)T_\rho$ satisfies properties i) and ii) of Theorem 2.6.1 with

$$\rho = \left(\frac{c_1 n}{\log(c_2 N/n)} \right)^{1/2}$$

where c_1, c_2 are positive absolute constants.

Proof. — We consider only the case of $T = B^N_{p,\infty}$, $0 < p < 1$. The case of B^N_1 is similar. Since the mechanism has already been developed in details, we will only indicate the different steps. Fix $\varepsilon_0 = 1/20$. To get i) we use Lemma 2.6.8 with $\varepsilon = \varepsilon_0/2$ and δ obtained from the equation $\varepsilon_0/2 = 2(2/p-1)^{-1/2}\delta^{1/p-1/2}$. Let

$1 \le m \le N$. We get that $S_2(\Sigma_{\lceil m/\delta \rceil})$ is an $(\varepsilon_0/2)$-net of $m^{1/p-1/2} B_{p,\infty}^N \cap S^{N-1}$ with respect to the Euclidean metric. Set $m' = \lceil m/\delta \rceil$. By Claim 2.2.9, we have

$$N(S_2(\Sigma_{m'}), \frac{\varepsilon_0}{2} B_2^N) \le \left(\frac{3eN}{m'(\varepsilon_0/2)} \right)^{m'} = \left(\frac{6eN}{m'\varepsilon_0} \right)^{m'}.$$

Thus, by the triangle inequality, we have

$$N(m^{1/p-1/2} B_{p,\infty}^N \cap S^{N-1}, \varepsilon_0 B_2^N) \le \left(\frac{6eN}{m'\varepsilon_0} \right)^{m'}$$

so that

$$N(m^{1/p-1/2} B_{p,\infty}^N \cap S^{N-1}, \varepsilon_0 B_2^N) \le \exp(c_0 n/2)$$

whenever

$$\left(\frac{6eN}{m'\varepsilon_0} \right)^{m'} \le \exp(c_0 n/2).$$

This shows that under this condition on m' (that is on m), the set $m^{1/p-1/2} B_{p,\infty}^N \cap S^{N-1}$ satisfies i).

In order to tackle ii), recall that $B_{p,\infty}$ is quasi-convex with constant $2^{1/p-1}$ (Claim 2.6.10. By symmetry, we have

$$B_{p,\infty}^N - B_{p,\infty}^N \subset 2^{1/p} B_{p,\infty}^N.$$

Let $r = (1/p - 1) m^{1/p-1/2}$. From Lemma 2.6.7, one has

$$r B_{p,\infty}^N \cap B_2^N \subset 2 \operatorname{conv} S_2(\Sigma_m).$$

As we saw previously,

$$N(S_2(\Sigma_m), \frac{1}{2} B_2^N) \le \left(\frac{3eN}{m(1/2)} \right)^{m} = \left(\frac{6eN}{m} \right)^{m}$$

and by Claim 2.2.9 there exists a subset $\Lambda' \subset S^{N-1}$ with $|\Lambda'| \le N(S_2(\Sigma_m), \frac{1}{2} B_2^N)$ such that $S_2(\Sigma_m) \subset 2 \operatorname{conv} \Lambda'$. We arrive at

$$\varepsilon_0 2^{-1/p} \left(r B_{p,\infty}^N - r B_{p,\infty}^N \right) \cap \varepsilon_0 B_2^N \subset \varepsilon_0 \left(r B_{p,\infty}^N \cap B_2^N \right) \subset$$
$$\subset 4\varepsilon_0 \operatorname{conv} \Lambda' \subset 2 \operatorname{conv} (\Lambda' \cup -\Lambda').$$

Therefore $\varepsilon_0 2^{-1/p} r B_{p,\infty}^N \cap S^{N-1}$ satisfies ii) whenever $2\,(6eN/m)^m \le \exp(c_0 n/2)$.

Finally $\varepsilon_0 2^{-1/p} r B_{p,\infty}^N \cap S^{N-1}$ satisfies i) and ii) whenever the two conditions on m are verified, that is when $cm \log(CN/m) \le c_0 n/2$ where $c, C > 0$ are absolute constants. We compute m and r and set $\rho = \varepsilon_0 2^{-1/p} r$ to conclude. □

Now we can apply Corollary 2.6.5, to conclude

Theorem 2.6.12. — *Let $\mathbb{P}$ be a probability satisfying (2.9) on the space of $n \times N$ matrices and let $0 < p < 1$. There exist c_p depending only on p, c' depending on c_0 and an absolute constant c such that the set Ω of $n \times N$ matrices A satisfying*

$$\operatorname{diam} \left(\ker A \cap B_p^N \right) \leqslant \operatorname{diam} \left(\ker A \cap B_{p,\infty}^N \right) \leqslant c_p \left(\frac{\log(cN/n)}{n} \right)^{1/p-1/2}.$$

has probability at least $1 - \exp(-c'n)$.

In particular, if $A \in \Omega$ *and if* $x', x \in B_{p,\infty}^n$ *are such that* $Ax' = Ax$ *then*

$$|x' - x|_2 \leqslant c_p' \left(\frac{\log(c_1 N/n)}{n}\right)^{1/p-1/2}.$$

An analogous result holds for the ball B_1^N.

2.7. An other complexity measure

We introduce a new parameter, $\ell_*(T)$ which is a *complexity measure* of a set $T \subset \mathbb{R}^N$. Let

$$\ell_*(T) = \mathbb{E} \sup_{t \in T} \left| \sum_{i=1}^N g_i t_i \right|, \tag{2.16}$$

where $t = (t_i)_{i=1}^N \in \mathbb{R}^N$ and $g_1, ..., g_N$ are independent $N(0,1)$ Gaussian random variables. This kind of parameter plays an important role in empirical processes (see Chapter 1) and in Geometry of Banach spaces (see [**Pis98**]).

The following result was proved in [**MPTJ08**] (Corollary 2.7).

Theorem 2.7.1. — *Let* $1 \leq n \leq N$ *and* $0 < \delta < 1$. *Let* Y *be an isotropic* ψ_2 *random vector on* $\mathbb{R}^N$ *with constant* α, *set* $Y_1, \dots, Y_n$ *to be independent copies of* Y, *put* A *the matrix with rows* $Y_1, \dots, Y_n$ *and let* $T \subset S^{N-1}$. *If* n *satisfies*

$$n \geq (c'\,\alpha^4/\delta^2)\,\ell_*(T)^2,$$

then with probability at least $1 - \exp(-\bar{c}\,\delta^2 n/\alpha^4)$, *one has for all* $t \in T$,

$$1 - \delta \leq \frac{|At|_2^2}{n} \leq 1 + \delta, \tag{2.17}$$

where $c', \bar{c} > 0$ *are absolute constants.*

Let us explain the meaning of Theorem 2.7.1, and for simplicity, assume that α is an absolute constant as in the situation of Gaussian or Bernoulli random vectors. The parameter $\ell_*(T)$ in this context, measures how probabilistic bounds on the concentration of each individual random variable of the form $|At|_2^2$ around its mean can be combined to form a bound that holds uniformly for every $t \in T$. Theorem 2.7.1 says that as long as $n \geqslant c\,\ell_*^2(T)/\delta^2$, the random operator $A/\sqrt{n}$ maps with overwhelming probability all the points of T in an almost norm preserving way.

We will not give here the proof of this result (see Theorem 3.2.1 in Chapter 3). Instead we will show how it applies to some sets T.

Theorem 2.7.2. — *There exist* $c_1, \bar{c}$ *such that the following holds: Let* N, δ, Y *and* A *be as in Theorem 2.7.1. Fix* $1 \leqslant n \leqslant N$, *let* $T \subset S^{N-1}$ *and assume that* $T \subset 2\,\mathrm{conv}\,\Lambda$ *for some* $\Lambda \subset B_2^N$ *with* $|\Lambda| \leq \exp(c_1 \delta^2 n/\alpha^4)$. *Then with probability at least* $1 - \exp(-\bar{c}\,\delta^2 n/\alpha^4)$, *for all* $t \in T$,

$$1 - \delta \leq \frac{|At|_2^2}{n} \leq 1 + \delta. \tag{2.18}$$

Remark 2.7.3. — *Constant 2 in the inclusion $T \subset 2\,\mathrm{conv}\,\Lambda$ is not significant.*

Proof. — The main point in the proof is that if $T \subset 2\,\mathrm{conv}\,\Lambda$, $\Lambda \subset B_2^n$ and we have a reasonable control of $|\Lambda|$, then $\ell_*(T)$ can be bounded from above. The rest is a direct application of Theorem 2.7.1. Note that the hypothesis is simpler than in theorem 2.6.1.

Let $c', \bar{c} > 0$ be constants from Theorem 2.7.1. It is well known (see Chapter 3) that there exists an absolute constant $c'' > 0$ such that for every $\Lambda \subset B_2^n$,

$$\ell_*(\mathrm{conv}\,\Lambda) = \ell_*(\Lambda) \leqslant c'' \sqrt{\log(|\Lambda|)},$$

and since $T \subset 2\,\mathrm{conv}\,\Lambda$,

$$\ell_*(T) \leq 2\ell_*(\mathrm{conv}\,\Lambda) \leqslant 2c'' \left(c(\delta^2/\alpha^4)n\right)^{1/2}.$$

Choosing $c_1 = 1/4c'c''^2$ we get (2.18) by applying Theorem 2.7.1. □

Using Theorem 2.7.2 we give another proof of the fact that subgaussian matrices are good sensing matrices.

Theorem 2.7.4. — *Let $1 \leq n \leq N$ and $0 < \delta < 1$. Let Y be an isotropic ψ_2 random vector on $\mathbb{R}^N$ with constant α, let $Y_1, \ldots, Y_n$ to be independent copies of Y, put A the matrix with rows $Y_1, \ldots, Y_n$. Then with probability at least $1 - \exp(-c_3 n)$*

$$(1-\delta)|t|^2 \leq \frac{|At|_2^2}{n} \leq (1+\delta)|t|^2 \tag{2.19}$$

holds for all $t \in \Sigma_m$ provided

$$m \leq c_1 n / \log(N/c_2 n)$$

where $c_1, c_2, c_3 > 0$ are of the form $c\delta^2/\alpha^4$ for some absolute constant c. In particular, the random matrix A satisfies $RIP_m(\delta)$ for $m = \lfloor c_1 n/\log(N/c_2 n)\rfloor$ with high probability.

Remark 2.7.5. — *Observe that the dependence in δ is of the form $c\delta^2/\alpha^4$ and is better than what was obtained in Theorem 2.5.5.*

CHAPTER 3

CHAINING

The restricted isometry property has been introduced in Chapter 2 in order to provide a simple way of showing that a $n \times N$ matrix A satisfies the exact reconstruction property. Indeed, if A is a $n \times N$ matrix such that for every $2m$-sparse vector $x \in \mathbb{R}^N$,

$$(1-\delta_{2m})|x|_2^2 \leqslant |Ax|_2^2 \leqslant (1+\delta_{2m})|x|_2^2$$

where $\delta_{2m} < \sqrt{2}-1$ then A satisfies the exact reconstruction property of order m by ℓ_1-minimization (cf. Chapter 2). In particular, if A is a random matrix with rows vectors $Y_1, \ldots, Y_n$, this property can be translated in terms of an empirical processes property since

$$\delta_{2m} = \sup_{x \in S_2(\Sigma_{2m})} \Big| \frac{1}{n} \sum_{i=1}^{n} \langle Y_i, x \rangle^2 - 1 \Big|. \tag{3.1}$$

If we show an upper bound on the supremum (3.1) smaller than $\sqrt{2}-1$, this will prove that the matrix A has the exact reconstruction property of order m by ℓ_1-minimization. In Chapter 2, matrices in the subgaussian Ensemble was showed to satisfy the restricted isometry property (with high probability) thanks to a technique called the epsilon-net argument. In this chapter, we present a technique called the chaining method used to obtain upper bounds on the supremum of stochastic processes.

3.1. The chaining method

The chaining mechanism is a technique used to obtain upper bounds on the supremum $\sup_{t \in T} X_t$ of a stochastic process $(X_t)_{t \in T}$ indexed by a set T. These upper bounds are usually expressed in terms of some metric complexity measure of T.

One key idea behind the chaining method is the trade-off between the deviation or concentration estimates of the increments of the process $(X_t)_{t \in T}$ and the complexity of T endowed with a metric structure connected with the stochastic process $(X_t)_{t \in T}$.

As an introduction, we show an upper bound on the supremum $\sup_{t \in T} X_t$ in terms of an entropy integral known as the *Dudley entropy integral*. This entropy integral is based on some metric quantities of T that were introduced in Chapter 1 and that we recall now.

Definition 3.1.1. — *Let (T, d) be a semi-metric space (that is for every x, y and z in T, $d(x, y) = d(y, x)$ and $d(x, y) \leqslant d(x, z) + d(z, y)$). For every $\varepsilon > 0$, the ε-covering number $N(T, d, \varepsilon)$ of (T, d) is the minimal number of balls for the semi-metric d of radius ε needed to cover T. The entropy is the logarithm of the ε-covering number as a function of ε.*

We develop the chaining argument under a subgaussian assumption on the increments of the process $(X_t)_{t \in T}$ saying that for every $s, t \in T$ and $u > 0$,

$$\mathbb{P}\Big[|X_s - X_t| > u d(s, t)\Big] \leqslant 2 \exp(-cu^2), \tag{3.2}$$

where d is a semi-metric on T and c is an absolute positive constant. To avoid some technical complications that are less important from our point of view, we will only consider processes indexed by finite sets T. To handle more general sets one may study the random variables $\sup_{T' \subset T: T' \text{ is finite}} \sup_{t \in T'} |X_t|$ or $\sup_{T' \subset T: T' \text{ is finite}} \sup_{t,s \in T'} |X_t - X_s|$ which suffices for our goals.

Theorem 3.1.2. — *There exist absolute constants c_0, c_1, c_2 and c_3 for which the following holds. Let (T, d) be a semi-metric space and assume that $(X_t)_{t \in T}$ is a stochastic process with increments satisfying the subgaussian condition (3.2). Then, for every $v \geqslant c_0$, with probability greater than $1 - c_1 \exp(-c_2 v^2)$*

$$\sup_{s,t \in T} |X_t - X_s| \leqslant c_3 v \int_0^\infty \sqrt{\log N(T, d, \varepsilon)} d\varepsilon.$$

In particular,

$$\mathbb{E} \sup_{s,t \in T} |X_t - X_s| \leqslant c_3 \int_0^\infty \sqrt{\log N(T, d, \varepsilon)} d\varepsilon.$$

Proof. — Put $\eta_{-1} = \operatorname{diam}(T, d)$ and for every integer $i \geqslant 0$ set

$$\eta_i = \inf \big\{ \eta > 0 : N(T, d, \eta) \leqslant 2^{2^i} \big\}.$$

Let $(T_i)_{i \geqslant 0}$ be a sequence of subsets of T where T_0 is a subset of T containing only one element and for every $i \geqslant 0$, by definition of η_i, we take T_{i+1} as a subset of T of cardinality smaller than $2^{2^{i+1}}$ such that

$$T \subset \bigcup_{x \in T_{i+1}} \Big(x + \eta_i B_d \Big),$$

where B_d is the unit ball associated with the semi-metric d. For every $t \in T$ and integer i, put $\pi_i(t)$ a nearest point to t in T_i. In particular, $d(t, \pi_i(t)) \leqslant \eta_{i-1}$.

Since T is finite, then for every $t \in T$,

$$X_t - X_{\pi_0(t)} = \sum_{i=0}^{\infty} X_{\pi_{i+1}(t)} - X_{\pi_i(t)}. \tag{3.3}$$

Let $i \in \mathbb{N}$ and $t \in T$. By the subgaussian assumption (3.2), for every $u > 0$, with probability greater than $1 - 2\exp(-cu^2)$,

$$|X_{\pi_{i+1}(t)} - X_{\pi_i(t)}| \leqslant ud(\pi_{i+1}(t), \pi_i(t)) \leqslant u(\eta_{i-1} + \eta_i) \leqslant 2u\eta_{i-1}. \tag{3.4}$$

To get this result uniformly over every links $(\pi_{i+1}(t), \pi_i(t))$ for $t \in T$ at level i, we use an union bound (note that there are at most $|T_{i+1}||T_i| \leqslant 2^{3.2^i}$ such links): with probability greater than $1 - 2|T_{i+1}||T_i|\exp(-cu^2) \geqslant 1 - 2\exp\big(3.2^i \log 2 - cu^2\big)\big)$, for every $t \in T$

$$|X_{\pi_{i+1}(t)} - X_{\pi_i(t)}| \leqslant 2u\eta_{i-1}.$$

To balance the "complexity" of the set of "links" with our deviation estimate, we take $u = v2^{i/2}$, where v is larger than $\sqrt{(6\log 2)/c}$. Thus, for the level i, we obtain with probability greater than $1 - 2\exp\big(-(c/2)v^2 2^i\big)$, for all $t \in T$,

$$|X_{\pi_{i+1}(t)} - X_{\pi_i(t)}| \leqslant 2v2^{i/2}\eta_{i-1},$$

for every v larger than an absolute constant.

By (3.3) and summing over all levels $i \in \mathbb{N}$, we have with probability greater than $1 - 2\sum_{i=0}^{\infty} \exp\big(-(c/2)v^2 2^i\big) \geqslant 1 - c_1\exp(-c_2 v^2)$, for every $t \in T$,

$$|X_t - X_{\pi_0(t)}| \leqslant 2v\sum_{i=0}^{\infty} 2^{i/2}\eta_{i-1} = 2^{3/2}v\sum_{i=-1}^{\infty} 2^{i/2}\eta_i. \tag{3.5}$$

Observe that if $i \in \mathbb{N}$ and $\eta < \eta_i$ then $N(T,d,\eta) > 2^{2^i}$. Hence $N(T,d,\eta) \geqslant 2^{2^i} + 1$ and thus

$$\sqrt{\log(1 + 2^{2^i})}(\eta_i - \eta_{i+1}) \leqslant \int_{\eta_{i+1}}^{\eta_i} \sqrt{\log N(T,d,\eta)}d\eta,$$

and since $\log(1 + 2^{2^i}) \geqslant 2^i \log 2$ then summing over all $i \geqslant -1$,

$$\sqrt{\log 2}\sum_{i=-1}^{\infty} 2^{i/2}(\eta_i - \eta_{i+1}) \leqslant \int_0^{\eta_{-1}} \sqrt{\log N(T,d,\eta)}d\eta$$

and

$$\sum_{i=-1}^{\infty} 2^{i/2}(\eta_i - \eta_{i+1}) = \sum_{i=-1}^{\infty} 2^{i/2}\eta_i - \sum_{i=0}^{\infty} 2^{(i-1)/2}\eta_i \geqslant \Big(1 - \frac{1}{\sqrt{2}}\Big)\sum_{i=-1}^{\infty} 2^{i/2}\eta_i.$$

This proves that

$$\sum_{i=-1}^{\infty} 2^{i/2}\eta_i \leqslant c_3\int_0^{\infty} \sqrt{\log N(T,d,\eta)}d\eta. \tag{3.6}$$

We conclude that, for every v larger than $\sqrt{(6\log 2)/c}$, with probability greater than $1 - c_1\exp(-c_2 v^2)$, we have

$$\sup_{t\in T}|X_t - X_{\pi_0(t)}| \leqslant c_4 v\int_0^{\infty} \sqrt{\log N(T,d,\eta)}d\eta.$$

Integrating the tail estimate,

$$\begin{aligned}\mathbb{E}\sup_{t\in T}|X_t - X_{\pi_0(t)}| &= \int_0^\infty \mathbb{P}\Big[\sup_{t\in T}|X_t - X_{\pi_0(t)}| > u\Big]du \\ &\leqslant c_5\int_0^\infty \sqrt{\log N(T,d,\varepsilon)}d\varepsilon.\end{aligned}$$

Finally, since $|T_0| = 1$, it follows that, for every $t, s \in T$,

$$|X_t - X_s| \leqslant |X_t - X_{\pi_0(t)}| + |X_s - X_{\pi_0(s)}|$$

and the theorem is showed. □

In the case of a stochastic process with subgaussian increments (cf. condition (3.2)), the entropy integral

$$\int_0^\infty \sqrt{\log N(T,d,\varepsilon)}d\varepsilon$$

is called the Dudley entropy integral.

A careful look at the previous proof reveals one potential source of looseness. At each level of the chaining mechanism, we used a uniform bound (depending only on the level) to control each link. Instead, one can use "individual" bounds for every link rather than the worst at every level. This idea is the basis of what is now called the generic chaining. The natural metric complexity measure coming out of this method is the γ_2-functional which is now introduced.

Definition 3.1.3. — *Let (T,d) be a semi-metric space. A sequence $(T_s)_{s\geqslant 0}$ of subsets of T is admissible if $|T_0| \leqslant 1$ and $|T_s| \leqslant 2^{2^s}$ for every $s \geqslant 1$. The γ_2-functional of (T,d) is*

$$\gamma_2(T,d) = \inf_{(T_s)}\sup_{t\in T}\Big(\sum_{s=0}^{\infty} 2^{s/2}d(t,T_s)\Big)$$

where the infimum is taken over all admissible sequences $(T_s)_{s\in\mathbb{N}}$ and $d(t,T_s) = \min_{y\in T_s} d(t,y)$ for every $t \in T$ and $s \in \mathbb{N}$.

We note that the γ_2-functional is upper bounded by the Dudley entropy integral:

$$\gamma_2(T,d) \leqslant c_0\int_0^\infty \sqrt{\log N(T,d,\varepsilon)}d\varepsilon, \tag{3.7}$$

where c_0 is an absolute positive constant. Indeed, we construct an admissible sequence $(T_s)_{s\in\mathbb{N}}$ in the following way: let T_0 be a subset of T containing one element and for every $s \in \mathbb{N}$, let T_{s+1} be a subset of T of cardinality smaller than $2^{2^{s+1}}$ such that for every $t \in T$ there exists $x \in T_{i+1}$ satisfying $d(t,x) \leqslant \eta_s$, where η_s is defined by

$$\eta_s = \inf\big(\eta > 0 : N(T,d,\eta) \leqslant 2^{2^s}\big).$$

Inequality (3.7) follows from (3.6) and

$$\sup_{t\in T}\Big(\sum_{s=0}^{\infty} 2^{s/2}d(t,T_s)\Big) \leqslant \sum_{s=0}^{\infty} 2^{s/2}\sup_{t\in T} d(t,T_s) \leqslant \sum_{s=0}^{\infty} 2^{s/2}\eta_{s-1}.$$

Now, we apply the generic chaining mechanism to show an upper bound on the supremum of processes whose increments satisfy the subgaussian assumption (3.2).

Theorem 3.1.4. — *There exist absolute constants c_0, c_1, c_2 and c_3 such that the following holds. Let (T, d) be a semi-metric space. Let $(X_t)_{t\in T}$ be a stochastic process satisfying the subgaussian condition (3.2). For every $v \geqslant c_0$, with probability greater than $1 - c_1 \exp(-c_2 v^2)$*

$$\sup_{s,t\in T} |X_t - X_s| \leqslant c_3 v \gamma_2(T, d)$$

and

$$\mathbb{E} \sup_{s,t\in T} |X_t - X_s| \leqslant c_3 \gamma_2(T, d).$$

Proof. — Let $(T_s)_{s\in\mathbb{N}}$ be an admissible sequence. For every $t \in T$ and $s \in \mathbb{N}$ denote by $\pi_s(t)$ one point in T_s such that $d(t, T_s) = d(t, \pi_s(t))$. Since T is finite, we can write for every $t \in T$,

$$|X_t - X_{\pi_0(t)}| \leqslant \sum_{s=0}^{\infty} |X_{\pi_{s+1}(t)} - X_{\pi_s(t)}|. \tag{3.8}$$

Let $s \in \mathbb{N}$. For every $t \in T$ and $v > 0$, with probability greater than $1 - 2\exp(-c_0 2^s v^2)$,

$$|X_{\pi_{s+1}(t)} - X_{\pi_s(t)}| \leqslant v 2^{s/2} d(\pi_{s+1}(t), \pi_s(t)).$$

We extend the last inequality to every link of the chains at level s by using an union bound: for every $v \geqslant c_1$, with probability greater than $1 - 2\exp(-c_2 2^s v^2)$, for every $t \in T$,

$$|X_{\pi_{s+1}(t)} - X_{\pi_s(t)}| \leqslant v 2^{s/2} d(\pi_{s+1}(t), \pi_s(t)).$$

An union bound on every level $s \in \mathbb{N}$ yields: for every $v \geqslant c_1$, with probability greater than $1 - 2\sum_{s=0}^{\infty} \exp(-c_2 2^s v^2)$, for every $t \in T$,

$$|X_t - X_{\pi_0(t)}| \leqslant c_2 v \sum_{s=0}^{\infty} 2^{s/2} d(\pi_s(t), \pi_{s+1}(t)) \leqslant c_3 v \sum_{s=0}^{\infty} 2^{s/2} d(t, T_s).$$

The claim follows since the sum in the last probability estimate is comparable to its first term. □

Note that for Gaussian processes, the upper bound in expectation obtained in Theorem 3.1.4 is sharp up to some absolute constants. This deep result, called the Majorizing measure theorem, makes an equivalence between two different quantities measuring the complexity of a set $T \subset \mathbb{R}^N$:

1. a metric complexity measure given by the γ_2 functional

$$\gamma_2(T, \ell_2^N) = \inf_{(T_s)} \sup_{t\in T} \sum_{s=0}^{\infty} 2^{s/2} d_{\ell_2^N}(t, T_s),$$

where the infimum is taken over all admissible sequences $(T_s)_{s\in\mathbb{N}}$ of T;

2. a probabilistic complexity measure given by the expectation of the supremum of the canonical Gaussian process indexed by T:

$$\ell_*(T) = \mathbb{E}\sup_{t\in T}\Big|\sum_{i=1}^N g_i t_i\Big|,$$

where $g_1, \ldots, g_n$ are n i.i.d. standard Gaussian variables.

***Theorem 3.1.5* (Majorizing measure Theorem).** — *There exist two absolute positive constants c_0 and c_1 such that for every subset T of $\mathbb{R}^N$,*

$$c_0\ell_*(T) \leqslant \gamma_2(T, \ell_2^N) \leqslant c_1\ell_*(T).$$

3.2. An example of a more sophisticated chaining argument

In this section, we show upper bounds on the supremum

$$\sup_{f\in F}\left|\frac{1}{n}\sum_{i=1}^n f^2(X_i) - \mathbb{E}f^2(X)\right|, \tag{3.9}$$

where $X_1, \ldots, X_n$ are n i.i.d. random variables with values in a measurable space $\mathcal{X}$ and F is a class of real-valued functions defined on $\mathcal{X}$.

In Chapter 2, this bound is used to show the restricted isometry property in Theorem 2.7.1. In this example, the class F is a class of linear functions indexed by a set of sparse vectors. In particular, for this example, the class F is not uniformly bounded.

In general, when $\|F\|_\infty = \sup_{f\in F}\|f\|_{L_\infty(\mu)} < \infty$, a bound on (3.9) follows from a symmetrization argument combined with the contraction principle. In the present study, we do not assume that F is uniformly bounded but we only assume that F has a finite diameter in $L_{\psi_2(\mu)}$ where μ is the probability distribution of $X \stackrel{d}{=} X_1$. This means that the norm

$$\|f\|_{\psi_2(\mu)} = \inf\Big(c > 0 : \mathbb{E}\exp\big(|f(X)|^2/c^2\big) \leqslant e\Big)$$

is uniformly bounded over every f in F. We denote this bound by α and thus we assume that

$$\alpha = \operatorname{diam}(F, \psi_2(\mu)) = \sup_{f\in F}\|f\|_{\psi_2(\mu)} < \infty. \tag{3.10}$$

In terms of random variables, Assumption (3.10) means that for all $f \in F$, $f(X)$ has a subgaussian behaviour and its ψ_2 norm is uniformly bounded over F.

Under (3.10), we can apply the classical generic chaining mechanism and obtain a bound on (3.9). Indeed, denote by $(X_f)_{f\in F}$ the empirical process where $X_f = n^{-1}\sum_{i=1}^n f^2(X_i) - \mathbb{E}f^2(X)$ for every $f \in F$. Assume that for every f and g in F, $\mathbb{E}f^2(X) = \mathbb{E}g^2(X)$. In this case, the increments of the process $(X_f)_{f\in F}$ are

$$X_f - X_g = \frac{1}{n}\sum_{i=1}^n f^2(X_i) - g^2(X_i)$$

and we have (cf. Chapter 1)

$$\left\|f^2 - g^2\right\|_{\psi_1(\mu)} \leqslant \|f+g\|_{\psi_2(\mu)} \|f-g\|_{\psi_2(\mu)} \leqslant 2\alpha \|f-g\|_{\psi_2(\mu)}. \tag{3.11}$$

In particular, the increment $X_f - X_g$ is a sum of i.i.d. mean-zero ψ_1 random variables. Hence, the concentration properties of the increments of $(X_f)_{f\in F}$ follow from Theorem 1.2.8. Provided that for some $f_0 \in F$, we have $X_{f_0} = 0$ or $(X_f)_{f\in F}$ is a symmetric process then running the classical generic chaining mechanism with this increment condition yields the following: for every $u \geqslant c_0$, with probability greater than $1 - c_1 \exp(-c_2 u)$,

$$\sup_{f\in F}\left|\frac{1}{n}\sum_{i=1}^n f^2(X_i) - \mathbb{E}f^2(X)\right| \leqslant c_3 u\alpha\left(\frac{\gamma_2(F,\psi_2(\mu))}{\sqrt{n}} + \frac{\gamma_1(F,\psi_2(\mu))}{n}\right) \tag{3.12}$$

for some absolute positive constants c_0, c_1, c_2 and c_3 and for

$$\gamma_1(F,\psi_2(\mu)) = \inf_{(F_s)}\sup_{f\in F}\Big(\sum_{s=0}^\infty 2^s d_{\psi_2(\mu)}(f,F_s)\Big)$$

where the infimum is taken over all admissible sequences $(F_s)_{s\in\mathbb{N}}$ and $d_{\psi_2(\mu)}(f,F_s) = \min_{g\in F_s}\|f-g\|_{\psi_2(\mu)}$ for every $f \in F$ and $s \in \mathbb{N}$. Result (3.12) can be derived from theorem 1.2.7 of [**Tal05**].

In some cases, computing $\gamma_1(F,d)$ for some metric d can be involved and only weak estimates can be showed. Obtaining upper bounds on (3.9) which does not require the computation of $\gamma_1(F,\psi_2(\mu))$ can be of importance. In particular, upper bounds depending only on $\gamma_2(F,\psi_2(\mu))$ can be useful when the metrics $L_{\psi_2}(\mu)$ and $L_2(\mu)$ are equivalent on F because of the Majorizing measure theorem (cf. Theorem 3.1.5). In the next result, we show an upper bound on the supremum (3.9) depending only on the $\psi_2(\mu)$ diameter of F and of the complexity measure $\gamma_2(F,\psi_2(\mu))$.

Theorem 3.2.1. — *There exists absolute constants c_0, c_1, c_2 and c_3 such that the following holds. Let F be a finite class of real-valued functions in $\mathcal{S}(L_2(\mu))$, the unit sphere of $L_2(\mu)$ and denote by α the diameter* $\mathrm{diam}(F,\psi_2)$. *Then, with probability at least* $1 - c_1\exp\Big(-(c_2/\alpha^2)\min\Big(n\alpha^2, \gamma_2(F,\psi_2)^2\Big)\Big)$,

$$\sup_{f\in F}\left|\frac{1}{n}\sum_{i=1}^n f^2(X_i) - \mathbb{E}f^2(X)\right| \leqslant c_3\max\left(\alpha\frac{\gamma_2(F,\psi_2)}{\sqrt{n}}, \frac{\gamma_2(F,\psi_2)^2}{n}\right).$$

Moreover, if F is a symmetric subset of $\mathcal{S}(L_2(\mu))$ then,

$$\mathbb{E}\sup_{f\in F}\left|\frac{1}{n}\sum_{i=1}^n f^2(X_i) - \mathbb{E}f^2(X)\right| \leqslant c_3\max\left(\alpha\frac{\gamma_2(F,\psi_2)}{\sqrt{n}}, \frac{\gamma_2(F,\psi_2)^2}{n}\right).$$

To show Theorem 3.2.1, we introduce the following notation. For every $f \in L_2(\mu)$, we set

$$Z(f) = \frac{1}{n}\sum_{i=1}^n f^2(X_i) - \mathbb{E}f^2(X) \text{ and } W(f) = \Big(\frac{1}{n}\sum_{i=1}^n f^2(X_i)\Big)^{1/2}. \tag{3.13}$$

Moreover, for the sake of shortness, in what follows, we omit to write the probability measure μ of the $L_2(\mu)$ norm, $\psi_1(\mu)$ norm and $\psi_2(\mu)$ norm.

To obtain upper bounds on the supremum (3.9) we study the deviation behaviour of the increments of the underlying process. Namely, we need deviation results for $Z(f) - Z(g)$ for every $f, g \in F$. Moreover, since the "end of the chains" will be analysed by different means, the deviation behaviour of the increments $W(f-g)$ will be of importance as well.

Lemma 3.2.2. — *There exists an absolute constant C_1 such that the following holds. Let $F \subset \mathcal{S}(L_2(\mu))$. Denote by α the diameter* $\mathrm{diam}(F, \psi_2)$. *For every $f, g \in F$, we have:*

1. *for every $u \geqslant 2$,*
$$\mathbb{P}\Big[W(f-g) \geqslant u \left\|f-g\right\|_{\psi_2}\Big] \leqslant 2\exp\big(-C_1 n u^2\big);$$
2. *for every $u > 0$,*
$$\mathbb{P}\Big[|Z(f)-Z(g)| \geqslant u\alpha \left\|f-g\right\|_{\psi_2}\Big] \leqslant 2\exp\big(-C_1 n \min(u, u^2)\big);$$
and for every $u > 0$,
$$\mathbb{P}\Big[|Z(f)| \geqslant u\alpha^2\Big] \leqslant 2\exp\big(-C_1 n \min(u, u^2)\big).$$

Proof. — Let $f, g \in F$. Since $f, g \in L_{\psi_2}$, we have $\left\|(f-g)^2\right\|_{\psi_1} = \left\|f-g\right\|_{\psi_2}^2$ and by Theorem 1.2.8, for every $t \geqslant 1$,
$$\mathbb{P}\Big[\frac{1}{n}\sum_{i=1}^{n}(f-g)^2(X_i) - \left\|f-g\right\|_{L_2}^2 \geqslant t\left\|f-g\right\|_{\psi_2}^2\Big] \leqslant 2\exp(-c_1 n t) \tag{3.14}$$
Using $\left\|f-g\right\|_{\psi_2} \geqslant \sqrt{e-1}\left\|f-g\right\|_{L_2}$ together with Equation (3.14), it is easy to get for every $u \geqslant 2$,
$$\begin{aligned}&\mathbb{P}\Big[W(f-g) \geqslant u\left\|f-g\right\|_{\psi_2}\Big]\\ &\leqslant \mathbb{P}\Big[\frac{1}{n}\sum_{i=1}^{n}(f-g)^2(X_i) - \left\|f-g\right\|_{L_2}^2 \geqslant (u^2-(e-1))\left\|f-g\right\|_{\psi_2}^2\Big]\\ &\leqslant 2\exp\big(-c_2 n u^2\big).\end{aligned}$$

For the second statement, since $\mathbb{E}f^2 = \mathbb{E}g^2$, the increments are
$$Z(f) - Z(g) = \frac{1}{n}\sum_{i=1}^{n} f^2(X_i) - g^2(X_i).$$
Thanks to (3.11), $Z(f) - Z(g)$ is a sum of mean-zero ψ_1 random variables and the result follows from Theorem 1.2.8. The last statement is also a consequence of Theorem 1.2.8 and (3.11). □

Once obtained the deviation properties of the increments of the underlying process(es) (that is $(Z(f))_{f\in F}$ and $(W(f))_{f\in F}$), we use the generic chaining mechanism to obtain a uniform bound on (3.9). Since we work in a special framework (sum of squares of ψ_2 random variables), we will perform a particular chaining argument which will allow us to avoid the $\gamma_1(F,\psi_2)$ term coming out of the classical generic chaining (cf. (3.12)).

If $\gamma_2(F,\psi_2)=\infty$ then the upper bound of Theorem 3.2.1 is trivial, otherwise consider an almost optimal admissible sequence $(F_s)_{s\in\mathbb{N}}$ of F with respect to $\psi_2(\mu)$. That is an admissible sequence $(F_s)_{s\in\mathbb{N}}$ such that

$$\gamma_2(F,\psi_2)\geqslant\frac{1}{2}\sup_{f\in F}\Big(\sum_{s=0}^{\infty}2^{s/2}d_{\psi_2}(f,F_s)\Big).$$

For every $f\in F$ and integer s, put $\pi_s(f)$ a nearest point to f in F_s.

The idea of the proof is for every $f\in F$ to analyze the links $\pi_{s+1}(f)-\pi_s(f)$ for $s\in\mathbb{N}$ of the chain $(\pi_s(f))_{s\in\mathbb{N}}$ in three different regions - values of the level s in $[0,s_1]$, $[s_1+1,s_0-1]$ or $[s_0,\infty)$ for some well chosen s_1 ans s_0 - depending on the deviation properties of the increments of the underlying process(es) at the s stage:

1. The end of the chain: we study the link $f-\pi_{s_0}(f)$. In this part of the chain, we work with the process $(W(f-\pi_{s_0}(f)))_{f\in F}$ which is subgaussian (cf. Lemma 3.2.2). Thanks to this remark, we can avoid the sub-exponential behaviour of the process $(Z(f))_{f\in F}$ and thus the term $\gamma_1(F,\psi_2)$ appearing in (3.12);
2. The middle of the chain: we work at these stages with the process $(Z(\pi_{s_0-1}(f))-Z(\pi_{s_1}(f)))_{f\in F}$ which has subgaussian increments in this range;
3. The beginning of the chain: we study the process $(Z(\pi_{s_1}(f))_{f\in F}$. For this part of the chain, the complexity of F_{s_1} is so small that a trivial comparison of the process with the ψ_2-diameter of F will be enough.

***Proposition 3.2.3* (End of the chain).** — *There exist absolute constant c_0,c_1,c_2 and c_3 for which the following holds. Let $F\subset\mathcal{S}(L_2(\mu))$ and $\alpha=\mathrm{diam}(F,\psi_2)$. For every $v\geqslant c_0$, with probability greater than $1-c_1\exp(-c_2nv)$,*

$$\sup_{f\in F}W(f-\pi_{s_0}(f))\leqslant c_3\sqrt{v}\frac{\gamma_2(F,\psi_2)}{\sqrt{n}},$$

where $s_0=\min\big(s\geqslant 0:2^s\geqslant n\big)$.

Proof. — Let f be in F. Since F is finite, we can write

$$f-\pi_{s_0}(f)=\sum_{s=s_0}^{\infty}\pi_{s+1}(f)-\pi_s(f),$$

and, since W is the empirical $L_2(P_n)$ norm (where P_n is the empirical distribution $n^{-1}\sum_{i=1}^n\delta_{X_i}$), it is sub-additive and so

$$W(f-\pi_{s_0}(f))\leqslant\sum_{s=s_0}^{\infty}W(\pi_{s+1}(f)-\pi_s(f)).$$

Now, fix a level $s \geqslant s_0$. Using a union bound on the set of links $\{(\pi_{s+1}(f), \pi_s(f)) : f \in F\}$ (note that there are at most $|F_{s+1}||F_s|$ such links) and the subgaussian property of W (i.e. Lemma 3.2.2), we get, for every $u \geqslant 2$, with probability greater than $1 - 2|F_{s+1}||F_s|\exp(-C_1 n u^2)$, for every $f \in F$,

$$W(\pi_{s+1}(f) - \pi_s(f)) \leqslant u \left\|\pi_{s+1}(f) - \pi_s(f)\right\|_{\psi_2}.$$

Then, note that for every $s \in \mathbb{N}$, $|F_{s+1}||F_s| \leqslant 2^{2^s} 2^{2^{s+1}} = 2^{3.2^s}$ so that a union bound over all the levels $s \geqslant s_0$ yields for every u such that $u^2 2^{s_0}$ is larger than some absolute constant, with probability greater than $1 - 2\sum_{s=s_0}^{\infty} |F_{s+1}||F_s| \exp(-C_1 n 2^s u^2) \geqslant 1 - c_1 \exp(-c_0 n u^2 2^{s_0})$, for every $f \in F$,

$$\begin{aligned} W(f - \pi_{s_0}(f)) &\leqslant \sum_{s=s_0}^{\infty} W(\pi_{s+1}(f) - \pi_s(f)) \leqslant \sum_{s=s_0}^{\infty} u 2^{s/2} \left\|\pi_{s+1}(f) - \pi_s(f)\right\|_{\psi_2} \\ &\leqslant 2u \sum_{s=s_0}^{\infty} 2^{s/2} d_{\psi_2}(f, F_s). \end{aligned}$$

We conclude with $v = u^2 2^{s_0}$ for v large enough, s_0 such that $2^{s_0} \sim n$ and with the quasi-optimality of the admissible sequence $(F_s)_{s \geqslant 0}$. □

***Proposition 3.2.4* (Middle of the chain).** — *There exist absolute constants c_0, c_1, c_2 and c_3 for which the following holds. Let $s_1 \in \mathbb{N}$ be such that $s_1 \leqslant s_0$ (where s_0 has been defined in Proposition 3.2.3). Let $F \subset \mathcal{S}(L_2(\mu))$ and $\alpha = \operatorname{diam}(F, \psi_2)$. For every $u \geqslant c_0$, with probability greater than $1 - c_1 \exp(-c_2 2^{s_1} u)$,*

$$\sup_{f \in F} \left| Z(\pi_{s_0}(f)) - Z(\pi_{s_1}(f)) \right| \leqslant c_3 u \alpha \frac{\gamma_2(F, \psi_2)}{\sqrt{n}}.$$

Proof. — For every $f \in F$, we write

$$Z(\pi_{s_0 - 1}(f)) - Z(\pi_{s_1}(f)) = \sum_{s=s_1+1}^{s_0-1} Z(\pi_s(f)) - Z(\pi_{s-1}(f)).$$

Let $s_1 \leqslant s \leqslant s_2$ and $u > 0$. Thanks to the second deviation result of Lemma 3.2.2, with probability greater than $1 - 2\exp\big(-C_1 n \min\big((u 2^{s/2}/\sqrt{n}), (u^2 2^s/n)\big)\big)$,

$$|Z(\pi_s(f)) - Z(\pi_{s-1}(f))| \leqslant \frac{u 2^{s/2}}{\sqrt{n}} \alpha \left\|\pi_s(f) - \pi_{s-1}(f)\right\|_{\psi_2}. \tag{3.15}$$

Moreover, $s \leqslant s_0$, thus $2^s/n \leqslant 2$ and so $\min\big(u 2^{s/2}/\sqrt{n}, u^2 2^s/n\big) \geqslant \min(u, u^2)(2^s/(2n))$. In particular, (3.15) holds with probability greater than $1 - 2\exp\big(-C_1 2^s \min(u, u^2)\big)\big)$.

Now, a union bound on the set of links for every levels $s = s_1, \ldots, s_0 - 1$ yields, for any $u > 0$, with probability greater than $1 - 2\sum_{s=s_1+1}^{s_0-1} |F_{s+1}||F_s| \exp\big(-C_1 2^s \min(u, u^2)\big)$, for every $f \in F$,

$$\left| Z(\pi_{s_0}(f)) - Z(\pi_{s_1}(f)) \right| \leqslant \sum_{s=s_1+1}^{s_0-1} \frac{u 2^{s/2}}{\sqrt{n}} \alpha \left\|\pi_s(f) - \pi_{s-1}(f)\right\|_{\psi_2}.$$

The result follows since $|F_s||F_{s+1}| \leqslant 2^{3.2^s}$ for every integer s and for u large enough. $\square$

***Proposition 3.2.5* (Beginning of the chain).** — *There exist absolute positive constants c_0 and c_1 such that the following holds. Let $w > 0$ and s_1 be such that $2^{s_1} < (C_1/2)n\min(w, w^2)$ (where C_1 is the constant appearing in Lemma 3.2.2). Let $F \subset \mathcal{S}(L_2(\mu))$ and $\alpha = \operatorname{diam}(F, \psi_2)$. With probability greater than $1 - c_0\exp(-c_1 n\min(w, w^2))$,*

$$\sup_{f\in F}\left|Z(\pi_{s_1}(f))\right| \leqslant w\alpha^2.$$

Proof. — It follows from the third deviation result of Lemma 3.2.2 and a union bound over F_{s_1}, that with probability greater than $1 - 2|F_{s_1}|\exp\big(-C_1 n\min(w, w^2)\big)$, for every $f \in F$,

$$|Z(\pi_{s_1}(f)| \leqslant w\alpha^2.$$

Since $|F_{s_1}| \leqslant 2^{2^{s_1}} < \exp\big((C_1/2)n\min(w, w^2)\big)$, the result follows. $\square$

Proof of Theorem 3.2.1. — Denote by $(F_s)_{s\in\mathbb{N}}$ an almost optimal admissible sequence of F with respect to the ψ_2-norm and, for every $s \in \mathbb{N}$ and $f \in F$, denote by $\pi_s(f)$ one of the closest point of f in F_s with respect to ψ_2. Let $s_0 \in \mathbb{N}$ be such that $s_0 = \min\big(s \geqslant 0 : 2^s \geqslant n\big)$. We have, for every $f \in F$,

$$\begin{aligned}
|Z(f)| &= \Big|\frac{1}{n}\sum_{i=1}^n f^2(X_i) - \mathbb{E}f^2(X)\Big| = \Big|\frac{1}{n}\sum_{i=1}^n (f - \pi_{s_0}(f) + \pi_{s_0}(f))^2(X_i) - \mathbb{E}f^2(X)\Big| \\
&= \Big|P_n(f - \pi_{s_0}(f))^2 + 2P_n(f - \pi_{s_0}(f))\pi_{s_0}(f) + P_n\pi_{s_0}(f)^2 - \mathbb{E}\pi_{s_0}(f)^2\Big| \\
&\leqslant W(f - \pi_{s_0}(f))^2 + 2W(f - \pi_{s_0}(f))W(\pi_{s_0}(f)) + |Z(\pi_{s_0}(f))| \\
&\leqslant W(f - \pi_{s_0}(f))^2 + 2W(f - \pi_{s_0}(f))\big(Z(\pi_{s_0}(f)) + 1\big)^{1/2} + |Z(\pi_{s_0}(f))| \\
&\leqslant 3W(f - \pi_{s_0}(f))^2 + 2W(f - \pi_{s_0}(f)) + 3|Z(\pi_{s_0}(f))| \qquad (3.16)
\end{aligned}$$

where we used $\|\pi_{s_0}(f)\|_{L_2} = 1 = \|f\|_{L_2}$ and the notation P_n stands for the empirical probability distribution $n^{-1}\sum_{i=1}^n \delta_{X_i}$.

Thanks to Proposition 3.2.3 for $v = 1$, with probability greater than $1 - c_0\exp(-c_1 n)$, for every $f \in F$,

$$W(f - \pi_{s_0}(f))^2 \leqslant c_2\frac{\gamma_2(F, \psi_2)^2}{n}. \qquad (3.17)$$

Let $w > 0$ to be chosen later and take $s_1 \in \mathbb{N}$ such that

$$s_1 = \max\Big(s \geqslant 0 : 2^s \leqslant \min\big(2^{s_0}, (C_1/2)n\min(w, w^2)\big)\Big) \qquad (3.18)$$

where C_1 is the constant defined in Lemma 3.2.2. We apply Proposition 3.2.4 with $u = 1$ and Proposition 3.2.5 to get, with probability greater than $1 - c_3\exp(-c_4 2^{s_1})$, for every $f \in F$,

$$\begin{aligned}|Z(\pi_{s_0}(f))| &\leqslant |Z(\pi_{s_0}(f)) - Z(\pi_{s_1}(f))| + |Z(\pi_{s_1}(f))| \\ &\leqslant c_5 \alpha \frac{\gamma_2(F,\psi_2)}{\sqrt{n}} + \alpha^2 w. \end{aligned} \tag{3.19}$$

We combine Equation (3.16), (3.17) and (3.19) to get, with probability greater than $1 - c_6 \exp(-c_7 2^{s_1})$, for every $f \in F$,

$$|Z(f)| \leqslant c_8 \frac{\gamma_2(F,\psi_2)^2}{n} + c_9 \frac{\gamma_2(F,\psi_2)}{\sqrt{n}} + c_{10}\alpha \frac{\gamma_2(F,\psi_2)}{\sqrt{n}} + 3\alpha^2 w.$$

First statement of Theorem 3.2.1 follows for

$$w = \max\Big(\frac{\gamma_2(F,\psi_2)}{\alpha\sqrt{n}}, \frac{\gamma_2(F,\psi_2)^2}{\alpha^2 n}\Big). \tag{3.20}$$

For the last statement, we use Proposition 3.2.3 to get

$$\mathbb{E} \sup_{f\in F} W(f - \pi_{s_0}(f))^2 = \int_0^\infty \mathbb{P}\big[\sup_{f\in F} W(f-\pi_{s_0}(f))^2 \geqslant t\big] dt \leqslant c_{11} \frac{\gamma_2(F,\psi_2)^2}{n} \tag{3.21}$$

and

$$\mathbb{E} \sup_{f\in F} W(f - \pi_{s_0}(f)) \leqslant c_{12} \frac{\gamma_2(F,\psi_2)}{\sqrt{n}}. \tag{3.22}$$

It follows from proposition 3.2.4 and 3.2.5 for s_1 and w defined in (3.18) and (3.20) that

$$\begin{aligned}&\mathbb{E} \sup_{f\in F} |Z(\pi_{s_0}(f))| \leqslant \mathbb{E} \sup_{f\in F} |Z(\pi_{s_0}(f)) - Z(\pi_{s_1}(f))| + \mathbb{E} \sup_{f\in F} |Z(\pi_{s_1}(f))| \\ &\leqslant \int_0^\infty \mathbb{P}\Big[\sup_{f\in F} |Z(\pi_{s_0}(f)) - Z(\pi_{s_1}(f))| \geqslant t\Big] dt + \int_0^\infty \mathbb{P}\Big[\sup_{f\in F} |Z(\pi_{s_0}(f))| \geqslant t\Big] dt \\ &\leqslant c\alpha \frac{\gamma_2(F,\psi_2)}{\sqrt{n}}. \end{aligned} \tag{3.23}$$

The claim follows by combining equations (3.21), (3.22) and (3.23) in Equation (3.16). □

3.3. Bibliographical references

Dudley entropy bound (cf. Theorem 3.1.2) can be found in [**Dud67**]. Other Dudley entropy bounds for processes $(X_t)_{t\in T}$ with Orlicz norm of the increments satisfying, for every $s, t \in T$,

$$\|X_t - X_s\|_\psi \leqslant d(s,t) \tag{3.24}$$

are obtained in [**Pis80**] and [**Kôn80**]. Under the increment condition (3.24), the Dudley entropy integral

$$\int_0^\infty \psi^{-1}\big(N(T,d,\epsilon)\big) d\epsilon,$$

where ψ^{-1} is the inverse function of the Orlicz function ψ, is an upper bound on $\mathbb{E}\sup_{t\in T}$ coming out of the chaining argument.

For the partition scheme method used in the generic chaining mechanism of Theorem 3.1.4, we refer to [**Tal05**] and [**Tal01**]. The generic chaining mechanism was first introduced using majorizing measures. This tool was introduced in [**Fer74, Fer75**] and is implicit in earlier work by Preston based on an important result of Garcia, Rodemich and Rumsey. In [**Tal87**], the author proves that majorizing measures are the key quantities to analyze the supremum of Gaussian processes. In particular, the majorizing measure theorem (cf. Theorem 3.1.5) is shown in [**Tal87**]. More about majorizing measures and majorizing measure theorems for other processes than Gaussian processes can be found in [**Tal96a**] and [**Tal95**]. Connections between the majorizing measures and partition schemes have been showed in [**Tal05**] and [**Tal01**].

The upper bounds on the process

$$\sup_{f\in F}\Big|\frac{1}{n}\sum_{i=1}^{n} f^2(X_i)-\mathbb{E}f^2(X)\Big| \tag{3.25}$$

developed in Section 3.2 follow the line of [**MPTJ07**]. Other bounds on (3.25) can be found in the next chapter (cf. Theorem 5.2.14).

CHAPTER 4

SINGULAR VALUES OF RANDOM MATRICES

The extremal singular values of a matrix are very natural geometrical quantities concentrating an essential information on the invertibility and stability of the matrix. This chapter aims to provide an accessible introduction to the notion of singular values of matrices and their behavior when the entries are random, including quite recent striking results from random matrix theory and high dimensional geometric analysis.

For every square matrix $A \in \mathcal{M}_{n,n}(\mathbb{C})$, we denote by $\lambda_1(A), \ldots, \lambda_n(A)$ the eigenvalues of A which are the roots in $\mathbb{C}$ of the characteristic polynomial $\det(A - ZI) \in \mathbb{C}[Z]$. We label the eigenvalues of A so that $|\lambda_1(A)| \geqslant \cdots \geqslant |\lambda_n(A)|$. In all this chapter, K stands for $\mathbb{R}$ or $\mathbb{C}$, and we say that $U \in \mathcal{M}_{n,n}(K)$ is K-unitary when $UU^* = I$.

4.1. Singular values of deterministic matrices

This section gathers a selection of classical results from linear algebra. We begin with the Singular Value Decomposition (SVD), a fundamental tool in matrix analysis, which expresses a diagonalization up to unitary transformations of the space.

Theorem 4.1.1 **(Singular Value Decomposition)**. — *For every $A \in \mathcal{M}_{m,n}(K)$, there exists a couple of K–unitary matrices U $(m \times m)$ and V $(n \times n)$ and a sequence of real numbers $s_1 \geqslant \cdots \geqslant s_{m\wedge n} \geqslant 0$ such that*

$$U^*AV = \operatorname{diag}(s_1, \ldots, s_{m\wedge n}) \in \mathcal{M}_{m,n}(K).$$

This sequence of real numbers does not depend on the particular choice of U, V.

Proof. — Let $v \in K^n$ be such that $|Av|_2 = \max_{|x|_2=1} |Ax|_2 = \|A\|_{2\to 2} = s$. If $|v|_2 = 0$ then $A = 0$ and the desired result is trivial. If $s > 0$ then let us define $u = Av/s$. One can find a K-unitary matrix U of size $m \times m$ with first column equal to u, and a K-unitary matrix V of size $n \times n$ with first column equal to v. It follows that

$$U^*AV = \begin{pmatrix} s & w^* \\ 0 & B \end{pmatrix} = A_1$$

for some $w \in \mathcal{M}_{n-1,1}(K)$ and $B \in \mathcal{M}_{m-1,n-1}(K)$. If t is the first row of A_1 then $|A_1 t^*|_2^2 \geqslant (s^2 + |w|_2^2)^2$ and therefore $\|A_1\|_{2\to 2}^2 \geqslant s^2 + |w|_2^2 \geqslant \|A\|_{2\to 2}^2$. On the other hand, since A and A_1 are unitary equivalent, we have $\|A_1\|_{2\to 2} = \|A\|_{2\to 2}$. Therefore $w = 0$, and the desired decomposition follows by a simple induction. □

The numbers $s_k(A) := s_k$ for $k \in \{1, \ldots, m \wedge n\}$ are called the *singular values* of A. The columns of U and V are the eigenvectors of AA^* ($m \times m$) and A^*A ($n \times n$). These two positive semidefinite Hermitian matrices share the same sequence of eigenvalues, up to the multiplicity of the eigenvalue 0, and for every $k \in \{1, \ldots, m \wedge n\}$,

$$s_k(A) = \lambda_k(\sqrt{AA^*}) = \sqrt{\lambda_k(AA^*)} = \sqrt{\lambda_k(A^*A)} = \lambda_k(\sqrt{A^*A}) = s_k(A^*).$$

Actually, if one sees the diagonal matrix $D := \operatorname{diag}(s_1(A)^2, \ldots, s_{m\wedge n}(A)^2)$ as an element of $\mathcal{M}_{m,m}(K)$ or $\mathcal{M}_{n,n}(K)$ by appending as much zeros as needed, we have

$$U^*AA^*U = D \quad \text{and} \quad V^*A^*AV = D.$$

When A is normal (i.e. $AA^* = A^*A$) then $m = n$ and $s_k(A) = |\lambda_k(A)|$ for every $k \in \{1, \ldots, n\}$. For any $A \in \mathcal{M}_{m,n}(K)$, the eigenvalues of the $(m+n) \times (m+n)$ Hermitian matrix

$$H = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \tag{4.1}$$

are given by

$$+s_1(A), -s_1(A), \ldots, +s_{m\wedge n}(A), -s_{m\wedge n}(A), 0, \ldots, 0$$

where the notation $0, \ldots, 0$ stands for a sequence of 0's of length

$$m + n - 2(m \wedge n) = (m \vee n) - (m \wedge n).$$

One may deduce the singular values of A from the eigenvalues of H. Note that when $m = n$ and $A_{i,j} \in \{0, 1\}$ for all i, j, then A is the adjacency matrix of an oriented graph, and H is the adjacency matrix of a compagnon nonoriented bipartite graph.

For any $A \in \mathcal{M}_{m,n}(K)$, the matrices A, $\bar{A}$, $A^\top$, A^*, WA, AW' share the same sequences of singular values, for any K–unitary matrices W, W'. If $u_1 \perp \cdots \perp u_m \in K^m$ and $v_1 \perp \cdots \perp v_n \in K^n$ are the columns of U, V then for every $k \in \{1, \ldots, m \wedge n\}$,

$$Av_k = s_k(A)u_k \quad \text{and} \quad A^*u_k = s_k(A)v_k \tag{4.2}$$

while $Av_k = 0$ and $A^*u_k = 0$ for $k > m \wedge n$. The SVD gives an intuitive geometrical interpretation of A and A^* as a dual correspondence/dilation between two orthonormal bases known as the left and right eigenvectors of A and A^*. Additionally, A has exactly $r = \operatorname{rank}(A)$ nonzero singular values $s_1(A), \ldots, s_r(A)$ and

$$A = \sum_{k=1}^{r} s_k(A) u_k v_k^* \quad \text{and} \quad \begin{cases} \operatorname{kernel}(A) & = \operatorname{span}\{v_{r+1}, \ldots, v_n\}, \\ \operatorname{range}(A) & = \operatorname{span}\{u_1, \ldots, u_r\}. \end{cases}$$

We have also $s_k(A) = |Av_k|_2 = |A^*u_k|_2$ for every $k \in \{1, \ldots, m \wedge n\}$. It is well known that the eigenvalues of a Hermitian matrix can be expressed in terms of the entries of the matrix via minimax variational formulas. The following theorem is the counterpart for the singular values, and can be deduced from its Hermitian cousin.

***Theorem 4.1.2* (Courant–Fischer variational formulas for singular values)** *For every $A \in \mathcal{M}_{m,n}(K)$ and every $k \in \{1,\dots,m\wedge n\}$,*

$$s_k(A) = \max_{V\in\mathcal{V}_k} \min_{\substack{x\in V\\ |x|_2=1}} |Ax|_2 = \min_{V\in\mathcal{V}_{n-k+1}} \max_{\substack{x\in V\\ |x|_2=1}} |Ax|_2$$

where $\mathcal{V}_k$ is the set of subspaces of K^n of dimension k. In particular, we have

$$s_1(A) = \max_{\substack{x\in K^n\\ |x|_2=1}} |Ax|_2 \quad \text{and} \quad s_{m\wedge n}(A) = \min_{\substack{x\in K^n\\ |x|_2=1}} |Ax|_2.$$

We have also the following alternative formulas, for every $k \in \{1,\dots,m\wedge n\}$,

$$s_k(A) = \max_{\substack{V\in\mathcal{V}_k\\ W\in\mathcal{V}_k}} \min_{\substack{(x,y)\in V\times W\\ |x|_2=|y|_2=1}} \langle Ax, y\rangle.$$

As an exercise, one can check that if $A \in \mathcal{M}_{m,n}(\mathbb{R})$ then the variational formulas for $K = \mathbb{C}$, if one sees A as an element of $\mathcal{M}_{m,n}(\mathbb{C})$, coincide actually with the formulas for $K = \mathbb{R}$. Geometrically, the matrix A maps the Euclidean unit ball to an ellipsoid, and the singular values of A are exactly the half lengths of the $m\wedge n$ largest principal axes of this ellipsoid, see figure 1. The remaining axes have a zero length. In particular, for $A \in \mathcal{M}_{n,n}(K)$, the variational formulas for the extremal singular values $s_1(A)$ and $s_n(A)$ correspond to the half length of the longest and shortest axes.

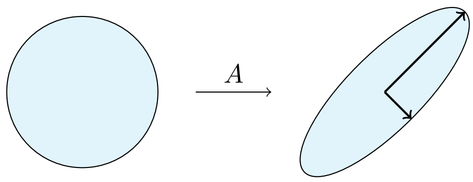

FIGURE 1. Largest and smallest singular values of $A \in \mathcal{M}_{2,2}(\mathbb{R})$.

From the Courant–Fischer variational formulas, the largest singular value is the operator norm of A for the Euclidean norm $|\cdot|_2$, namely

$$s_1(A) = \|A\|_{2\to 2}\,.$$

The map $A \mapsto s_1(A)$ is Lipschitz and convex. In the same spirit, if U, V are the couple of K–unitary matrices from an SVD of A, then for any $k \in \{1,\dots,\mathrm{rank}(A)\}$,

$$s_k(A) = \min_{\substack{B\in\mathcal{M}_{m,n}(K)\\ \mathrm{rank}(B)=k-1}} \|A-B\|_{2\to 2} = \|A-A_k\|_{2\to 2} \quad \text{where} \quad A_k = \sum_{i=1}^{k-1} s_i(A)u_iv_i^*$$

with u_i, v_i as in (4.2). Let $A \in \mathcal{M}_{n,n}(K)$ be a square matrix. If A is invertible then the singular values of A^{-1} are the inverses of the singular values of A, in other words

$$\forall k \in \{1,\dots,n\}, \quad s_k(A^{-1}) = s_{n-k+1}(A)^{-1}.$$

Moreover, a square matrix $A \in \mathcal{M}_{n,n}(K)$ is invertible iff $s_n(A) > 0$, and in this case

$$s_n(A) = s_1(A^{-1})^{-1} = \left\|A^{-1}\right\|_{2\to 2}^{-1}.$$

Contrary to the map $A \mapsto s_1(A)$, the map $A \mapsto s_n(A)$ is Lipschitz but is not convex. Regarding the Lipschitz nature of the singular values, the Courant–Fischer variational formulas provide the following more general result, which has a Hermitian couterpart.

Theorem 4.1.3 **(Weyl additive perturbations)**. — *If $A, B \in \mathcal{M}_{m,n}(K)$ then for every $i, j \in \{1, \ldots, m \wedge n\}$ with $i + j \leqslant 1 + (m \wedge n)$,*

$$s_{i+j-1}(A) \leqslant s_i(B) + s_j(A - B).$$

In particular, the singular values are uniformly Lipschitz functions since

$$\max_{1\leqslant k\leqslant m\wedge n} |s_k(A) - s_k(B)| \leqslant \|A - B\|_{2\to 2}.$$

From the Courant–Fischer variational formulas we obtain also the following result.

Theorem 4.1.4 **(Cauchy interlacing by rows deletion)**

Let $A \in \mathcal{M}_{m,n}(K)$ and $k \in \{1, 2, \ldots\}$ with $1 \leqslant k \leqslant m \leqslant n$ and let $B \in \mathcal{M}_{m-k,n}(K)$ be a matrix obtained from A by deleting k rows. Then for every $i \in \{1, \ldots, m - k\}$,

$$s_i(A) \geqslant s_i(B) \geqslant s_{i+k}(A).$$

In particular we have $[s_{m-k}(B), s_1(B)] \subset [s_m(A), s_1(A)]$. Row deletions produce a sort of compression of the singular values interval. Another way to express this phenomenon consists in saying that if we add a row to B then the largest singular value increases while the smallest singular value is diminished. From this point of view, the worst case corresponds to square matrices. Closely related, the following result on finite rank additive perturbations can be proved by using interlacing inequalities for the eigenvalues of Hermitian matrices and their principal submatrices.

Theorem 4.1.5 **(Interlacing for finite rank additive perturbations [Tho76])**

For any $A, B \in \mathcal{M}_{n,n}(K)$ with $\operatorname{rank}(A - B) \leqslant k$, we have, for any $i \in \{1, \ldots, n\}$,

$$s_{i-k}(A) \geqslant s_i(B) \geqslant s_{i+k}(A)$$

where $s_r = +\infty$ if $r \leqslant 0$ and $s_r = 0$ if $r \geqslant n + 1$. Conversely, any sequences of non negative real numbers which satisfy to these interlacing inequalities are the singular values of matrices A and B with $\operatorname{rank}(A - B) \leqslant k$.

In particular, we have $[s_{n-k}(B)\,,\, s_{k+1}(B)] \subset [s_n(A)\,,\, s_1(A)]$. It is worthwhile to observe that the interlacing inequalities of theorem 4.1.5 give neither an upper bound for the largest singular values $s_1(B), \ldots, s_k(B)$ nor a lower bound for the smallest singular values $s_{n-k+1}(B), \ldots, s_n(B)$, even when $k = 1$.

Remark 4.1.6 **(Hilbert-Schmidt norm)**. — *For every $A \in \mathcal{M}_{m,n}(K)$ we set*

$$\|A\|_{\mathrm{HS}}{}^2 := \operatorname{Tr}(AA^*) = \operatorname{Tr}(A^*A) = \sum_{i,j=1}^{n} |A_{i,j}|^2 = s_1(A)^2 + \cdots + s_{m\wedge n}(A)^2.$$

This defines the so called Hilbert–Schmidt or Frobenius norm $\|\cdot\|_{\mathrm{HS}}$. *We have always*

$$\|A\|_{2\to 2} \leqslant \|A\|_{\mathrm{HS}} \leqslant \sqrt{\mathrm{rank}(A)}\ \|A\|_{2\to 2}$$

where equalities are achieved when $\mathrm{rank}(A) = 1$ *and* $A = I \in \mathcal{M}_{m,n}(K)$ *respectively. The advantage of* $\|\cdot\|_{\mathrm{HS}}$ *over* $\|\cdot\|_{2\to 2}$ *lies in its convenient expression in terms of the matrix entries. Actually, the Frobenius norm is Hilbertian for the Hermitian product*

$$\langle A, B\rangle = \mathrm{Tr}(AB^*).$$

Let us mention a result on the Frobenius Lipschitz norm of the singular values, due to Wielandt and Hoffman [**HW53**]*, which says that if* $A, B \in \mathcal{M}_{m,n}(K)$ *then*

$$\sum_{k=1}^{m\wedge n} (s_k(A) - s_k(B))^2 \leqslant \|A - B\|_{\mathrm{HS}}{}^2.$$

We end up by a result related to the Frobenius norm, due to Eckart and Young [**EY39**]. *We have seen that a matrix* $A \in \mathcal{M}_{m,n}(K)$ *has exactly* $r = \mathrm{rank}(A)$ *non zero singular values. More generally, if* $k \in \{0, 1, \ldots, r\}$ *and if* $A_k \in \mathcal{M}_{m,n}(K)$ *is obtained from the SVD of* A *by forcing* $s_i = 0$ *for all* $i > k$ *then*

$$\min_{\substack{B\in\mathcal{M}_{m,n}(K)\\ \mathrm{rank}(B)=k}} \|A - B\|_{\mathrm{HS}}{}^2 = \|A - A_k\|_{\mathrm{HS}}{}^2 = s_{k+1}(A)^2 + \cdots + s_r(A)^2.$$

***Remark 4.1.7* (Norms and unitary invariance).** — *For every* $k \in \{1, \ldots, m \wedge n\}$ *and any real number* $p \geqslant 1$, *the map* $A \in \mathcal{M}_{m,n}(K) \mapsto (s_1(A)^p + \cdots + s_k(A)^p)^{1/p}$ *is a unitary invariant norm on* $\mathcal{M}_{m,n}(K)$. *We recover the operator norm* $\|A\|_{2\to 2}$ *for* $k = 1$ *and the Frobenius norm* $\|A\|_{\mathrm{HS}}$ *for* $(k, p) = (m \wedge n, 2)$. *The special case* $(k, p) = (m \wedge n, 1)$ *is known as the Ky Fan norm of order* k, *while the special case* $k = m\wedge n$ *is known as the Schatten* p*-norm. For more material, see* [**Bha97, Zha02**].

4.1.1. Condition number. — The condition number of $A \in \mathcal{M}_{n,n}(K)$ is given by

$$\kappa(A) = \|A\|_{2\to 2} \left\|A^{-1}\right\|_{2\to 2} = \frac{s_1(A)}{s_n(A)}.$$

The condition number quantifies the numerical sensitivity of linear systems involving A. For instance, if $x \in K^n$ is the solution of the linear equation $Ax = b$ then $x = A^{-1}b$. If b is known up to precision $\delta \in K^n$ then x is known up to precision $A^{-1}\delta$. Therefore, the ratio of relative errors for the determination of x is given by

$$R(b, \delta) = \frac{\left|A^{-1}\delta\right|_2 / \left|A^{-1}b\right|_2}{|\delta|_2/|b|_2} = \frac{\left|A^{-1}\delta\right|_2}{|\delta|_2} \frac{|b|_2}{\left|A^{-1}b\right|_2}.$$

Consequently, we obtain

$$\max_{b\neq 0,\delta\neq 0} R(b, \delta) = \left\|A^{-1}\right\|_{2\to 2} \|A\|_{2\to 2} = \kappa(A).$$

Geometrically, $\kappa(A)$ measures the “spherical defect” of the ellipsoid in figure (1).

4.1.2. Basic relationships between eigenvalues and singular values. — We know that if $A \in \mathcal{M}_{n,n}(K)$ is normal (i.e. $AA^* = A^*A$) then $s_k(A) = |\lambda_k(A)|$ for every $k \in \{1, \ldots, n\}$. Beyond normal matrices, for every $A \in \mathcal{M}_{n,n}(K)$ with rows $R_1, \ldots, R_n$, we have, by viewing $|\det(A)|$ as the volume of a hyperparallelogram,

$$|\det(A)| = \prod_{k=1}^{n} |\lambda_k(A)| = \prod_{k=1}^{n} s_k(A) = \prod_{k=1}^{n} \mathrm{dist}(R_k, \mathrm{span}\{R_1, \ldots, R_{k-1}\}) \tag{4.3}$$

The following result, due to Weyl, is less global and more subtle.

Theorem 4.1.8 **(Weyl inequalities [Wey49]).** — *If $A \in \mathcal{M}_{n,n}(K)$, then*

$$\forall k \in \{1, \ldots, n\}, \quad \prod_{i=1}^{k} |\lambda_i(A)| \leqslant \prod_{i=1}^{k} s_i(A) \quad \text{and} \quad \prod_{i=k}^{n} s_i(A) \leqslant \prod_{i=k}^{n} |\lambda_i(A)| \tag{4.4}$$

Moreover, for every increasing function φ from $(0, \infty)$ to $(0, \infty)$ such that $t \mapsto \varphi(e^t)$ is convex on $(0, \infty)$ and $\varphi(0) := \lim_{t \to 0^+} \varphi(t) = 0$, we have

$$\forall k \in \{1, \ldots, n\}, \quad \sum_{i=1}^{k} \varphi(|\lambda_i(A)|^2) \leqslant \sum_{i=1}^{k} \varphi(s_i(A)^2). \tag{4.5}$$

Observe that from (4.5) with $\varphi(t) = t$ for every $t > 0$ and $k = n$, we obtain

$$\sum_{k=1}^{n} |\lambda_k(A)|^2 \leqslant \sum_{k=1}^{n} s_k(A)^2 = \mathrm{Tr}(AA^*) = \sum_{i,j=1}^{n} |A_{i,j}|^2 = \mathrm{Tr}(AA^*) = \|A\|_{\mathrm{HS}}{}^2. \tag{4.6}$$

The following result, due to Horn, constitutes a converse to Weyl inequalities 4.3. It explains why so many generic relationships between eigenvalues and singular values are consequences of (4.3), for instance via majorization inequalities and techniques.

Theorem 4.1.9 **(Sherman inverse problem [Hor54]).** — *Let $(\lambda, s) \in \mathbb{C}^n \times \mathbb{R}^n$ be such that $|\lambda_1| \geqslant \cdots \geqslant |\lambda_n|$ and $s_1 \geqslant \cdots \geqslant s_n \geqslant 0$. If these numbers satisfy additionally to all the Weyl relationships (4.3) then there exists $A \in \mathcal{M}_{n,n}(\mathbb{C})$ such that $\lambda_i(A) = \lambda_i$ and $s_i(A) = s_i$ for every $i \in \{1, \ldots, n\}$.*

From (4.3) we get $s_n(A) \leqslant |\lambda_n(A)| \leqslant |\lambda_1(A)| \leqslant s_1(A)$ for any $A \in \mathcal{M}_{n,n}(K)$. In particular, we have the following comparison between the spectral radius and the operator norm:

$$\rho(A) = |\lambda_1(A)| \leqslant s_1(A) = \|A\|_{2\to 2}\,.$$

In this spirit, the following result, due to Gelfand, allows to estimate the spectral radius $\rho(A)$ with the singular values of the powers of A.

Theorem 4.1.10 **(Gelfand spectral radius formula [Gel41])**

Let $\|\cdot\|$ be a submultiplicative matrix norm on $\mathcal{M}_{n,n}(K)$ such as the operator norm $\|\cdot\|_{2\to 2}$ or the Frobenius norm $\|\cdot\|_{\mathrm{HS}}$. Then for every matrix $A \in \mathcal{M}_{n,n}(K)$ we have

$$\rho(A) := |\lambda_1(A)| = \lim_{k\to\infty} \sqrt[k]{\|A^k\|}.$$

The eigenvalues of non normal matrices are far more sensitive to perturbations than the singular values, and this is captured by the notion of pseudo spectrum, which bridges eigenvalues and singular values, see for instance the book [**TE05**].

4.1.3. Relation with rows distances. — The following couple of lemmas relate the singular values of matrices to distances between rows (or columns). For square random matrices, they provide a convenient control on the operator norm and Frobenius norm of the inverse respectively. The first lemma can be found in the work of Rudelson and Vershynin while the second appears in the work of Tao and Vu.

Lemma 4.1.11 **(Rudelson-Vershynin [RV09])**. — *If $A \in \mathcal{M}_{m,n}(K)$ has rows $R_1, \ldots, R_m$, then, denoting $R_{-i} = \mathrm{span}\{R_j : j \neq i\}$, we have*

$$m^{-1/2} \min_{1 \leqslant i \leqslant m} \mathrm{dist}_2(R_i, R_{-i}) \leqslant s_{m \wedge n}(A) \leqslant \min_{1 \leqslant i \leqslant m} \mathrm{dist}_2(R_i, R_{-i}).$$

Proof. — Since A and $A^\top$ have same singular values, we can prove the statement for the columns of A. For every vector $x \in K^n$ and every $i \in \{1, \ldots, n\}$, the triangle inequality and the identity $Ax = x_1 C_1 + \cdots + x_n C_n$ give

$$|Ax|_2 \geqslant \mathrm{dist}_2(Ax, C_{-i}) = \min_{y \in C_{-i}} |Ax - y|_2 = \min_{y \in C_{-i}} |x_i C_i - y|_2 = |x_i| \mathrm{dist}_2(C_i, C_{-i}).$$

If $|x|_2 = 1$ then necessarily $|x_i| \geqslant n^{-1/2}$ for some $i \in \{1, \ldots, n\}$, and therefore

$$s_{m \wedge n}(A) = \min_{|x|_2 = 1} |Ax|_2 \geqslant n^{-1/2} \min_{1 \leqslant i \leqslant n} \mathrm{dist}_2(C_i, C_{-i}).$$

Conversely, for any $i \in \{1, \ldots, n\}$, there exists a vector $y \in K^n$ with $y_i = 1$ such that

$$\mathrm{dist}_2(C_i, C_{-i}) = |y_1 C_1 + \cdots + y_n C_n|_2 = |Ay|_2 \geqslant |y|_2 \min_{|x|_2 = 1} |Ax|_2 \geqslant s_{m \wedge n}(A)$$

where we used the fact that $|y|_2^2 = |y_1|^2 + \cdots + |y_n|^2 \geqslant |y_i|^2 = 1$. □

Lemma 4.1.12 **(Tao-Vu [TV10])**. — *Let $1 \leqslant m \leqslant n$ and $A \in \mathcal{M}_{m,n}(K)$ with rows $R_1, \ldots, R_m$. If $\mathrm{rank}(A) = m$ then, denoting $R_{-i} = \mathrm{span}\{R_j : j \neq i\}$, we have*

$$\sum_{i=1}^m s_i^{-2}(A) = \sum_{i=1}^m \mathrm{dist}_2(R_i, R_{-i})^{-2}.$$

Proof. — The orthogonal projection of R_i on R_{-i} is $B^*(BB^*)^{-1}BR_i^*$ where B is the $(m-1) \times n$ matrix obtained from A by removing the row R_i. In particular, we have

$$|R_i|_2^2 - \mathrm{dist}_2(R_i, R_{-i})^2 = \left|B^*(BB^*)^{-1}BR_i^*\right|_2^2 = (BR_i^*)^*(BB^*)^{-1}BR_i^*$$

by the Pythagoras theorem. On the other hand, the Schur bloc inversion formula states that if M is an $m \times m$ matrix then for every partition $\{1, \ldots, m\} = I \cup I^c$,

$$(M^{-1})_{I,I} = (M_{I,I} - M_{I,I^c}(M_{I^c,I^c})^{-1}M_{I^c,I})^{-1}.$$

Now we take $M = AA^*$ and $I = \{i\}$, and we note that $(AA^*)_{i,j} = R_i \cdot R_j$, which gives

$$((AA^*)^{-1})_{i,i} = (R_i \cdot R_i - (BR_i^*)^*(BB^*)^{-1}BR_i^*)^{-1} = \mathrm{dist}_2(R_i, R_{-i})^{-2}.$$

The desired formula follows by taking the sum over $i \in \{1, \ldots, m\}$. □

4.1.4. Algorithm for the computation of the SVD. — To compute the SVD of $A \in \mathcal{M}_{m,n}(K)$ one can diagonalize both AA^* and A^*A or diagonalize the matrix H defined in (4.1). Unfortunately, this approach can lead to a loss of information numerically. In practice, and up to machine precision, the SVD is better computed with a two step algorithm such as (the real world algorithm is a bit more involved):

1. Unitary bidiagonalization. Compute a couple of K–unitary matrices W, W' such that $B = WAW'$ is bidiagonal. Both W, W' are product of Householder reflections, see [**GVL96**]. One can also use Gram–Schmidt orthonormalization of the rows. It is worthwhile to mention that a very similar method allows also the tridiagonalization of Hermitian matrices (in this case we have $W = W'$).
2. Iterative algorithm for bidiagonal matrices. Compute the SVD of B up to machine precision with a variant of the QR algorithm due to Golub and Kahan. Note that the standard QR iterative algorithm allows the iterative numerical computation of the eigenvalues of arbitrary square matrices.

The `svd` command of Matlab, GNU Octave, GNU R, and Scilab allows the numerical computation of the SVD. At the time of writing, the GNU Octave and GNU R version is based on LAPACK. The GNU Scientific Library (GSL) offers an algorithm based on Jacobi orthogonalization. There exists many other algorithms/variants for the numerical computation of the SVD, see [**GVL96**, sections 5.4.5 and 8.6].

```
octave> A = rand(5,3) % Generate a random 5x3 matrix
A = 0.3479368    0.7948432    0.0011214
    0.4912752    0.6836159    0.8509682
    0.0315889    0.9831456    0.3328946
    0.3665785    0.9985220    0.6228932
    0.2481886    0.5890069    0.2542045
octave> [U,D,V] = svd(A) % Compute SVD up to machine prec.
U = -0.351343   -0.557528    0.667944   -0.165446   -0.303643
    -0.509631    0.708933    0.144001   -0.438839   -0.156123
    -0.448938   -0.414874   -0.704250   -0.353062   -0.075585
    -0.563423    0.084485   -0.082364    0.808135   -0.124705
    -0.312799   -0.085546    0.174251   -0.048095    0.928527
D = Diagonal Matrix
      2.18534          0          0
            0    0.61541          0
            0          0    0.31967
            0          0          0
            0          0          0
V = -0.30703    0.24525    0.91956
    -0.83093   -0.54016   -0.13337
    -0.46400    0.80503   -0.36963
octave> norm(U*D*V'-A,"fro") % Quality check (Frobenius)
ans =   6.0189e-16
octave> norm(U*U'-eye(5,5),"fro") % Quality check (Frobenius)
ans =   8.2460e-16
octave> norm(V*V'-eye(3,3),"fro") % Quality check (Frobenius)
ans =   3.3309e-16
```

4.1.5. Some concrete applications of the SVD. — The SVD is typically used for dimension reduction and for regularization. For instance, the SVD allows to construct the so called Moore–Penrose pseudoinverse [**Moo20, Pen56**] of a matrix by replacing the non null singular values by their inverse while leaving in place the null singular values. Generalized inverses of integral operators were introduced earlier by Fredholm in [**Fre03**]. Such generalized inverse of matrices provide for instance least squares solutions to degenerate systems of linear equations. A diagonal shift in the SVD is used in the so called Tikhonov regularization [**Tik43, Tar05**] or ridge regression for solving over determined systems of linear equations. The SVD is at the heart of the so called principal component analysis (PCA) technique in applied statistics for multivariate data analysis, see for instance the book [**Jol02**]. The partial least squares (PLS) regression technique is also connected to PCA/SVD. In the last decade, the PCA was used together with the so called kernel methods in learning theory. Certain generalizations of the SVD are used for the regularization of ill posed inverse problems such as X ray tomography, emission tomography, inverse diffraction and inverse source problems, and the linearized inverse scattering problem, see for instance the book [**BB98**]. The application of the SVD to compressed sensing is under development and some few devoted books will appear in the near future.

4.2. Singular values of Gaussian random matrices

In the sequel, the standard Gaussian on K is $\mathcal{N}(0,1)$ if $K=\mathbb{R}$ and $\mathcal{N}(0,\frac{1}{2}I_2)$ if $K=\mathbb{C}\equiv\mathbb{R}^2$. If Z is a standard Gaussian random variable on K then

$$\mathrm{Var}(Z) := \mathbb{E}(|Z-\mathbb{E}Z|^2) = \mathbb{E}(|Z|^2) = 1.$$

4.2.1. Matrix model. — Let $(G_{i,j})_{i,j\geqslant 1}$ be an infinite matrix of i.i.d. standard Gaussian random variables on K. For every $m,n\in\{1,2,\ldots\}$, the random $m\times n$ matrix

$$G := (G_{i,j})_{1\leqslant i\leqslant m,\,1\leqslant j\leqslant n}$$

has Lebesgue density, in $\mathcal{M}_{m,n}(K)\equiv K^{nm}$, proportional to

$$G\mapsto \exp\left(-\frac{\beta}{2}\sum_{i=1}^m\sum_{j=1}^n |G_{i,j}|^2\right) = \exp\left(-\frac{\beta}{2}\mathrm{Tr}(GG^*)\right) = \exp\left(-\frac{\beta}{2}\|G\|_{\mathrm{HS}}{}^2\right)$$

where

$$\beta := \begin{cases} 1 & \text{if } K=\mathbb{R}, \\ 2 & \text{if } K=\mathbb{C}. \end{cases}$$

The law of G is K–unitary invariant since $UGV \stackrel{d}{=} G$ for every deterministic K–unitary matrices U $(m\times m)$ and V $(n\times n)$. For $K=\mathbb{C}$ we have

$$G = \frac{1}{\sqrt{2}}\,(G_1+\sqrt{-1}\,G_2)$$

where G_1 and G_2 are i.i.d. copies of the case $K = \mathbb{R}$. The law of G is also known as the K Ginibre ensemble, see [**Gin65, Meh04**]. The symplectic case where K is the quaternions ($\beta = 4$) is not considered in these notes. The columns $C_1, \dots, C_n$ of the random matrix G are i.i.d. standard Gaussian random column vectors of K^m with i.i.d. standard Gaussian coordinates. Their *empirical covariance matrix* is

$$\frac{1}{n}\sum_{k=1}^{n} C_k C_k^* = \frac{1}{n} GG^*.$$

The strong law of large numbers gives $\lim_{n\to\infty} n^{-1}GG^* = I_m$ a.s. We are interested in the sequel in asymptotics when both n and m tend to infinity. The random matrix GG^* is $m \times m$ Hermitian positive semidefinite. If $m > n$ then the random matrix GG^* is singular with probability one, as a linear combination of $n < m$ rank one $m \times m$ matrices $C_1C_1^*, \dots, C_nC_n^*$. If $m \leqslant n$ then the random matrix GG^* is invertible with probability one (comes from the diffuse nature of Gaussian measures), and

$$\forall k \in \{1, \dots, m\}, \quad s_k(G)^2 = \lambda_k(GG^*) = n\lambda_k\left(\frac{1}{n}GG^*\right).$$

4.2.2. Unitary bidiagonalization. — Let us consider the K–unitary bidiagonalization of section 4.1.4, for the Gaussian matrix G. Assume for convenience that $m \leqslant n$. One can find random K–unitary matrices W ($m \times m$) and W' ($n \times n$) such that $B := WGW'$ is bidiagonal with

$$B = \frac{1}{\sqrt{\beta}}\begin{pmatrix} S_n & 0 & 0 & 0 & & & \cdots & & & 0 \\ T_{m-1} & S_{n-1} & 0 & 0 & & & \cdots & & & 0 \\ 0 & T_{m-2} & S_{n-2} & 0 & & & \cdots & & & 0 \\ 0 & 0 & \vdots & \vdots & & & \cdots & & & 0 \\ \vdots & \vdots & & & & & \cdots & & & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & T_1 & S_{n-(m-1)} & 0 \; \cdots & 0 \end{pmatrix}. \tag{4.7}$$

Following Silverstein [**Sil85**] the random variables $S_n, \dots, S_{n-(m-1)}, T_{m-1}, \dots, T_1$ are independent with laws given by $S_k^2 \sim \chi^2(\beta k)$ for every $k \in \{n-(m-1), \dots, n\}$ and $T_k^2 \sim \chi^2(\beta k)$ for every $k \in \{1, \dots, m-1\}$. The random matrices B and G share the same sequence of singular values. Such an explicit bidiagonalization has an amazing consequence for the simulation of the singular values of G. It allows to reduce the dimension from nm to $2(n \wedge m) - 1$.

4.2.3. Densities. — The random Hermitian positive semidefinite $m \times m$ matrix GG^* can be seen as a random vector of $\mathbb{R}^m \times K^{(m^2-m)/2}$. If $m \leqslant n$, the law of GG^* is a Wishart distribution with Lebesgue density in $\mathbb{R}^m \times K^{(m^2-m)/2}$ proportional to

$$W \mapsto \det(W^{\beta(n-m+1)/2-1}) \exp\left(-\frac{\beta}{2}\mathrm{Tr}(W)\right) \tag{4.8}$$

on the cone of Hermitian positive semidefinite matrices. This Wishart law is also known as the β–Laguerre ensemble or Laguerre Orthogonal Ensemble (LOE) for $\beta = 1$

and Laguerre Unitary Ensemble (LUE) for $\beta = 2$. The correlation between the entries is captured by the determinental term, which surprisingly vanishes when $n = m + 2\beta^{-1} - 1$. In the SVD of G, one can take U, V distributed according to the normalized Haar measure on the K–unitary group, and independent of the singular values. As a consequence, the same holds true for the K–unitary diagonalization of the positive semidefinite Hermitian matrix GG^*. When $m \leqslant n$, this diagonalization, seen as a change of variable, followed by the partial integration over the K–unitary group of (4.8) with respect to the eigenvectors, gives the expression of the density of $\lambda_1(GG^*), \ldots, \lambda_m(GG^*)$, which turns out to be proportional to

$$\lambda \mapsto \exp\left(-\frac{\beta}{2}\sum_{i=1}^m \lambda_i\right) \prod_{i=1}^m \lambda_i^{\beta(n-m+1)/2-1} \prod_{1\leqslant i<j\leqslant m} |\lambda_i - \lambda_j|^\beta \qquad (4.9)$$

on $\{\lambda \in [0,\infty)^m : \lambda_1 \geqslant \cdots \geqslant \lambda_n\}$. The normalizing constant is a Selberg integral, and can be explicitly computed [**Meh04**]. The (repulsive) correlation is captured by the Vandermonde determinant, which comes from the Jacobian of the change of variable (unitary diagonalization). If $m = n = 1$ then (4.8,4.9) are identical (χ^2 law). The formulas (4.8,4.9) were considered by e.g. Wishart [**Wis28**] and James [**Jam60**]. For a modern presentation, see e.g. Edelman and Rao [**ER05**] or Haagerup and Thorbjørnsen [**HT03**].

4.2.4. Orthogonal polynomials. — Set $K = \mathbb{C}$. If $m \leqslant n$ then the density (4.9) of the eigenvalues of the $m \times m$ random matrix GG^* turns out to be proportional to

$$\lambda \mapsto \det\left[(S(\lambda_i, \lambda_j))_{1\leqslant i,j\leqslant m}\right] \quad \text{with} \quad S(x,y) := \sqrt{g(x)g(y)}\sum_{k=0}^{m-1} P_k(x)P_k(y) \quad (4.10)$$

where $(P_k)_{k\geqslant 0}$ are the Laguerre orthonormal polynomials [**Sze75**] relative to the Gamma law on $[0,\infty)$ with density g proportional to $x \mapsto x^{n-m}\exp(-x)$. Both g and $(P_k)_{k\geqslant 0}$ depend on m, n. These determinental/polynomial formulas appear in various works, see e.g. Deift [**Dei99**], Forrester [**For10**], and Mehta [**Meh04**], Haagerup and Thorbjørnsen [**HT03**] and Ledoux [**Led04**]. When $m \leqslant n$, and at the formal level, it follows from this determinental/polynomial expression of the density that for any Borel symmetric function $F : [0,\infty)^m \to \mathbb{R}$, the expectation

$$\mathbb{E}[F(\lambda_1(GG^*), \ldots, \lambda_m(GG^*))]$$

can be expressed in terms of the determinant (4.10). The behavior of such averaged symmetric functions when n and m tend to infinity is related to the asymptotics of the Laguerre polynomials $(P_k)_{k\geqslant 0}$. Useful symmetric functions of the eigenvalues include

1. $F(\lambda_1, \ldots, \lambda_m) = f(\lambda_1) + \cdots + f(\lambda_m)$ for some fixed $f : [0,\infty) \to \mathbb{R}$;
2. $F(\lambda_1, \ldots, \lambda_m) = \min(\lambda_1, \ldots, \lambda_m)$;
3. $F(\lambda_1, \ldots, \lambda_m) = \max(\lambda_1, \ldots, \lambda_m)$.

The real case $K = \mathbb{R}$ is similar but trickier due to $\beta = 1$ in (4.9), see [**For10**].

4.2.5. Behavior of the singular values. — We begin our tour of horizon with the behavior of the counting probability measure of the eigenvalues of $n^{-1}GG^*$. It is customary in random matrix theory to speak about the "bulk behavior", in contrast with the "edge behavior" which concerns the extremal eigenvalues. When $m \leqslant n$, this corresponds to the counting probability measure of the squared singular values of $n^{-1/2}G$. The first version of the following theorem is due to Marchenko and Pastur [**MP67**].

Theorem 4.2.1 **(Bulk behavior).** — *If $m = m_n \to \infty$ with $\lim_{n\to\infty} m_n/n = y \in (0,\infty)$ then a.s. the spectral counting probability measure*

$$\mu_{n^{-1}GG^*} := \frac{1}{m}\sum_{k=1}^{m} \delta_{\lambda_k(n^{-1}GG^*)}$$

converges narrowly to the Marchenko–Pastur law

$$\mathcal{L}_{\mathrm{MP}} = \left(1 - \frac{1}{y}\right)^{+} \delta_0 + \frac{1}{2\pi y}\frac{\sqrt{(b-x)(x-a)}}{x}\mathbb{1}_{[a,b]}(x)\,dx$$

where

$$a = (1-\sqrt{y})^2 \quad \textit{and} \quad b = (1+\sqrt{y})^2.$$

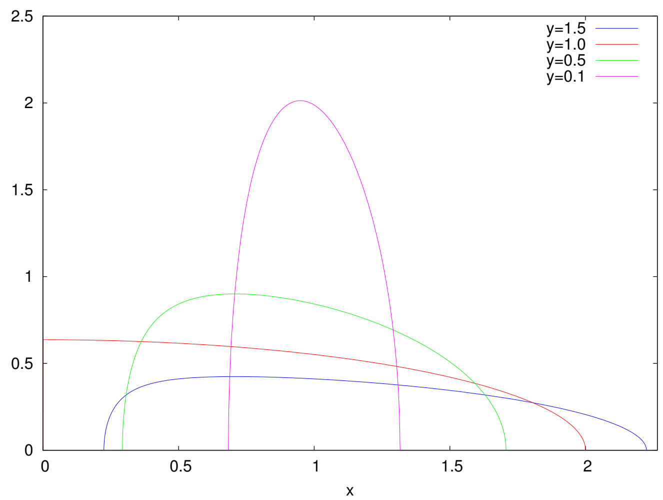

FIGURE 2. Density of the limiting law of the empirical singular values distribution $\frac{1}{m}\sum_{k=1}^{m}\delta_{s_k(n^{-1/2}G)}$ when $m = m_n$ with $\lim_{n\to\infty} m_n/n = y$, for different values of y (theorem 4.2.1). This is nothing else but the density of the absolutely continuous part of the image law of $\mathcal{L}_{\mathrm{MP}}$ by the map $x \mapsto \sqrt{x}$. The case $y = 1$ corresponds to the so called quartercircular law. This graphics was obtained by using the wxMaxima software.

Idea of the proof. — When $K=\mathbb{C}$, the result can be obtained by using the determinental/polynomial approach. Namely, following Haagerup and Thorbjørnsen [**HT03**] or Ledoux [**Led04**], for every Borel function $f:[0,\infty)\to\mathbb{R}$, we have,

$$\int f(x)\,d\mu_{n^{-1}GG^*}(x)=\Big(1-\frac{n}{m}\Big)^+f(0)+\int_0^\infty f(x)\,\frac{1}{m}\,S(x,x)\,dx$$

where S is as in (4.10). Note that the left hand side is a symmetric function of the eigenvalues. The Dirac mass at point 0 in the first term of the right hand side above comes from the fact that if $m>n$ then $m-n$ eigenvalues of GG^* are necessarily zero (the remaining eigenvalues are the square of the singular values of G). The convergence to $\mathcal{L}_{\mathrm{MP}}$ is a consequence of the behavior of $m^{-1}S(x,x)$ related to classical equilibrium measures of orthogonal polynomials, see [**Led04**, pages 191–192]. In this approach, $\mathcal{L}_{\mathrm{MP}}$ is recovered as a mixture of uniform and arcsine laws.

Another approach is the so called trace/moments method, based on the identity

$$\int x^r\,d\mu_{n^{-1}GG^*}(x)=\frac{1}{nm^r}\mathrm{Tr}((GG^*)^r)$$

valid for every $r\in\{0,1,2,\ldots\}$. The expansion of the right hand side in terms of the entries of G allows to show that the moments of $\mu_{n^{-1}GG^*}$ converge to the moments of $\mathcal{L}_{\mathrm{MP}}$. The Gaussian nature of the entries allows to use the Wick formula in order to simplify the computations. There is also an approach based on the Cauchy–Stieltjes transform, or equivalently the trace–resolvent, see [**HP00**] or [**Bai99**]. This gives a recursive equation obtained by bloc matrix inversion, which leads to a fixed point problem. Here again, the Gaussian integration by parts may help. The trace/moment and the Cauchy–Stieltjes trace/resolvent methods are “universal” in the sense that they are still available when G is replaced by a random matrix with non Gaussian i.i.d. entries. The determinental/polynomial approach is rigid, and relies on the determinental nature of the law of G, which comes from the unitary invariance of G. It remains available beyond the Gaussian case, provided that G has a unitary invariant density proportional to $G\mapsto\exp(-\mathrm{Tr}(V(GG^*)))$ for a potential $V:\mathbb{R}\to\mathbb{R}$.

There is finally a more original approach based on large deviations via a Varadhan like lemma, which exploits the explicit expression of the law of the eigenvalues. We recover $\mathcal{L}_{\mathrm{MP}}$ as a minimum of the logarithmic energy with Laguerre external field. □

The limiting distribution is a mixture of a Dirac mass at zero (when $y>1$) with an absolutely continuous compactly supported distribution known as the Marchenko–Pastur law. The presence of this Dirac mass is due to the fact that if $y>1$ then a.s. the random matrix $n^{-1}GG^*$ is not of full rank for large enough n. The a.s. weak convergence in theorem 4.2.1 says that for any interval $I\subset[0,\infty)$,

$$\frac{\big|\big\{k\in\{1,\ldots,m\}\text{ such that }\lambda_k(n^{-1}GG^*)\in I\big\}\big|}{m}\xrightarrow[n\to\infty]{\text{a.s.}}\mathcal{L}_{\mathrm{MP}}(I).$$

This convergence implies immediately the following corollary.

Corollary 4.2.2 **(Edge behavior implied by bulk behavior)**

If $m = m_n \to \infty$ with $\lim_{n\to\infty} m_n/n = y \in (0,\infty)$ then a.s.

$$\liminf_{n\to\infty} \lambda_1(n^{-1}GG^*) \geqslant (1+\sqrt{y})^2.$$

Moreover, if $y \leqslant 1$ then a.s.

$$\limsup_{n\to\infty} \lambda_{m_n}(n^{-1}GG^*) \leqslant (1-\sqrt{y})^2.$$

In particular, if $m_n = n$ then $y = 1$ and a.s.

$$\frac{1}{\sqrt{n}}\, s_n(G) = \sqrt{\lambda_n(n^{-1}GG^*)} \xrightarrow[n\to\infty]{a.s.} 0.$$

It is then natural to ask about the convergence of the extremal eigenvalues of $n^{-1}GG^*$ to the edge of the limiting support. In a sense, the left edge a is "soft" if $y < 1$ and "hard" if $y = 1$. The term "soft" means that the fluctuation may hold in both sides while "hard" means that the fluctuation is confined in a single side. The right edge b is "soft" regardless of y. We will see that the nature of the fluctuations of the extremal singular values depends on the hard/soft nature of the edge.

***Theorem 4.2.3* (Convergence of smallest singular value)**
If $m = m_n \to \infty$ with $m_n \leqslant n$ and $\lim_{n\to\infty} m_n/n = y \in (0,1]$ then

$$\left(\frac{1}{\sqrt{n}}\, s_{m_n}(G)\right)^2 = \lambda_{m_n}(n^{-1}GG^*) \xrightarrow[n\to\infty]{a.s.} (1-\sqrt{y})^2.$$

Idea of the proof. — Corollary 4.2.2 reduces immediately the problem to show that a.s.

$$\liminf_{n\to\infty} \lambda_m(n^{-1}GG^*) \geqslant (1-\sqrt{y})^2.$$

Following Silverstein [**Sil85**], we have $\lambda_m(GG^*) = \lambda_m(BB^*)$ where B is as in (4.7). Observe that BB^* is tridiagonal. One can then control $\lambda_m(BB^*)$ by using the law of B and the Geršgorin disks theorem which states that if $A \in \mathcal{M}_{n,n}(K)$ then

$$\{\lambda_1(A), \ldots, \lambda_n(A)\} \subset \bigcup_{i=1}^{n} \{z \in \mathbb{C}; |z - A_{i,i}| \leqslant r_i\} \quad \text{where} \quad r_i := \sum_{j\neq i} |A_{i,j}|.$$

When $K = \mathbb{C}$, an alternative approach is based on the determinental/polynomial formula for the density of the eigenvalues of GG^*, and can be found in [**Led04**]. □

***Theorem 4.2.4* (Fluctuation of smallest singular value for hard edge)**
Assume that $m = n$.

- *If $K = \mathbb{C}$ then for every $n \in \{1, 2, \ldots\}$, the random variable $(\sqrt{n}\, s_n(G))^2$ follows an exponential law of unit mean with Lebesgue density $x \mapsto \exp(-x)$. In other words, for every $n \in \{1, 2, \ldots\}$ and any real number $t \geqslant 0$,*

$$\mathbb{P}\big(\sqrt{n}\, s_n(G) \geqslant t\big) = \exp\left(-t^2\right).$$

– *If* $K = \mathbb{R}$ *then the random variable* $(\sqrt{n}\, s_n(G))^2$ *converges in distribution as* $n \to \infty$ *to the law with Lebesgue density density*

$$x \mapsto \frac{1+\sqrt{x}}{2\sqrt{x}} \exp\left(-\frac{1}{2}x - \sqrt{x}\right).$$

In other words, for every real number $t \geqslant 0$,

$$\lim_{n\to\infty} \mathbb{P}\big(\sqrt{n}\, s_n(G) \geqslant t\big) = \exp\left(-\frac{1}{2}t^2 - t\right).$$

Idea of the proof. — When $K = \mathbb{C}$, it suffices to integrate (4.9) over all but the smallest eigenvalue. This gives that the random variable $n\lambda_n(GG^*) = ns_n(G)^2$ follows an exponential law with unit mean. This is immediate when $n = 1$ from (4.9). When $K = \mathbb{R}$, one can proceed as for the complex case, but with this time $\beta = 1$. This makes the computations non explicit for a fixed n due to the factors $\lambda_i^{-1/2}$ which were not present for $K = \mathbb{C}$. However, following Edelman [**Ede88**], for every n,

$$\lambda_n(GG^*) = s_n(G)^2 \quad \text{has density proportional to} \quad x \mapsto \frac{1}{\sqrt{x}} U_n\Big(\frac{x}{2}\Big) \exp\left(-\frac{1}{2}nx\right)$$

where U_n is the Tricomi function, unique solution of the Kummer differential equation

$$2xU_n''(x) - (1+2x)U_n'(x) - (n-1)U_n(x) = 0$$

with boundary conditions $2U_n(0)\Gamma(1+n/2) = \sqrt{\pi}$ and $U_n(\infty) = 0$. The Tricomi function admits an integral representation, and is also known as the Gordon function or the confluent hypergeometric function of the second kind, see [**AS64**, Chapter 13.6]. The behavior of the Tricomi function gives the limiting law of $\sqrt{n}\, s_n(G)$. □

***Theorem 4.2.5* (Convergence of largest singular value)**
If $m = m_n \to \infty$ *with* $\lim_{n\to\infty} m_n/n = y \in (0,\infty)$ *then*

$$\left(\frac{1}{\sqrt{n}}\, s_1(G)\right)^2 = \frac{1}{n}\lambda_1(GG^*) \xrightarrow[n\to\infty]{a.s.} (1+\sqrt{y})^2.$$

Idea of the proof. — Corollary 4.2.2 reduces the problem to show that

$$\limsup_{n\to\infty} \lambda_1(n^{-1}GG^*) \leqslant (1+\sqrt{y})^2.$$

This was proved in turn by Geman [**Gem80**], following an idea of Grenander. The method, which can be seen as an instance of the so called power method, consists in the control of the expected operator norm of a power of $n^{-1}GG^*$ with the expected Frobenius norm, and then in the usage of expansions in terms of the matrix entries via the trace formula for the Frobenius norm. This method does not rely on explicit Gaussian computations. When $K = \mathbb{C}$, one can use the determinental/polynomial formula for the density of the eigenvalues of GG^* as in the work of Ledoux [**Led04**]. □

Gaussian exponential bounds for the tail of the singular values of G are also available, and can be found for instance in the work of Szarek [**Sza91**], Davidson and Szarek [**DS01**], Haagerup and Thorbjørnsen [**HT03**], and Ledoux [**Led04**].

The fluctuation of the smallest singular value in the hard edge case given by theorem 4.2.4 can be also expressed in terms of a Bessel kernel, see for instance the work of Forrester [**For10**]. Let us consider now the fluctuation of the largest singular value around its limit. The famous Tracy–Widom laws TW_1 and TW_2 are known to describe the fluctuation of the largest eigenvalue in the ensembles of square Hermitian Gaussian random matrices (GOE for $K = \mathbb{R}$ and GUE for $K = \mathbb{C}$), see [**TW02**]. One can ask if these laws still describe the fluctuations of the largest singular values of the Gaussian matrix G. By definition, TW_1 and TW_2 are the probability distributions on $\mathbb{R}$ with cumulative distribution functions F_1 and F_2 given for every $s \in \mathbb{R}$ by

$$F_2(s) = \exp\left(-\int_s^\infty (x-s)q(x)^2\,dx\right) \quad \text{and} \quad F_1(s)^2 = F_2(s)\exp\left(-\int_s^\infty q(x)\,dx\right)$$

where q is the solution of the Painlevé II differential equation

$$q''(x) = xq(x) + 2q^3(x)$$

with boundary condition $q(x) \sim \mathrm{Ai}(x)$ as $x \to \infty$, where Ai is the Airy function

$$\mathrm{Ai}(x) := \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right)dt.$$

The Airy function Ai is also uniquely defined by the properties

$$\mathrm{Ai}''(x) = x\mathrm{Ai}(x) \quad \text{and} \quad \mathrm{Ai}(x) \sim_{x\to\infty} \frac{1}{2\sqrt{\pi}x^{1/4}}\exp\left(-\frac{2}{3}x^{3/2}\right).$$

The function F_2 can be expressed as a Fredholm determinant: $F_2(s) = \det(I - A_s)$ where A_s is the Airy operator on square integrable functions on (s,∞), with kernel

$$A_s(x,y) := \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}.$$

See for instance [**Dei99, Dei07**] and [**For10**] for more information, and [**ER05**] and [**Joh01**] for the numerical evaluation of F_1 and F_2.

Theorem 4.2.6 **(Fluctuation of largest singular value)**
If $m = m_n \to \infty$ with $m_n \leqslant n$ and $\lim_{n\to\infty} m_n/n = y \in (0,1]$ then, by denoting

$$\mu_{\beta,n} := \left(\sqrt{n+\beta-2} + \sqrt{m}\right)^2 \quad \textit{and} \quad \sigma_{\beta,n} := \sqrt{\mu_{\beta,n}}\left(\frac{1}{\sqrt{n+\beta-2}} + \frac{1}{\sqrt{m}}\right)^{1/3},$$

the random variable

$$\frac{s_1(G)^2 - \mu_{\beta,n}}{\sigma_{\beta,n}}$$

converges narrowly as $n \to \infty$ to the Tracy–Widom law TW_β. Moreover, if $y > 1$ then the result remains true up to the swap of the roles of m and n in the formulas (recall that G and G^* have same singular values).

For $K = \mathbb{C}$ and $m = n$, we have $\beta = 2$ and $\mu_{2,n} = 4n$ while $\sigma_{2,n} = (16n)^{1/3}$. The Tracy–Widom fluctuation based on the Airy kernel describes also the fluctuation of the smallest singular value in the soft edge regime ($y < 1$), see for instance the book by Forrester [**For10**] and the approach of Ledoux [**Led04**].

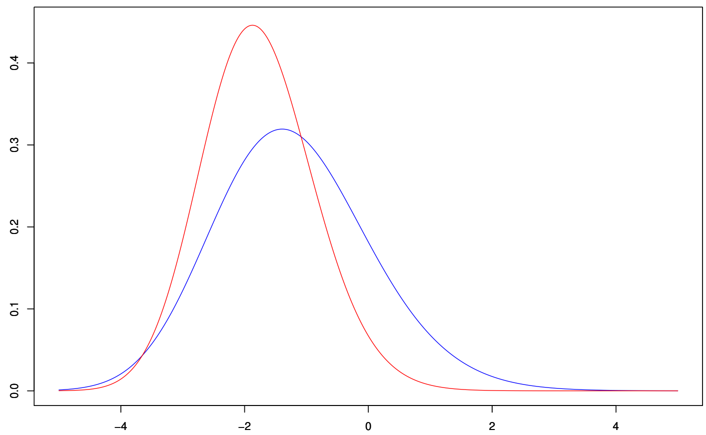

FIGURE 3. Density of TW_β for $\beta = 1$ (blue) and $\beta = 2$ (red), obtained by using the GNU-R package RMTstat.

Idea of the proof. — The proofs of Johnstone [**Joh01**] and Johansson [**Joh00**] are based on the determinental/polynomial approach. Let us give the first steps when $K = \mathbb{C}$. If S is as in (4.10), then for every Borel function $f : [0, \infty) \to \mathbb{R}$,

$$\mathbb{E}\left[\prod_{k=1}^{m}(1 + f(\lambda_k(GG^*)))\right] = c_{n,m} \det(I + Sf)$$

where $c_{n,m}$ is a normalizing constant. Here one must see S as an integral operator. For the particular choice $f = -\mathbb{1}_{[t,\infty)}$ for some fixed $t \geqslant 0$, this gives

$$\mathbb{P}\left(\max_{1\leqslant k\leqslant m} \lambda_k(GG^*) \geqslant t\right) = c_{n,m} \det(I - S\mathbb{1}_{[t,\infty)}).$$

Now the Tracy and Widom [**TW02**] heuristics says that the determinant in the right hand side satisfies to a differential equation, which is Painlevé II as $n \to \infty$. See also the work of Borodin and Forrester [**BF03, For10**], and the work of Ledoux [**Led04**] inspired from the work of Haagerup and Thorbjørnsen [**HT03**].

An alternative approach, based on the bidiagonalization trick (4.7), was provided by Ramírez, Rider, and Virág [**RRV09**]. This can be viewed as the β–Laguerre (LUE and LOE) analogue of the work of Edelman and Sutton [**ES05**] for β–Hermite ensembles (GUE and GOE). In particular, it provides the convergence of the rescaled extremal singular values to a Schrödinger operator. □

The largest eigenvalue of such matrices can be seen as the maximum of a random vector with correlated coordinates (Vandermonde repulsion). Here the asymptotic fluctuation is not captured by classical extreme values theory for i.i.d. samples

(Gnedenko–Fréchet–Fisher–Tippett–Gumbel theorem, see [**Res08**]). The laws TW_1 and TW_2 are unimodal, asymmetric, with exponentially light tails. For instance, TW_1 has a left tail $\approx \exp\left(-\frac{1}{24}|x|^3\right)$ and a right tail $\approx \exp\left(-\frac{2}{3}|x|^{3/2}\right)$, see [**Joh01**].

The study of the extremal singular values $s_1(G), s_n(G)$ and the condition number $\kappa(G) = s_1(G)/s_n(G)$ of the random Gaussian matrix G was motivated at the origin by the behavior of numerical algorithms with random inputs. This goes back at least to von Neumann and his collaborators [**vN63, vNG47**], Smale [**Sma85**], Demmel [**Dem88**], and Kostlan [**Kos88**]. Note that if $m = n$ then

$$\kappa(G) = \frac{s_1(G)}{s_n(G)} = \sqrt{\frac{\lambda_1(GG^*)}{\lambda_n(GG^*)}} = \sqrt{\kappa(GG^*)} = \sqrt{n}\,\kappa\left(\frac{1}{n}\,GG^*\right).$$

An elementary result on $\kappa(G)$ is captured by the following corollary. For sharp estimates on the tails of $\kappa(G)$, see for instance the work of Edelman and Sutton [**ES05**], Szarek [**Sza91**], Azaïs and Wschebor [**AW05**], and also Chen and Dongarra [**CD05**]. These sharp bounds involve the control of the joint law of the extremal singular values. This joint law can be expressed with zonal polynomials and hypergeometric functions [**Mui82, RVA05**]. This expression is difficult to exploit. The approach of Azaïs and Wschebor [**AW05**] is based on Rice formulas for Gaussian processes extrema, see [**AW09**]. For the case $\beta \notin \{1,2\}$, see for instance [**DK08**] and references therein.

Corollary 4.2.7 **(Condition number, $m = n$).** — *If $m = n$ then $n^{-1}\,\kappa(G)$ converges in distribution as $n \to \infty$ to a law with Lebesgue density*

$$x \mapsto \begin{cases} \dfrac{2(x+1)}{x^3}\,\exp\left(-\dfrac{1}{2x^2} - \dfrac{1}{x}\right) & \text{if } K = \mathbb{R}, \\[2ex] \dfrac{4}{x^3}\,\exp\left(-\dfrac{1}{x^2}\right) & \text{if } K = \mathbb{C}. \end{cases}$$

Proof. — Theorem 4.2.5 gives that a.s. $s_1(G) = (2+o(1))\sqrt{n}$ as $n \to \infty$. We conclude by using the Slutsky lemma and the limiting law of $\sqrt{n}\,s_n(G)$ (theorem 4.2.4). □

4.3. Universality of the Gaussian case

Gaussian random matrices with i.i.d. entries such as G have the advantage to allow explicit computations. But one can ask if such Gaussian matrices are good enough for the modelling of random inputs of algorithms. For instance, the support of such random matrices is essentially concentrated on a centered Frobenius ball, which can be seen as a drawback. More generally, let us consider a sample of n i.i.d. random column vectors of K^m. One can ask about the behavior of the eigenvalues of their empirical covariance matrix in the situation where the common law of the vectors…

- is not centered
- is not a tensor product
- has only few finite moments (heavy tails)
- does not have a density (for instance Bernoulli or Rademacher entries).

It is rather difficult to give a comprehensive account on the available literature in few pages. Regarding independent column vectors, a whole line of research is based on tools and concepts from high dimensional geometric analysis, such as the work of Mendelson and Pajor [**MP06**] and the work of Adamczak, Guédon, Litvak, Pajor, and Tomczak-Jaegermann [**AGL$^+$08**]. In the sequel, we restrict our attention on some few results regarding the singular values of random matrices with i.i.d. entries.

Many results for random matrices with i.i.d. Gaussian entries remain valid for non Gaussian entries when the moments match the Gaussian moments up to some order. This is referred as "universality". Let $(X_{i,j})_{i,j\geqslant 1}$ be an infinite matrix with i.i.d. entries in K. We consider in the sequel the $m \times n$ random matrix

$$X := (X_{i,j})_{1\leqslant i\leqslant m,\ 1\leqslant j\leqslant n}.$$

When $X_{1,1}$ is a standard Gaussian random variable then $X \overset{d}{=} G$ where G is the Gaussian random matrix of the preceding section. Note that if the law of $X_{1,1}$ has atoms, then XX^* is singular with positive probability, even if $m \leqslant n$. Moreover, if $X_{1,1}$ is not standard Gaussian, the law of X is no longer K–unitary invariant, and the law of the eigenvalues of XX^* is not explicit in general. One of the first universal version of theorem 4.2.1 is due to Wachter [**Wac78**]. See also the review article of Bai [**Bai99**]. For the version given below, see the book of Bai and Silverstein [**BS10**], and the article by Bai and Yin [**BY93**] on the behavior at the edge.

***Theorem 4.3.1* (Universality for bulk and edges convergence)**
If $X_{1,1}$ has mean $\mathbb{E}[X_{1,1}] \in K$ and variance $\mathbb{E}[|X_{1,1} - \mathbb{E}[X_{1,1}]|^2] = 1$, and if $m = m_n \to \infty$ with $m_n/n \to y \in (0,\infty)$, then the conclusion of theorem 4.2.1 remain valid if we replace G by X. Moreover, if $\mathbb{E}[X_{1,1}] = 0$ and $\mathbb{E}[|X_{1,1}|^4] < \infty$ then the conclusion of theorem 4.2.3 (when $y \in (0,1)$) and theorem 4.2.5 remain valid if we replace G by X. If however $\mathbb{E}[|X_{1,1}|^4] = \infty$ or $\mathbb{E}[X_{1,1}] \neq 0$ then a.s.

$$\limsup_{n\to\infty} \lambda_1(n^{-1}XX^*) = \infty.$$

The bulk behavior is not sensitive to the mean $\mathbb{E}[X]$, and this can be understood from the decomposition $X = X - \mathbb{E}[X] + \mathbb{E}[X]$ where $\mathbb{E}[X] = \mathbb{E}[X_{1,1}](1 \otimes 1)$ has rank at most 1, by using the Thompson theorem 4.1.5. Regarding empirical covariance matrices, many other situations are considered in the literature, for instance in the work of Bai and Silverstein [**BS98**], Dozier and Silverstein [**DS07a, DS07b**], Hachem, Loubaton, and Najim [**HLN06**], with various concrete motivations ranging from asymptotic statistics to information theory and signal processing.

The universality of the fluctuation of the smallest and largest eigenvalues of empirical covariances matrices was studied for instance by Soshnikov [**Sos02**], Baik, Ben Arous, and Péché [**BBAP05**], Ben Arous and Péché [**BAP05**], El Karoui [**EK07**], Féral and Péché [**FP09**], Péché [**Péc09**], Tao and Vu [**TV09b**], and by Feldheim and Sodin [**FS10**]. The following theorem on the soft edge case, due to Feldheim and Sodin [**FS10**], includes partly the results of Soshnikov [**Sos02**] and Péché [**Péc09**].

***Theorem 4.3.2* (Universality for soft edges).** — *If the law of $X_{1,1}$ is symmetric about 0, with sub-Gaussian tails, and first two moments identical to the ones of $G_{1,1}$,* *and if $m = m_n \to \infty$ with $m_n \leqslant n$ and $\lim_{n\to\infty} m_n/n = y \in (0,\infty)$, then $s_1(X)^2$ has the Tracy–Widom rates and fluctuations of $s_1(G)^2$ as in the Gaussian theorem 4.2.6. If moreover $y < 1$ then the same holds true for the smallest singular value $s_{m_n}(X)^2$.*

The following theorem concerns the universality of the fluctuations of the smallest singular value in the hard edge regime, recently obtained by Tao and Vu [**TV09b**].

***Theorem 4.3.3* (Universality for hard edge).** — *If $m = n$ and if $X_{1,1}$ has first two moments identical to the ones of $G_{1,1}$, and if $\mathbb{E}[|X_{1,1}|^{10000}] < \infty$, then the random variable $(\sqrt{n}\, s_n(X))^2$ converges as $n \to \infty$ to the limiting law of the Gaussian case which appears in theorem 4.2.4.*

Actually, a stronger version of theorems 4.3.2 and 4.3.3 is available, expressing the fact that for every fixed k, the top and bottom k singular values are identical in law asymptotically to the corresponding quantities for the Gaussian model.

Following [**TV09b**], theorem 4.3.3 implies in particular that if $X_{1,1}$ follows the symmetric Rademacher law $\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ then, for $m = n$, for all $t > 0$,

$$\mathbb{P}(\sqrt{n}\, s_n(X) \leqslant t) = \int_0^{t^2} \frac{1+\sqrt{x}}{2\sqrt{x}} e^{-\frac{1}{2}x-\sqrt{x}}\, dx + o(1) = 1 - e^{-\frac{1}{2}t^2 - t} + o(1).$$

In [**TV09b**], the $o(1)$ error term is shown to be of the form $O(n^{-c})$ uniformly over t. This is close to the statement of a conjecture by Spielman and Teng on the invertibility of random sign matrices stating the existence of a constant $c \in (0,1)$ such that

$$\mathbb{P}(\sqrt{n}\, s_n(X) \leqslant t) \leqslant t + c^n \tag{4.11}$$

for every $t \geqslant 0$. The c^n is due to the fact that X has a positive probability of being singular (e.g. equality of two rows). In 2008, Spielman and Teng were awarded the Gödel Prize for their work on smoothed analysis of algorithms [**ST03, ST02**]. Actually, it has been conjectured years ago that

$$\mathbb{P}(s_n(X) = 0) = \left(\frac{1}{2} + o(1)\right)^n.$$

This intuition comes from the probability of equality of two rows, which implies that $\mathbb{P}(s_n(X) = 0) \geqslant (1/2)^n$. Many authors contributed to this difficult nonlinear discrete problem, such as Komlós [**Kom67**], Kahn, Komlós, and Szemerédi [**KKS95**], Rudelson [**Rud08**], Bruneau and Germinet [**BG09**], Tao and Vu [**TV06, TV07, TV09a**], and Bourgain, Vu, and Wood [**BVW10**] who proved that

$$\mathbb{P}(s_n(X) = 0) \leqslant \left(\frac{1}{\sqrt{2}} + o(1)\right)^n \quad \text{for large enough } n.$$

Back to theorem 4.3.1, the bulk behavior when $X_{1,1}$ has an infinite variance was recently investigated by Belinschi, Dembo, and Guionnet [**BDG09**], using (4.1). They considered heavy tailed laws similar to α–stable laws ($0 < \alpha \leqslant 2$), with polynomial tails. For simulations, if U and ε are independent random variables with U uniform on $[0,1]$ and ε Rademacher $\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$, then the random variable $T = \varepsilon(U^{-1/\alpha} - 1)$ has a symmetric bounded density and $\mathbb{P}(|T| > t) = (1+t)^{-\alpha}$ for any $t \geqslant 0$. In this situation, the normalization n^{-1} in $n^{-1}XX^*$ must be replaced by $n^{-2/\alpha}$. The limiting spectral distribution is no longer a Marchenko–Pastur distribution, and has heavy tails. In the case where $X_{1,1}$ is Cauchy distributed, it is known that the largest eigenvalues are distributed according to a Poisson statistics, see the work of Soshnikov and Fyodorov [**SF05**] and the review article by Soshnikov [**Sos06**]. On can ask about the invertibility of such random matrices with heavy tailed i.i.d. entries. The following lemma gives a rather crude lower bound on the smallest singular value of random matrices with i.i.d. entries with bounded density. However, it shows that the invertibility of these random matrices can be controlled without moments assumptions. Arbitrary heavy tails are therefore allowed, but Dirac masses are not allowed.

***Lemma 4.3.4* (Polynomial lower bound on s_n for bounded densities)**
Assume that $X_{1,1}$ is absolutely continuous with bounded density f. If $m = n$ then there exists an absolute constant $c > 0$ such that for every $n \in \{1, 2, \dots\}$ and $u \geqslant 0$,

$$\mathbb{P}(\sqrt{n}\, s_n(X) \leqslant u) \leqslant cn^{\frac{3}{2}} \left\| f \right\|_\infty u^\beta.$$

From the first Borel–Cantelli lemma, it follows that there exists $b > 0$ such that a.s. for large enough n, we have $s_n(X) > n^{-b}$.

Proof. — Let $R_1, \dots, R_n$ be the rows of X. From lemma 4.1.11 we have

$$\min_{1 \leqslant i \leqslant n} \mathrm{dist}_2(R_i, R_{-i}) \leqslant \sqrt{n}\, s_n(X).$$

Consequently, by the union bound and the exchangeability, for any $u \geqslant 0$,

$$\mathbb{P}(\sqrt{n}\, s_n(X) \leqslant u) \leqslant n\mathbb{P}(\mathrm{dist}_2(R_1, R_{-1}) \leqslant u).$$

Let Y be a unit normal vector to R_{-1}. Such a vector is not unique, but we just pick one which is measurable with respect to $R_2, \dots, R_n$. This defines a random variable on the unit sphere $S_2(K^n) = \{x \in K^n : \left|x\right|_2 = 1\}$, independent of R_1. By the Cauchy–Schwarz inequality, we have $|R_1 \cdot Y| \leqslant \left|p(R_1)\right|_2 \left|Y\right|_2 = \mathrm{dist}_2(R_1, R_{-1})$ where $p(\cdot)$ is the orthogonal projection on the orthogonal space of R_{-1}. Actually, since the law of $X_{1,1}$ is diffuse, the matrix X is a.s. invertible, the subspace R_{-1} is a hyperplane, and $|R_1 \cdot Y| = \mathrm{dist}_2(R_1, R_{-1})$, but this is useless in the sequel. Let ν be the distribution of Y on $S_2(K^n)$. Since Y and R_1 are independent, for any $u \geqslant 0$,

$$\mathbb{P}(\mathrm{dist}_2(R_1, R_{-1}) \leqslant u) \leqslant \mathbb{P}(|R_1 \cdot Y| \leqslant u) = \int_{S_2(K^n)} \mathbb{P}(|R_1 \cdot y| \leqslant u)\, d\nu(y).$$

Consider some $y \in S_2(K^n)$. Since $\left|y\right|_2 = 1$, there exists some $i \in \{1, \dots, n\}$ such that $|y_i| > 0$ and $\left|y_i\right|^{-1} \leqslant \sqrt{n}$. The random variable $X_{i,i}\, \overline{y_i}$ is absolutely continuous with density $|y_i|^{-1} f(\overline{y_i}^{-1} \cdot)$. Now, the random variable $R_1 \cdot y$ is a sum of independent random variables $X_{1,1}\, \overline{y_1}, \dots, X_{1,n}\, \overline{y_n}$, and one of them is absolutely continuous with a density bounded above by $\sqrt{n}\, \left\| f \right\|_\infty$. Consequently, by a basic property of convolutions of probability distributions, the random variable $R_1 \cdot y$ is itself absolutely continuous with a density φ bounded above by $\sqrt{n}\, \left\| f \right\|_\infty$. Therefore, we have,

$$\mathbb{P}(|R_1 \cdot y| \leqslant u) = \int_{\{z \in K; |z| \leqslant u\}} \varphi(s)\, ds \leqslant \begin{cases} 2u\sqrt{n}\, \left\| f \right\|_\infty & \text{if } K = \mathbb{R}, \\ \pi u^2 \sqrt{n}\, \left\| f \right\|_\infty & \text{if } K = \mathbb{C}. \end{cases}$$

□

The proof of lemma 4.3.4 above is quite instructive. Let us focus on the control of $\mathbb{P}(|R_1 \cdot y| \leqslant u)$ when u is small. In the case where $\mathcal{L}$ is Gaussian, the rotational invariance of the distribution of R_1 implies that the quantity $\mathbb{P}(|R_1 \cdot y| \leqslant u)$ does not depend on y and is of order u (take for y an element of the canonical basis and use the fact that $\mathcal{L}$ has a bounded density). However, when $\mathcal{L}$ is not Gaussian, the quantity $\mathbb{P}(|R_1 \cdot y| \leqslant u)$ depends heavily on $\mathcal{L}$. Recall that the simple lemma above does not allow atoms in $\mathcal{L}$. In particular, it does not cover the case where $\mathcal{L}$ is Rademacher $\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}$. Such discrete matrices have a positive probability of being singular. In this discrete case, the quantity $\mathbb{P}(|R_1 \cdot y| \leqslant u)$ depends heavily on the arithmetic and sparsity structure of the coordinates of y. For instance, if $y = n^{-1/2}(e_1 + \cdots + e_n)$, then by the central limit theorem, the quantity $\mathbb{P}(|R_1 \cdot y| \leqslant u)$ is of order u as $n \to \infty$, whereas if $y = 2^{-1/2}(e_1 + e_2)$ then we get a completely different behavior:

$$\mathbb{P}(|R_1 \cdot y| \leqslant u) \geqslant \mathbb{P}(X_{1,1} = 0) = 1/2.$$

Also, one should restart from $s_n(X) = \min_{|x|_2=1} |Xx|_2$ and partition the unit sphere into "compressible" and "incompressible" vectors. This leads to the use of ε–nets techniques and to the consideration of Littlewood–Offord type problems for the control of small balls probabilities. In this direction, an important step was first made by Rudelson [**Rud06**]. Later, Rudelson and Vershynin [**RV08b, RV08a**] have shown that if $X_{1,1}$ has zero mean, unit variance, and finite fourth moment, then for any fixed $t > 0$ (recall that $m = n$),

$$\mathbb{P}(\sqrt{n}\, s_n(X) \leqslant t) \leqslant f(t) + o(1) \quad \text{and} \quad \mathbb{P}(\sqrt{n}\, s_n(X) \geqslant t) \leqslant g(t) + o(1)$$

where f, g do not depend on n and $f(t), g(t) \to 0$ as $t \to 0$, and where $o(1)$ is relative to $n \to \infty$. Moreover, if the entries are additionally sub-Gaussian then there exist constants $C > 0$, $c \in (0,1)$ depending only on the moments such that for any $t \geqslant 0$,

$$\mathbb{P}(\sqrt{n}\, s_n(X) \leqslant t) \leqslant C\, t + c^n. \tag{4.12}$$

Since the Rademacher law is sub-Gaussian, the remarkable bound (4.12) proves, up to the multiplicative constant C, the conjecture of Spielman and Teng (4.11). The proof of Rudelson and Vershynin has many ingredients, including an upper bound on the right tail of the largest singular value and a lower bound on the smallest singular value of rectangular matrices obtained in [**LPRTJ05**] (see also the more recent work [**RV09**]). Regarding moments, Tao and Vu have shown [**TV08, TV09a**] that under the sole assumption that $X_{1,1}$ has non zero finite variance then for any constants $a, c > 0$ there exists $b > 1/2$ depending on a, c and the law of $X_{1,1}$ such that for any $n \times n$ deterministic matrix Y with $\|Y\|_{2\to 2} \leqslant n^c$,

$$\mathbb{P}(\sqrt{n}\, s_n(X + Y) \leqslant n^{-b}) \leqslant n^{-a}. \tag{4.13}$$

Actually, one can find in [**TV09c**] many bounds of this flavor. For instance, under the sole assumptions that $X_{1,1}$ has zero mean and unit variance, for any fixed $a > 0$,

$$\mathbb{P}(\sqrt{n}\, s_n(X) \leqslant n^{-\frac{1}{2} - \frac{5}{2}a - a^2}) \leqslant n^{-a+o(1)}. \tag{4.14}$$

The bounds (4.13–4.14) are less precise than the bound (4.12) but do not rely on moments assumptions beyond the finite variance. With lemma 4.3.4 in mind, one can ask if the finite moment assumption in (4.13) can be weakened in order to allow for instance heavy tailed non centered discrete laws such as the Zipf type law

$$\zeta(s)^{-1}\sum_{n=0}^{\infty} n^{-s}\delta_n$$

where $s > 0$, and where ζ is the Riemann zeta function.

4.4. Comments

The singular values of deterministic matrices are studied in many books such as [**HJ90, HJ94**], [**Bha97**], and [**Zha02**]. For the algorithmic aspects, we recommend [**GVL96**] and [**CG05**]. The singular values of random matrices are studied in the books [**Meh04**], [**Dei99**], [**For10**], [**AGZ09**].

CHAPTER 5

EMPIRICAL METHODS AND SELECTION OF CHARACTERS

The purpose of this chapter is to present the connections between two different topics. The first one is a recent subject about reconstruction of signals with small supports from a small amount of linear measurements, called also compressed sensing. A big amount of work was recently made to develop some strategy to construct an encoder (to compress a signal) and an associate decoder (to reconstruct exactly or approximately the original signal). Several deterministic methods are known but recently, some random methods allow the reconstruction of signal with much larger size of support. A lot of ideas are common with a subject of harmonic analysis, going back to the construction of $\Lambda(p)$ sets which are not $\Lambda(q)$ for $q > p$. The most powerful method was to select a random choice of characters via the method of selectors. In a different direction, we will discuss about the problem of selecting a large part of a bounded orthonormal system such that on the vectorial span of this family, the L_2 and the L_1 norms are as close as possible. The study of some empirical processes is at the heart of these proofs and it is the main connection between these subjects.

5.1. Reconstruction of signals with small support by random methods.

5.1.1. Presentation of the problem. — Let $U \in \mathbb{R}^N$ (or $\mathbb{C}^N$) be an unknown signal. We receive ΦU where Φ is an $n \times N$ matrix with rows $Y_1, \ldots, Y_n \in \mathbb{R}^N$ (or $\mathbb{C}^N$) which means that

$$\Phi = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} \qquad \text{and} \qquad \Phi U = (\langle Y_i, U\rangle)_{1\le i\le n}$$

and we assume that $n \le N-1$. In this case the linear system to reconstruct U is ill-posed. However, the main information is that U has a small support in the canonical basis chosen at the beginning, that is $|\operatorname{supp} U| \le m$. We also say that U is m-sparse and we denote by Σ_m the set of m-sparse vectors. Our aim is to find conditions on

Φ, m, n and N such that the solution of the problem

$$\min_{t\in\mathbb{R}^N} \{|t|_1 : \Phi U = \Phi t\} \tag{5.1}$$

is unique and equal to U. This minimization problem is called the basis pursuit algorithm and we refer to Chapter 2 for more details about the description of this problem. Let us recall (see Proposition 2.2.11) that the property "for every signal $U \in \Sigma_m$, the solution of (5.1) is unique and equal to U" is equivalent to the following

$$\forall h \in \ker\Phi, h \neq 0, \forall I \subset [N], |I| \leq m, \sum_{i\in I} |h_i| < \sum_{i\notin I} |h_i|.$$

This property is also called the null space property. Let $\mathcal{C}_m$ be the cone

$$\mathcal{C}_m = \{h \in \mathbb{R}^N, \exists I \subset [N] \text{ with } |I| \leq m, |h_{I^c}|_1 \leq |h_I|_1\}.$$

The null space property is therefore equivalent to $\ker\Phi \cap \mathcal{C}_m = \{0\}$. Introducing an intersection with the Euclidean sphere S^{N-1} the following picture makes the situation clear:

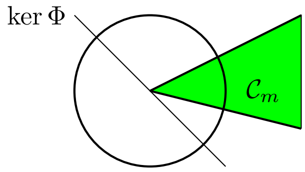

FIGURE 1. The null space property.

In conclusion, we can say that

"for every signal $U \in \Sigma_m$, the solution of (5.1) is unique and equal to U"
if and only if
$\ker\Phi \cap \mathcal{C}_m \cap S^{N-1} = \emptyset$.

We observe the following simple fact: if $t \in \mathcal{C}_m \cap S^{N-1}$ then

$$|t|_1 = \sum_{i=1}^N |t_i| = \sum_{i\in I} |t_i| + \sum_{i\notin I} |t_i| \leq 2\sum_{i\in I} |t_i| \leq 2\sqrt{m}$$

since $|I| \leq m$ and $|t|_2 = 1$. This implies that

$$\mathcal{C}_m \cap S^{N-1} \subset 2\sqrt{m}B_1^N \cap S^{N-1}$$

from which we conclude that if

$$\ker\Phi \cap 2\sqrt{m}B_1^N \cap S^{N-1} = \emptyset$$

then "for every $U \in \Sigma_m$, the solution of (5.1) is unique and equal to U". We can now state the conclusion of this introduction.

Proposition 5.1.1. — *Denote by* diam$T$ *the half-diameter of a set* T *with respect to the Euclidean distance:*

$$\operatorname{diam} T = \sup_{t \in T} |t|_2.$$

If

$$\operatorname{diam}(\ker \Phi \cap B_1^N) < \frac{1}{2\sqrt{m}} \tag{5.2}$$

then "for every $U \in \Sigma_m$*, the solution of the basis pursuit algorithm* (5.1) *is unique and equal to* U*".*

In this chapter, instead of considering the RIP property (as discussed in Chapter 2), we will focus on the condition about the half-diameter of the section of the unit ball of ℓ_1^N with the kernel of the matrix Φ. The main reasons are that it is a very simple condition that was deeply studied in the so called Local Theory of Banach Spaces during the seventies and the eighties, that it will give sharp results and that this condition is very stable and allows also approximate reconstructions. Moreover we will meet exactly the same question when studying another problem of harmonic analysis.

5.1.2. Notations. — We briefly indicate some notations that will be used in this section. For any $p > 0$ and $t \in \mathbb{R}^N$, we define its ℓ_p-norm by

$$|t|_p = \left(\sum_{i=1}^{N} |t_i|^p \right)^{1/p}$$

and its L_p-norm by

$$\|t\|_p = \left(\frac{1}{N} \sum_{i=1}^{N} |t_i|^p \right)^{1/p}.$$

For $p = \infty$, $|t|_\infty = \|t\|_\infty = \max\{|t_i| : i = 1, \ldots, n\}$. We denote by B_p^N the unit ball of the ℓ_p-norm in $\mathbb{R}^N$. The half-diameter of a set $T \subset \mathbb{R}^N$ is

$$\operatorname{diam} T = \sup_{t \in T} |t|_2.$$

More generally, if μ is a probability measure on a measurable space Ω, for any $p > 0$ and any measurable function f, we denote its L_p-norm and its L_∞-norm by

$$\|f\|_p = \left(\int |f|^p d\mu \right)^{1/p} \quad \text{and} \quad \|f\|_\infty = \sup |f|.$$

The unit ball of $L_p(\mu)$ is denoted by B_p and the unit sphere by S_p. If $T \subset L_2(\mu)$ then its half-diameter with respect to $L_2(\mu)$ is defined by

$$\operatorname{Diam} T = \sup_{t \in T} \|t\|_2.$$

Observe that it μ is the probability counting measure on $\mathbb{R}^N$, $B_p = N^{1/p} B_p^N$ and for a subset $T \subset L_2(\mu)$, $\sqrt{N} \operatorname{Diam} T = \operatorname{diam} T$.

The letters c, C are used for numerical constants which do not depend on any parameter (dimension, size of sparsity, ...). Since the dependence of these parameters is important in this study, it will be always indicated (as precisely as we can). Sometimes, the value of these numerical constants can change from line to line.

5.1.3. Approximate reconstruction via the study of the half-diameter of sections.— We explain in the following theorem why the same type of condition than (5.2) about the size of the half-diameter of the section allows also approximate reconstruction of the original unknown signal.

Theorem 5.1.2. — *Let $U \in \mathbb{R}^N$ be an unknown signal, Φ be an $n \times N$ matrix (encoder) and x be a solution of the minimization problem* (5.1)

$$\min_{t \in \mathbb{R}^N} \left\{ |t|_1 : \Phi U = \Phi t \right\}.$$

Let m be an integer such that $1 \leq m \leq N$. If

$$\operatorname{diam}\left(\ker \Phi \cap B_1^N\right) \leq \rho < \frac{1}{2\sqrt{m}}$$

then for any set $I \subset \{1, \ldots, N\}$ of cardinality less than m

$$|x - U|_2 \leq \rho\, |x - U|_1 \leq \frac{2\rho}{1 - 2\rho\sqrt{m}}\, |U_{I^c}|_1.$$

In particular: if $\operatorname{diam}\left(\ker \Phi \cap B_1^N\right) \leq 1/4\sqrt{m}$ *then for any subset I of cardinality less than m,*

$$|x - U|_2 \leq \frac{|x - U|_1}{4\sqrt{m}} \leq \frac{|U_{I^c}|_1}{\sqrt{m}}.$$

Moreover if $U \in B_{p,\infty}^N$ i.e. if for all $s > 0$, $|\{i, |U_i| \geq s\}| \leq s^{-p}$ then

$$|x - U|_2 \leq \frac{|x - U|_1}{4\sqrt{m}} \leq \frac{1}{(1-p)\, m^{\frac{1}{p} - \frac{1}{2}}}.$$

Proof. — Since $\operatorname{diam}\left(\ker \Phi \cap B_1^N\right) \leq \rho$ we get that for any $h \in \ker \Phi$, $|h|_2 \leq \rho |h|_1$. Since $x - U \in \ker \Phi$ then $|x - U|_2 \leq \rho\, |x - U|_1$. The set I has cardinality less than m hence by Hölder inequality, for any $h \in \ker \Phi$,

$$|h_I|_1 \leq \sqrt{m} |h_I|_2 \leq \sqrt{m} |h|_2 \leq \rho\sqrt{m} |h|_1.$$

By definition of the ℓ_1-norm, $|h|_1 = |h_I|_1 + |h_{I^c}|_1$ and we conclude that for any $h \in \ker \Phi$, if $1 - \rho\sqrt{m} > 0$,

$$|h_I|_1 \leq \frac{\rho\sqrt{m}}{1 - \rho\sqrt{m}} |h_{I^c}|_1. \tag{5.3}$$

Moreover, x is a solution of the minimization problem (5.1) hence $x - U \in \ker \Phi$ and $|x|_1 \leq |U|_1$. Since $|x|_1 = |x_I|_1 + |x_{I^c}|_1$ and $|U|_1 = |U_I|_1 + |U_{I^c}|_1$, we deduce that $|x_{I^c}|_1 \leq |U_I|_1 - |x_I|_1 + |U_{I^c}|_1 \leq |x_I - U_I|_1 + |U_{I^c}|_1$. Therefore,

$$|x_{I^c} - U_{I^c}|_1 \leq |x_{I^c}|_1 + |U_{I^c}|_1 \leq |x_I - U_I|_1 + 2|U_{I^c}|_1.$$

Since $x - U \in \ker \Phi$, we can combine this inequality with (5.3) to conclude that

$$|x_I - U_I|_1 \leq \frac{\rho\sqrt{m}}{1-\rho\sqrt{m}}|x_{I^c} - U_{I^c}|_1 \leq \frac{\rho\sqrt{m}}{1-\rho\sqrt{m}}|x_I - U_I|_1 + \frac{2\rho\sqrt{m}}{1-\rho\sqrt{m}}|U_{I^c}|_1.$$

The general conclusion of the Theorem follows. The particular case is obtained by taking $\rho = 1/4\sqrt{m}$. Moreover, if $U \in B_{p,\infty}^N$, we denote by $(U_i^\star)_{i=1}^N$ the non-increasing rearrangement of $(|U_i|)_{i=1}^N$ and we get by definition that for all $i = 1, \ldots, N$, $U_i^\star \leq i^{-1/p}$. Let I be a set of indices of the m-th largest coordinates of U therefore

$$|U_{I^c}|_1 = \sum_{m+1}^{N} U_i^\star \leq \sum_{m+1}^{N} i^{-1/p} \leq \int_m^{+\infty} s^{-1/p} ds = \frac{1}{(1-p)\, m^{\frac{1}{p}-1}}.$$

□

Remark 5.1.3. — *This Theorem gives an alternative proof of the exact reconstruction property discussed in Proposition 5.1.1. Indeed, we apply the general conclusion taking the set I as the support of the unknown signal which means that $|U_{I^c}|_1 = 0$.*

5.1.4. Approximation numbers and local theory of Banach spaces. — Let $u : X \to Y$ be an operator between Banach spaces. Let $k \geq 1$ be a natural integer. Recall the definition of the approximation numbers of u

$$a_k(u) = \inf\{\|u - v\| : \ v : X \to Y, \mathrm{rank}\,(v) < k\}.$$

The Gelfand numbers of u are defined as

$$\begin{aligned} c_k(u) &= \inf\{\| \ u_{|S} \ \| : S \subset X, \mathrm{codim}\, S < k\} \\ &= \inf_{S\subset X, \mathrm{codim}\, S<k} \ \sup_{x\in S, \|x\|_X \leq 1} \|u(x)\|_Y. \end{aligned}$$

For any closed subspace $S \subset X$, we denote by Q_S the quotient mapping from X onto X/S. We define the Kolmogorov numbers of u as

$$d_k(u) = \inf\{\|Q_S u\| : S \subset X, \dim S < k\}.$$

The sequences $(a_k(u))_{k\geq 1}$, $(c_k(u))_{k\geq 1}$, $(d_k(u))_{k\geq 1}$ are all non-increasing and satisfy $a_1(u) = c_1(u) = d_1(u) = \|u\|$, $c_k(u) \leq a_k(u)$, $d_k(u) \leq a_k(u)$. Moreover, we have $c_k(u) = d_k(u^*)$ and if u is compact, we have $d_k(u) = c_k(u^*)$ and $a_k(u) = a_k(u^*)$. In the finite dimensional setting, we are interested in the optimal dependance in k and the dimension. For example, considering the identity map from ℓ_1^N to ℓ_2^N, the Gelfand numbers are

$$c_k(id : \ell_1^N \to \ell_2^N) = \inf_{\mathrm{codim}\, S<k} \mathrm{diam}\,(S \cap B_1^N)$$

and any upper bound on these numbers gives a solution to (5.2) for a matrix Φ such that $S = \ker \Phi$. The study of these numbers attracted a lot of attention during the seventies and the eighties. An important result is the following

Theorem 5.1.4. — *There exist numbers $c, C > 0$ such that for any integer $N \geq 1$ and $k \in \{1, \ldots, N\}$,*

$$c \min\left\{1, \sqrt{\frac{\log(N/k)}{k}}\right\} \leq c_k(id : \ell_1^N \to \ell_2^N) \leq C \min\left\{1, \sqrt{\frac{\log(N/k)}{k}}\right\}.$$

The upper bound follows from the fact that if $g_{i,j}$ *are independent Gaussian standard random variables and* $\Phi = (g_{i,j})_{1\le i\le k, 1\le j\le N} : \mathbb{R}^N \to \mathbb{R}^k$*, then*

$$\mathbb{P}\left(\operatorname{diam}(\ker\Phi \cap B_1^N) \le C \min\left\{1, \sqrt{\frac{\log(N/k)}{k}}\right\}\right) \ge 1 - \exp(-ck).$$

Using Proposition 5.1.1, the following result is an immediate corollary about the basis pursuit algorithm and the exact reconstruction property of random Gaussian matrices.

Corollary 5.1.5. — *For every natural integers* N *and* $n \le N$*, if* $g_{i,j}$ *are independent Gaussian standard random variables and* $\Phi = (g_{i,j})_{1\le i\le n, 1\le j\le N} : \mathbb{R}^N \to \mathbb{R}^n$*, if*

$$m \le C \frac{n}{\log(N/n)}$$

then with probability greater than $1 - \exp(-cn)$*, for every* $U \in \Sigma_m$*, the solution of the basis pursuit algorithm* (5.1) *is unique and equal to* U*.*

The estimate of a lower bound of the Gelfand numbers is based on a volumetric and entropic argument. We will not prove it and will focus on the study of the upper bound. The first important and simple idea to study the diameter of a section of a star shape body by a vectorial subspace is the following.

Proposition 5.1.6. — *Let* T *be a star shape body with respect to the origin that is a compact subset* T *of* $\mathbb{R}^N$ *such that for any* $x \in T$*, the segment* $[0, x]$ *is contained in* T*. Let* Φ *be an* $n \times N$ *matrix with rows denoted by* $Y_1, \ldots, Y_n$*.*

$$\text{If} \quad \inf_{y \in T \cap \rho S^{N-1}} \sum_{i=1}^n \langle Y_i, y\rangle^2 > 0 \quad \text{then} \quad \operatorname{diam}(\ker\Phi \cap T) < \rho.$$

Remark 5.1.7. — *By a simple compacity argument, the reciprocal of this statement holds true. We can also replace the Euclidean norm* $|\Phi z|_2$ *by any other norm* $\|\Phi z\|$*.*

Proof. — The argument is geometric. Indeed, if $z \in T \cap \rho S^{N-1}$ then $|\Phi z|_2^2 > 0$ so $z \notin \ker\Phi$. Since T is star shaped, if $y \in T$ and $|y|_2 \ge \rho$ then $z = \rho y/|y|_2 \in T \cap \rho S^{N-1}$ so z and y do not belong to $\ker\Phi$. □

The vectors $Y_1, \ldots, Y_n$ will now be chosen at random and we will find the good conditions such that, in average, the key inequality of this proposition holds true. In the case of Theorem 5.1.4, the rows of the matrix Φ are independent copies of a standard random Gaussian vector in $\mathbb{R}^N$.

5.1.5. A way to construct a good random decoder. — The setting of the study is the following. We start with a square $N \times N$ orthogonal matrix and we would like to select n rows of this matrix such that the $n \times N$ matrix Φ is a good decoder for every m-sparse vectors. In view of Proposition 5.1.1, we want to find the conditions on n, N and m such that

$$\operatorname{diam}(\ker\Phi \cap B_1^N) \le \frac{1}{2\sqrt{m}}.$$

The main examples are the discrete Fourier matrix with

$$\phi_{k\ell} = \frac{1}{\sqrt{N}}\,\omega^{k\ell} \quad 1 \le k, \ell \le N \quad \text{where } \omega = \exp\left(-2i\pi/N\right),$$

and the Walsh matrix defined by induction: $W_1 = 1$ and for any $p \ge 2$,

$$W_p = \frac{1}{\sqrt{2}} \begin{pmatrix} W_{p-1} & W_{p-1} \\ -W_{p-1} & W_{p-1} \end{pmatrix}.$$

The matrix W_p is an orthogonal matrix of size $N = 2^p$ with entries $\frac{\pm 1}{\sqrt{N}}$. In each case, the column vectors form an orthonormal basis of ℓ_2^N, with ℓ_∞-norm bounded by $1/\sqrt{N}$. We will consider more generally a system of vectors $\phi_1, \ldots, \phi_N$ such that

$$(H) \quad \begin{cases} \text{it is an orthogonal system of } \ell_2^N, \\ \forall i \le N, |\phi_i|_\infty \le 1/\sqrt{N} \text{ and } |\phi_i|_2 = K \text{ where } K \text{ is a fixed number.} \end{cases}$$

5.1.5.1. The empirical method. — The first definition of randomness is an empirical one. Let Y be the random vector defined by $Y = \phi_i$ with probability $1/N$ and let $Y_1, \ldots, Y_n$ be independent copies of Y. We define the random matrix Φ by

$$\Phi = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}.$$

We have the following properties:

$$\mathbb{E}\langle Y, y\rangle^2 = \frac{1}{N}\sum_{i=1}^{N} \langle \phi_i, y\rangle^2 = \frac{K^2}{N}|y|_2^2 \quad \textit{and} \quad \mathbb{E}|\Phi y|_2^2 = \frac{K^2\, n}{N}|y|_2^2. \tag{5.4}$$

In view of Proposition 5.1.6, we would like to find ρ such that

$$\mathbb{E} \inf_{y \in T \cap \rho S^{N-1}} \sum_{i=1}^{n} \langle Y_i, y\rangle^2 > 0.$$

However it is difficult to study the infimum of an empirical process. We shall prefer to study

$$\mathbb{E} \sup_{y \in T \cap \rho S^{N-1}} \left| \sum_{i=1}^{n} \langle Y_i, y\rangle^2 - \frac{K^2\, n\rho^2}{N} \right|$$

that is the supremum of the deviation of the empirical process to its mean (because of (5.4)). We will focus our attention on the following problem.

Problem 5.1.8. — *What are the conditions on ρ such that we have*

$$\mathbb{E} \sup_{y \in T \cap \rho S^{N-1}} \left| \sum_{i=1}^{n} \langle Y_i, y\rangle^2 - \frac{K^2\, n\rho^2}{N} \right| \le \frac{2}{3}\frac{K^2\, n\rho^2}{N} \; ?$$

Indeed if this inequality is satisfied then there exists a choice of vectors $(Y_i)_{1 \le i \le n}$ such that

$$\forall y \in T \cap \rho S^{N-1}, \left| \sum_{i=1}^{n} \langle Y_i, y\rangle^2 - \frac{K^2\, n\rho^2}{N} \right| \le \frac{2}{3}\frac{K^2\, n\rho^2}{N},$$

from which we deduce that

$$\forall y \in T \cap \rho S^{N-1}, \sum_{i=1}^{n} \langle Y_i, y\rangle^2 \geq \frac{1}{3} \frac{K^2\, n\rho^2}{N} > 0.$$

Therefore, by Proposition 5.1.6, we conclude that $\operatorname{diam}(\ker \Phi \cap T) < \rho$. Doing this with $T = B_1^N$, we will conclude by Proposition 5.1.1 that if

$$m \leq \frac{1}{4\rho^2}$$

then the matrix Φ is a good decoder, that is for every $U \in \Sigma_m$, the solution of the basis pursuit algorithm (5.1) is unique and equal to U.

Remark 5.1.9. — *The number* $2/3$ *can be replaced by any number strictly less than* 1.

5.1.5.2. The method of selectors. — The second definition of randomness uses the notion of selectors. Let $\delta \in (0,1)$ and let δ_i be i.i.d. random variables taking the value 1 with probability δ and 0 with probability $1-\delta$. We start from the orthogonal matrix with rows $\phi_1, \ldots, \phi_N$ and we select randomly some rows to construct a matrix Φ with rows $\delta_1\phi_1, \ldots, \delta_N\phi_N$. This matrix will contain some rows equal to zero that we will "erase" and we will keep the rows that we have selected from the starting orthogonal matrix. The number of non-zero lines will be highly concentrated around δN. The problem 5.1.8 can be stated in the following way:

Problem 5.1.10. — *What are the conditions on* ρ *such that we have*

$$\mathbb{E} \sup_{y \in T \cap \rho S^{N-1}} \left| \sum_{i=1}^{N} \delta_i \langle \phi_i, y\rangle^2 - \delta K^2 \rho^2 \right| \leq \frac{2}{3}\delta K^2 \rho^2 \ ?$$

The same argument as before shows that if this inequality is satisfied for $T = B_1^N$, then there exists a choice of selectors such that $\operatorname{diam}(\ker \Phi \cap B_1^N) < \rho$ and we will conclude as before that the matrix Φ is a good decoder.

Before we state and explain the main results, we will need some tools from the theory of empirical processes to solve Problems 5.1.8 and 5.1.10 and also to get deviation inequality in order to deduce some probabilistic estimate (instead of just the existence of a matrix Φ).

5.2. Empirical processes

5.2.1. Classical tools. — A lot is known about the supremum of empirical processes and the connection with Rademacher averages. We refer to the chapter 4 of [**LT91**] for a detailed description. We recall the important comparison theorem for Rademacher average.

Theorem 5.2.1. — *Let $F : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing convex function, let $h_i : \mathbb{R} \to \mathbb{R}$ be contractions such that $h_i(0) = 0$. Then for any separable bounded set $T \subset \mathbb{R}^n$,*

$$\mathbb{E}F\left(\frac{1}{2}\sup_{t\in T}\left|\sum_{i=1}^n \varepsilon_i h_i(t_i)\right|\right) \leq \mathbb{E}F\left(\sup_{t\in T}\left|\sum_{i=1}^n \varepsilon_i t_i\right|\right).$$

The proof of this theorem is however beyond the scope of this chapter. We will concentrate on the study of the average of the supremum of some empirical processes. Consider n independent random vectors $Y_1, \ldots, Y_n$ taking values in a measurable space Ω and $\mathcal{F}$ be a class of measurable functions, and define

$$Z = \sup_{f\in\mathcal{F}}\left|\sum_{i=1}^n (f(Y_i) - \mathbb{E}f(Y_i))\right|.$$

The situation will be different from Chapter 1 because the control on the ψ_α norm of $f(Y_i)$ will not be relevant in our situation. In this case, a classical strategy consists to "symmetrize" the variable and to introduce Rademacher averages.

Theorem 5.2.2. — *Consider n independent random vectors $Y_1, \ldots, Y_n$ taking values in a measurable space Ω, $\mathcal{F}$ be a class of measurable functions and $\varepsilon_1, \ldots, \varepsilon_n$ be independent Bernoulli random variables, independent of the Y_i's. Denote by $\mathbb{E}_\varepsilon$ the expectation with respect to these Bernoulli random variables. Then the following inequalities hold true.*

$$\mathbb{E}\sup_{f\in\mathcal{F}}\left|\sum_{i=1}^n (f(Y_i) - \mathbb{E}f(Y_i))\right| \leq 2\mathbb{E}\mathbb{E}_\varepsilon \sup_{f\in\mathcal{F}}\left|\sum_{i=1}^n \varepsilon_i f(Y_i)\right|. \tag{5.5}$$

$$\mathbb{E}\sup_{f\in\mathcal{F}}\sum_{i=1}^n |f(Y_i)| \leq \sup_{f\in\mathcal{F}}\sum_{i=1}^n \mathbb{E}|f(Y_i)| + 4\mathbb{E}\mathbb{E}_\varepsilon \sup_{f\in\mathcal{F}}\left|\sum_{i=1}^n \varepsilon_i f(Y_i)\right|. \tag{5.6}$$

If for every $f \in \mathcal{F}$, $\mathbb{E}f(Y_i) = 0$ then

$$\mathbb{E}\mathbb{E}_\varepsilon \sup_{f\in\mathcal{F}}\left|\sum_{i=1}^n \varepsilon_i f(Y_i)\right| \leq 2\mathbb{E}\sup_{f\in\mathcal{F}}\left|\sum_{i=1}^n f(Y_i)\right|. \tag{5.7}$$

Proof. — Let $Y_1', \ldots, Y_n'$ be independent copies of $Y_1, \ldots, Y_n$. We replace $\mathbb{E}f(Y_i)$ by $\mathbb{E}'f(Y_i')$ where $\mathbb{E}'$ denotes the expectation with respect to the random vectors $Y_1', \ldots, Y_n'$ then by Jensen inequality,

$$\mathbb{E}\sup_{f\in\mathcal{F}}\left|\sum_{i=1}^n (f(Y_i) - \mathbb{E}f(Y_i))\right| \leq \mathbb{E}\mathbb{E}'\sup_{f\in\mathcal{F}}\left|\sum_{i=1}^n (f(Y_i) - f(Y_i'))\right|.$$

The variables $(f(Y_i) - f(Y_i'))_{1\leq i\leq n}$ are independent symmetric hence $(f(Y_i) - f(Y_i'))_{1\leq i\leq n}$ has the same law as $(\varepsilon_i(f(Y_i) - f(Y_i')))_{1\leq i\leq n}$ where $\varepsilon_1, \ldots, \varepsilon_n$ are independent Bernoulli random variables. We deduce that

$$\mathbb{E}\sup_{f\in\mathcal{F}}\left|\sum_{i=1}^n (f(Y_i) - \mathbb{E}f(Y_i))\right| \leq \mathbb{E}\mathbb{E}'\mathbb{E}_\varepsilon \sup_{f\in\mathcal{F}}\left|\sum_{i=1}^n \varepsilon_i(f(Y_i) - f(Y_i'))\right|.$$

We conclude the proof of (5.5) by using the triangle inequality.
Inequality (5.6) is a consequence of (5.5) when applying it to $|f|$ instead of f, using the triangle inequality and Theorem 5.2.1 (in the case $F(x) = x$ and $h_i(x) = |x|$) to deduce that

$$\mathbb{E}_\varepsilon \sup_{f\in\mathcal{F}} \left| \sum_{i=1}^n \varepsilon_i |f(Y_i)| \right| \le 2\mathbb{E}_\varepsilon \sup_{f\in\mathcal{F}} \left| \sum_{i=1}^n \varepsilon_i f(Y_i) \right|.$$

For the proof of (5.7), we compute the expectation conditionally with respect to the Bernoulli random variables. Let $I = I(\varepsilon) = \{i, \varepsilon_i = 1\}$ then

$$\begin{aligned}\mathbb{E}\mathbb{E}_\varepsilon \sup_{f\in\mathcal{F}} \left| \sum_{i=1}^n \varepsilon_i f(Y_i) \right| &\le \mathbb{E}_\varepsilon \mathbb{E} \sup_{f\in\mathcal{F}} \left| \sum_{i\in I} f(Y_i) - \sum_{i\notin I} f(Y_i) \right| \\ &\le \mathbb{E}_\varepsilon \mathbb{E} \sup_{f\in\mathcal{F}} \left| \sum_{i\in I} f(Y_i) \right| + \mathbb{E}_\varepsilon \mathbb{E} \sup_{f\in\mathcal{F}} \left| \sum_{i\notin I} f(Y_i) \right|.\end{aligned}$$

However, since for every $i \le n$, $\mathbb{E}f(Y_i) = 0$ we deduce from Jensen inequality that for any $I \subset \{1, \dots, n\}$

$$\mathbb{E} \sup_{f\in\mathcal{F}} \left| \sum_{i\in I} f(Y_i) \right| = \mathbb{E} \sup_{f\in\mathcal{F}} \left| \sum_{i\in I} f(Y_i) + \sum_{i\notin I} \mathbb{E}f(Y_i) \right| \le \mathbb{E} \sup_{f\in\mathcal{F}} \left| \sum_{i=1}^n f(Y_i) \right|$$

which ends the proof of (5.7). □

Another simple fact about Rademacher averages is the following comparison between the supremum of Rademacher processes and the supremum of the same Gaussian processes.

Proposition 5.2.3. — *Let $\varepsilon_1, \dots, \varepsilon_n$ be independent Bernoulli random variables and $g_1, \dots, g_n$ be independent Gaussian $\mathcal{N}(0,1)$ random variables, then for any set $T \subset \mathbb{R}^n$*

$$\mathbb{E} \sup_{t\in T} \left| \sum_{i=1}^n \varepsilon_i t_i \right| \le \sqrt{\frac{2}{\pi}} \mathbb{E} \sup_{t\in T} \left| \sum_{i=1}^n g_i t_i \right|.$$

Proof. — Indeed, $(g_1, \dots, g_n)$ has the same law as $(\varepsilon_1|g_1|, \dots, \varepsilon_n|g_n|)$ and by Jensen inequality,

$$\mathbb{E}_\varepsilon \mathbb{E}_g \sup_{t\in T} \left| \sum_{i=1}^n \varepsilon_i |g_i| t_i \right| \ge \mathbb{E}_\varepsilon \sup_{t\in T} \left| \mathbb{E}_g \sum_{i=1}^n \varepsilon_i |g_i| t_i \right| = \sqrt{\frac{\pi}{2}} \mathbb{E} \sup_{t\in T} \left| \sum_{i=1}^n \varepsilon_i t_i \right|.$$

□

To conclude this part, we state an important result about the concentration of the supremum of empirical processes around its mean. This motivates the fact that we will focus on the estimation of the expectation of the supremum of such empirical process.

Theorem 5.2.4. — *Consider n independent vectors $Y_1, \dots, Y_n$ taking values in a measurable space Ω, and $\mathcal{G}$ a class of measurable functions. Let*

$$Z = \sup_{g\in\mathcal{G}} \left| \sum_{i=1}^{n} g(Y_i) \right|, \; M = \sup_{g\in\mathcal{G}} \|g\|_\infty, \; V = \mathbb{E} \sup_{g\in\mathcal{G}} \sum_{i=1}^{n} g(Y_i)^2.$$

Then for any $t > 0$, we have

$$\mathbb{P}\left(|Z - \mathbb{E}Z| > t\right) \le C \exp\left(-c\,\frac{t}{M}\,\log\left(1 + \frac{tM}{V}\right)\right).$$

Sometimes, we need a more simple quantity than V in this concentration inequality. Let $\mathcal{F}$ be a class of measurable functions, and define the function g by $g(Y) = f(Y) - \mathbb{E}f(Y)$ for any $f \in \mathcal{F}$. In this situation, we have a very useful estimate for V.

Proposition 5.2.5. — *Consider n independent vectors $Y_1, \dots, Y_n$ taking values in a measurable space Ω, and $\mathcal{F}$ a class of measurable functions. Let*

$$Z = \sup_{f\in\mathcal{F}} \left| \sum_{i=1}^{n} f(Y_i) - \mathbb{E}f(Y_i) \right|, \; u = \sup_{f\in\mathcal{F}} \|f\|_\infty, \text{ and}$$

$$v = \sup_{f\in\mathcal{F}} \sum_{i=1}^{n} \mathrm{Var} f(Y_i) + 32\,u\,\mathbb{E} \sup_{f\in\mathcal{F}} \left| \sum_{i=1}^{n} f(Y_i) - \mathbb{E}f(Y_i) \right|.$$

Then for any $t > 0$, we have

$$\mathbb{P}\left(|Z - \mathbb{E}Z| > t\right) \le C \exp\left(-c\,\frac{t}{u}\,\log\left(1 + \frac{tu}{v}\right)\right).$$

Proof. — It is a typical use of the symmetrization principle. Let $\mathcal{G}$ be the set of functions defined by $g(Y) = f(Y) - \mathbb{E}f(Y)$ where $f \in \mathcal{F}$. Using Theorem 5.2.4, the conclusion will follow when estimating

$$M = \sup_{g\in\mathcal{G}} \|g\|_\infty \text{ and } V = \mathbb{E} \sup_{g\in\mathcal{G}} \sum_{i=1}^{n} g(Y_i)^2.$$

It is clear that $M \le 2u$ and by the triangle inequality we get

$$\mathbb{E} \sup_{g\in\mathcal{G}} \sum_{i=1}^{n} g(Y_i)^2 \;\le\; \mathbb{E} \sup_{g\in\mathcal{G}} \left| \sum_{i=1}^{n} g(Y_i)^2 - \mathbb{E}g(Y_i)^2 \right| + \sup_{g\in\mathcal{G}} \sum_{i=1}^{n} \mathbb{E}g(Y_i)^2.$$

Using inequality (5.5), we deduce that

$$\mathbb{E} \sup_{g\in\mathcal{G}} \left| \sum_{i=1}^{n} g(Y_i)^2 - \mathbb{E}g(Y_i)^2 \right| \le 2\,\mathbb{E}\mathbb{E}_\varepsilon \sup_{g\in\mathcal{G}} \left| \sum_{i=1}^{n} \varepsilon_i g(Y_i)^2 \right| = 2\,\mathbb{E}\mathbb{E}_\varepsilon \sup_{t\in T} \left| \sum_{i=1}^{n} \varepsilon_i t_i^2 \right|$$

where T is the random set $\{t = (t_1, \dots, t_n) = (g(Y_1), \dots, g(Y_n)) : g \in \mathcal{G}\}$. Since $T \subset [-2u, 2u]^n$, we deduce that the function $h(x) = x^2$ is $4u$-Lipschitz on T. By the comparison Theorem 5.2.1, we get that

$$\mathbb{E}_\varepsilon \sup_{t\in T} \left| \sum_{i=1}^{n} \varepsilon_i t_i^2 \right| \le 8\,u\,\mathbb{E}_\varepsilon \sup_{t\in T} \left| \sum_{i=1}^{n} \varepsilon_i t_i \right|.$$

Since for every $i \leq n$, $\mathbb{E} g(Y_i) = 0$, we deduce from (5.7) that

$$\mathbb{E}\mathbb{E}_\varepsilon \sup_{g\in\mathcal{G}} \left| \sum_{i=1}^n \varepsilon_i g(Y_i)^2 \right| \leq 16\, u\, \mathbb{E} \sup_{g\in\mathcal{G}} \left| \sum_{i=1}^n g(Y_i) \right|.$$

This allows to conclude that

$$V \leq 32\, u\, \mathbb{E} \sup_{f\in\mathcal{F}} \left| \sum_{i=1}^n f(Y_i) - \mathbb{E} f(Y_i) \right| + \sup_{f\in\mathcal{F}} \sum_{i=1}^n \mathrm{Var} f(Y_i).$$

This ends the proof of the proposition. □

5.2.2. The study of the expectation of the supremum of some empirical processes. — We go back to the study of Problem 5.1.8 with a definition of randomness given by the empirical method. The situation is similar if we worked with the method of selectors. For any star-shape body $T \subset \mathbb{R}^N$, we define the class of functions $\mathcal{F}$ in the following way:

$$\mathcal{F} = \left\{ \begin{array}{rcl} f_y : \mathbb{R}^N & \to & \mathbb{R} \\ Y & \mapsto & \langle Y, y\rangle \end{array} \quad : \quad y \in T \cap \rho S^{N-1} \right\}.$$

Therefore

$$Z = \sup_{f\in\mathcal{F}} \left| \sum_{i=1}^n (f^2(Y_i) - \mathbb{E} f^2(Y_i)) \right| = \sup_{y\in T\cap\rho S^{N-1}} \left| \sum_{i=1}^n \langle Y_i, y\rangle^2 - \frac{n\rho^2}{N} \right|.$$

Applying the symmetrization procedure to Z (cf (5.5)) and comparing Rademacher and Gaussian processes, we conclude that

$$\begin{aligned} \mathbb{E} \sup_{y\in T\cap\rho S^{N-1}} \left| \sum_{i=1}^n \langle Y_i, y\rangle^2 - \frac{n\rho^2}{N} \right| &\leq 2\mathbb{E}\mathbb{E}_\varepsilon \sup_{y\in T\cap\rho S^{N-1}} \left| \sum_{i=1}^n \varepsilon_i \langle Y_i, y\rangle^2 \right| \\ &\leq \sqrt{2\pi}\mathbb{E}\mathbb{E}_g \sup_{y\in T\cap\rho S^{N-1}} \left| \sum_{i=1}^n g_i \langle Y_i, y\rangle^2 \right|. \end{aligned}$$

We will first get a bound for the Rademacher average (or the Gaussian one) and then we will take the expectation with respect to the Y_i's. Before to work with these difficult processes, we present a result of Rudelson where the supremum is taken on the unit sphere S^{N-1}.

Theorem 5.2.6. — *For any fixed vectors $Y_1, \ldots, Y_n$ in $\mathbb{R}^N$,*

$$\mathbb{E}_\varepsilon \sup_{y\in S^{N-1}} \left| \sum_{i=1}^n \varepsilon_i \langle Y_i, y\rangle^2 \right| \leq C\sqrt{\log n} \max_{1\leq i\leq n} |Y_i|_2 \sup_{y\in S^{N-1}} \left(\sum_{i=1}^n \langle Y_i, y\rangle^2 \right)^{1/2}.$$

Proof. — We define the self-adjoint rank 1 operators

$$T_i = Y_i \otimes Y_i : \left\{ \begin{array}{rcl} \mathbb{R}^N & \to & \mathbb{R}^N \\ y & \mapsto & \langle Y_i, y\rangle Y_i \end{array} \right.$$

in such a way that

$$\sup_{y\in S^{N-1}}\left|\sum_{i=1}^n \varepsilon_i\langle Y_i,y\rangle^2\right| = \sup_{y\in S^{N-1}}\left|\langle\sum_{i=1}^n \varepsilon_i T_i y, y\rangle\right| = \left\|\sum_{i=1}^n \varepsilon_i T_i\right\|_{2\to 2}.$$

Let $(\lambda_i)_{1\le i\le N}$ be the eigenvalues of a self-adjoint operator S. By definition of the S_q^N norms for any $q>0$,

$$\|S\|_{2\to 2} = \|S\|_{S_\infty^N} = \max_{1\le i\le n}|\lambda_i| \quad \text{and} \quad \|S\|_{S_q^N} = \left(\sum_{i=1}^N |\lambda_i|^q\right)^{1/q}.$$

Assume that the operator has rank less than n then for $i\ge n+1$, $\lambda_i=0$ and we deduce by Hölder inequality that

$$\|S\|_{S_\infty^N} \le \|S\|_{S_q^N} \le n^{1/q}\|S\|_{S_\infty^N} \le e\|S\|_{S_\infty^N} \quad \text{for} \quad q\ge \log n.$$

The non-commutative Khinchine inequality of Lust-Piquard and Pisier states that for any operator $T_1,\dots,T_n$,

$$\mathbb{E}_\varepsilon\left\|\sum_{i=1}^n \varepsilon_i T_i\right\|_{S_q^N} \le C\sqrt{q}\max\left\{\left\|\left(\sum_{i=1}^n T_i^* T_i\right)^{1/2}\right\|_{S_q^N}, \left\|\left(\sum_{i=1}^n T_i T_i^*\right)^{1/2}\right\|_{S_q^N}\right\}.$$

In our situation, $T_i^*T_i = T_iT_i^* = |Y_i|_2^2 T_i$ and $S = \left(\sum_{i=1}^n T_i^*T_i\right)^{1/2}$ has rank less than n, hence for $q=\log n$,

$$\left\|\left(\sum_{i=1}^n T_i^*T_i\right)^{1/2}\right\|_{S_q^N} \le e\left\|\left(\sum_{i=1}^n |Y_i|_2^2 T_i\right)^{1/2}\right\|_{S_\infty^N} \le e\max_{1\le i\le n}|Y_i|_2\left\|\sum_{i=1}^n T_i\right\|_{S_\infty^N}^{1/2}.$$

Combining all these inequalities, we conclude that for $q=\log n$

$$\begin{aligned}\mathbb{E}_\varepsilon\left\|\sum_{i=1}^n \varepsilon_i T_i\right\|_{S_\infty^N} &\le C\sqrt{\log n}\left\|\left(\sum_{i=1}^n T_i^*T_i\right)^{1/2}\right\|_{S_{\log n}^N}\\ &\le C\,e\sqrt{\log n}\max_{1\le i\le n}|Y_i|_2\sup_{y\in S^{N-1}}\left(\sum_{i=1}^n\langle Y_i,y\rangle^2\right)^{1/2}.\end{aligned}$$

□

Remark 5.2.7. — *Since the non-commutative Khinchine inequality holds true for independent Gaussian standard random variables, this result is also valid for Gaussian instead of Bernoulli.*

The proof that we presented here is based on an expression related to some operator norms and our original question can not be expressed with these tools. The original proof of Rudelson used the majorizing measure theory. Several improvements are known and the statements of these results need some definition from the theory of Banach spaces.

Definition 5.2.8. — *A Banach space X is of type 2 if there exists a constant $c > 0$ such that for every n and every $x_1, \dots, x_n \in X$,*

$$\left(\mathbb{E}_\varepsilon \left\| \sum_{i=1}^n \varepsilon_i x_i \right\|^2 \right)^{1/2} \leq c \left(\sum_{i=1}^n \|x_i\|^2 \right)^{1/2}.$$

The smallest constant $c > 0$ satisfying this statement is called the type 2 constant of X and is denoted by $T_2(X)$.

Classical examples are Hilbert spaces and L_q space for $2 \leq q < +\infty$. From Theorem 1.2.1 in Chapter 1, we know also that L_{ψ_2} has type 2.

Definition 5.2.9. — *A Banach space X has modulus of convexity of power type 2 with constant λ if*

$$\forall x, y \in X, \quad \left\| \frac{x+y}{2} \right\|^2 + \lambda^{-2} \left\| \frac{x-y}{2} \right\|^2 \leq \frac{1}{2} \left(\|x\|^2 + \|y\|^2 \right).$$

The modulus of convexity of a Banach space X is defined for every $\varepsilon \in (0, 2)$ by

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \left\| \frac{x+y}{2} \right\| : \quad \|x\| = \|y\| = 1 \text{ and } \|x - y\| \leq \varepsilon \right\}.$$

It is obvious that if X has modulus of convexity of power type 2 with constant λ then $\delta_X(\varepsilon) \geq \varepsilon^2/2\lambda^2$ and it is well known that the reverse holds true (with a different constant than 2).

Definition 5.2.10. — *A Banach space Y has modulus of smoothness of power type 2 with constant μ if*

$$\forall x, y \in Y, \quad \left\| \frac{x+y}{2} \right\|^2 + \mu^2 \left\| \frac{x-y}{2} \right\|^2 \geq \frac{1}{2} \left(\|x\|^2 + \|y\|^2 \right).$$

The modulus of smoothness of a Banach space Y is defined for every $\tau > 0$ by

$$\rho_Y(\tau) = \sup \left\{ \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : \quad \|x\| = \|y\| = 1 \right\}.$$

It is clear that if Y has modulus of smoothness of power type with constant μ then for every $\tau \in (0, 1), \rho_Y(\tau) \leq 2\tau^2\mu^2$ and it is well known that the reverse holds true (with a different constant than 2).
More generally, a Banach space X is uniformly convex if for every $\varepsilon > 0$, $\delta_X(\varepsilon) > 0$ and a Banach space Y is uniformly smooth if $\lim_{\tau \to 0} \rho_Y(\tau)/\tau = 0$. We have the following simple relationship between these notions.

Proposition 5.2.11. — *For every Banach space X, $X^\star$ being its dual, we have*
(i) For every $\tau > 0$, $\rho_{X^\star}(\tau) = \sup\{\tau\varepsilon/2 - \delta_X(\varepsilon), 0 < \varepsilon \leq 2\}$.
(ii) X is uniformly convex if and only if $X^\star$ is uniformly smooth.
(iii) For any Banach space X, if X has modulus of convexity of power type 2 with constant λ then $X^\star$ has modulus of smoothness of power type 2 with constant $c\lambda$ and $T_2(X^\star) \leq c\lambda$.

Proof. — The proof of (i) is straightforward, using the definition of duality. We have for $\tau > 0$,

$$\begin{aligned}
2\rho_{X^\star}(\tau) &= \sup\{\|x^\star + \tau y^\star\| + \|x^\star - \tau y^\star\| - 2 : \ \|x^\star\| = 1, \|y^\star\| = 1\} \\
&= \sup\{x^\star(x) + \tau y^\star(x) + x^\star(y) - \tau y^\star(y) - 2 : \ \|x^\star\| = 1, \|y^\star\| = 1, \|x\| = 1, \|y\| = 1\} \\
&= \sup\{x^\star(x+y) + \tau y^\star(x-y) - 2 : \ \|x^\star\| = 1, \|y^\star\| = 1, \|x\| = 1, \|y\| = 1\} \\
&= \sup\{\|x+y\| + \tau\|x-y\| - 2 : \ \|x\| = 1, \|y\| = 1\} \\
&= \sup\{\|x+y\| + \tau\varepsilon - 2 : \ \|x\| = 1, \|y\| = 1, \|x-y\| \le \varepsilon, \varepsilon \in (0,2]\} \\
&= \sup\{\tau\varepsilon - 2\delta_X(\varepsilon) : \ \varepsilon \in (0,2]\}.
\end{aligned}$$

The proof of (ii) follows directly from (i). We will just prove (iii). If X has modulus of convexity of power type 2 with constant λ then $\delta_X(\varepsilon) \ge \varepsilon^2/2\lambda^2$. By (i) we deduce that $\rho_{X^\star}(\tau) \ge \tau^2\lambda^2/4$. It implies that for any $x^\star$, $y^\star \in X^\star$,

$$\left\|\frac{x^\star + y^\star}{2}\right\|_\star^2 + (c\lambda)^2 \left\|\frac{x^\star - y^\star}{2}\right\|_\star^2 \ge \frac{1}{2}\left(\|x^\star\|_\star^2 + \|y^\star\|_\star^2\right)$$

where c is a positive number. We deduce that for any $u^\star$, $v^\star \in X^\star$,

$$\mathbb{E}_\varepsilon \|\varepsilon u^\star + v^\star\|_\star^2 = \frac{1}{2}\left(\|u^\star + v^\star\|_\star^2 + \|-u^\star + v^\star\|_\star^2\right) \le \|v^\star\|_\star^2 + (c\lambda)^2\|u^\star\|_\star^2.$$

We conclude by induction that for any integer n and any vectors $x_1^\star, \ldots, x_n^\star \in X^\star$,

$$\mathbb{E}_\varepsilon \left\|\sum_{i=1}^n \varepsilon_i x_i^\star\right\|_\star^2 \le (c\lambda)^2 \left(\sum_{i=1}^n \|x_i^\star\|_\star^2\right)$$

which proves that $T_2(X^\star) \le c\lambda$. □

It is now possible to state the results about the estimate of the average of the supremum of empirical processes.

Theorem 5.2.12. — *If X is a Banach space with modulus of convexity of power type 2 with constant λ then for any integer n and $\xi_1, \ldots \xi_n \in X^\star$,*

$$\mathbb{E}_g \sup_{\|x\|\le 1} \left|\sum_{i=1}^n g_i \langle \xi_i, x\rangle^2\right| \le C\,\lambda^5\,\sqrt{\log n}\,\max_{1\le i\le n} \|\xi_i\|_\star \sup_{\|x\|\le 1}\left(\sum_{i=1}^n \langle \xi_i, x\rangle^2\right)^{1/2}$$

where $g_1, \ldots, g_n$ are independent $\mathcal{N}(0,1)$ Gaussian random variables and C is a numerical constant.

Corollary 5.2.13. — *Let X be a Banach space with modulus of convexity of power type 2 with constant λ. Let $Y_1, \ldots, Y_n \in X^\star$ be independent random vectors and denote*

$$K(n,Y) = 2\sqrt{\frac{2}{\pi}} C\lambda^5 \sqrt{\log n}\left(\mathbb{E}\max_{1\le i\le n}\|Y_i\|_\star^2\right)^{1/2} \quad \textit{and} \quad \sigma^2 = \sup_{\|y\|\le 1}\sum_{i=1}^n \mathbb{E}\langle Y_i, y\rangle^2$$

where C is the numerical constant of Theorem 5.2.12. Then we have

$$\mathbb{E}\sup_{\|y\|\le 1}\left|\sum_{i=1}^n \langle Y_i,y\rangle^2-\mathbb{E}\langle Y_i,y\rangle^2\right|\le K(n,Y)^2\,+\,K(n,Y)\,\sigma.$$

Proof. — Denote by V_2 the expectation of the supremum of the empirical process, that is

$$V_2=\mathbb{E}\sup_{\|y\|\le 1}\left|\sum_{i=1}^n\left(\langle Y_i,y\rangle^2-\mathbb{E}\langle Y_i,y\rangle^2\right)\right|.$$

We start with a symmetrization argument. By (5.5) and Proposition 5.2.3 we have

$$V_2\le 2\;\mathbb{E}\mathbb{E}_\varepsilon\sup_{\|y\|\le 1}\left|\sum_{i=1}^n\varepsilon_i\langle Y_i,y\rangle^2\right|\le 2\sqrt{\frac{2}{\pi}}\;\mathbb{E}\mathbb{E}_g\sup_{\|y\|\le 1}\left|\sum_{i=1}^n g_i\langle Y_i,y\rangle^2\right|.$$

In view of Theorem 5.2.12, we observe that the crucial quantity in the estimate is $\sup_{\|x\|\le 1}\left(\sum_{i=1}^n\langle Y_i,x\rangle^2\right)^{1/2}$. Indeed, by the triangle inequality,

$$\mathbb{E}\sup_{\|x\|\le 1}\sum_{i=1}^n\langle Y_i,x\rangle^2\le \mathbb{E}\sup_{\|y\|\le 1}\left|\sum_{i=1}^n\left(\langle Y_i,y\rangle^2-\mathbb{E}\langle Y_i,y\rangle^2\right)\right|+\sup_{\|y\|\le 1}\sum_{i=1}^n\mathbb{E}\langle Y_i,y\rangle^2=V_2+\sigma^2.$$

Therefore, applying Theorem 5.2.12 and the Cauchy Schwarz inequality, we get

$$\begin{aligned}V_2&\le 2\sqrt{\frac{2}{\pi}}\,C\lambda^5\,\sqrt{\log n}\,\mathbb{E}\left(\max_{1\le i\le n}\|Y_i\|_\star\sup_{\|x\|\le 1}\left(\sum_{i=1}^n\langle Y_i,x\rangle^2\right)^{1/2}\right)\\&\le 2\sqrt{\frac{2}{\pi}}\,C\lambda^5\,\sqrt{\log n}\,\left(\mathbb{E}\max_{1\le i\le n}\|Y_i\|_\star^2\right)^{1/2}\left(\mathbb{E}\sup_{\|x\|\le 1}\sum_{i=1}^n\langle Y_i,x\rangle^2\right)^{1/2}\\&\le K(n,Y)\left(V_2+\sigma^2\right)^{1/2}.\end{aligned}$$

We get that

$$V_2^2-K(n,Y)^2V_2-K(n,Y)^2\sigma^2\le 0$$

from which it is easy to conclude that

$$V_2\le K(n,Y)\;(K(n,Y)\,+\,\sigma)\,.$$

□

The proof of Theorem 5.2.12 is slightly complicate and uses the tools of majorizing measure theory and deep results about the duality of covering numbers (it is where the notion of type is used). We will not present it. However, using simpler ideas, we can also prove a general result where the assumption that X has a good modulus of convexity is not needed.

Theorem 5.2.14. — *Let X be a Banach space and $Y_1,\dots,Y_n$ be independent random vectors in $X^\star$. Let $\mathcal{F}$ be a set of functionals on $X^\star$. Denote by $d_{\infty,n}$ the random quasi-metric on $\mathcal{F}$ defined for every $f,\overline{f}$ in $\mathcal{F}$ by*

$$d_{\infty,n}(f,\overline{f})=\max_{1\le i\le n}\left|f(Y_i)-\overline{f}(Y_i)\right|.$$

We have

$$\mathbb{E}\sup_{f\in\mathcal{F}}\left|\sum_{i=1}^{n}\left(f(Y_i)^2-\mathbb{E}f(Y_i)^2\right)\right|\le\max(\sigma_{\mathcal{F}}U_n,U_n^2)$$

where for a numerical constant C,

$$U_n=C\left(\mathbb{E}\gamma_2^2(\mathcal{F},d_{\infty,n})\right)^{1/2}\quad\textit{and}\quad\sigma_{\mathcal{F}}=\left(\sup_{f\in\mathcal{F}}\sum_{i=1}^{n}\mathbb{E}f(Y_i)^2\right)^{1/2}.$$

We refer to Definition 3.1.3 in Chapter 3 for the precise definition of $\gamma_2(\mathcal{F},d_{\infty,n})$.

Proof. — As in the proof of Corollary 5.2.13, we need first to get a bound of

$$\mathbb{E}_g\sup_{f\in\mathcal{F}}\left|\sum_{i=1}^{n}g_if(Y_i)^2\right|.$$

Let $(X_f)_{f\in\mathcal{F}}$ be the Gaussian process defined conditionally with respect to the Y_i's, $X_f=\sum_{i=1}^{n}g_if(Y_i)^2$ and indexed by $f\in\mathcal{F}$. The quasi-metric d associated to this process is given for any $f,\overline{f}\in\mathcal{F}$ by

$$\begin{aligned}d(f,\overline{f})^2=\mathbb{E}_g|X_f-X_{\overline{f}}|^2&=\sum_{i=1}^{n}\left(f(Y_i)^2-\overline{f}(Y_i)^2\right)^2\\&=\sum_{i=1}^{n}\left(f(Y_i)-\overline{f}(Y_i)\right)^2\left(f(Y_i)+\overline{f}(Y_i)\right)^2\\&\le 2\sum_{i=1}^{n}\left(f(Y_i)-\overline{f}(Y_i)\right)^2\left(f(Y_i)^2+\overline{f}(Y_i)^2\right)\\&\le 4\sup_{f\in\mathcal{F}}\left(\sum_{i=1}^{n}f(Y_i)^2\right)\max_{1\le i\le n}(f(Y_i)-\overline{f}(Y_i))^2.\end{aligned}$$

In conclusion, we have proved that for any $f,\overline{f}\in\mathcal{F}$,

$$d(f,\overline{f})\le 2\sup_{f\in\mathcal{F}}\left(\sum_{i=1}^{n}f(Y_i)^2\right)^{1/2}d_{\infty,n}(f,\overline{f}).$$

By definition of the γ_2 functionals, see Chapter 3, we conclude that for every vectors $Y_1,\dots,Y_n\in X^\star$,

$$\mathbb{E}_g\sup_{f\in\mathcal{F}}\left|\sum_{i=1}^{n}g_if(Y_i)^2\right|\le C\sup_{f\in\mathcal{F}}\left(\sum_{i=1}^{n}f(Y_i)^2\right)^{1/2}\gamma_2(\mathcal{F},d_{\infty,n})$$

where C is a universal constant. We repeat the proof of Corollary 5.2.13. Let

$$V_2=\mathbb{E}\sup_{f\in\mathcal{F}}\left|\sum_{i=1}^{n}\left(f(Y_i)^2-\mathbb{E}f(Y_i)^2\right)\right|.$$

By a symmetrization argument and the Cauchy-Schwarz inequality,

$$V_2 \le 2\sqrt{\frac{2}{\pi}}\mathbb{E}\mathbb{E}_g \sup_{f\in\mathcal{F}} \left| \sum_{i=1}^n g_i f(Y_i)^2 \right| \le C \left(\mathbb{E}\gamma_2(\mathcal{F}, d_{\infty,n})^2\right)^{1/2} \left(\mathbb{E} \sup_{f\in\mathcal{F}} \sum_{i=1}^n f(Y_i)^2\right)^{1/2}$$
$$\le C \left(\mathbb{E}\gamma_2(\mathcal{F}, d_{\infty,n})^2\right)^{1/2} \left(V_2 + \sigma_{\mathcal{F}}^2\right)^{1/2}.$$

where the last inequality follows from the triangle inequality: $\mathbb{E}\sup_{f\in\mathcal{F}} \sum_{i=1}^n f(Y_i)^2 \le (V_2 + \sigma_{\mathcal{F}}^2)^2$. This shows that V_2 satisfies an inequality of degree 2 from which it is easy to conclude that

$$V_2 \le \max(\sigma_{\mathcal{F}} U_n, U_n^2), \text{ where } U_n = C \left(\mathbb{E}\gamma_2(\mathcal{F}, d_{\infty,n})^2\right)^{1/2}.$$

□

5.3. Selection of characters

5.3.1. Reconstruction property. — We are now able to state one main theorem concerning the reconstruction property of a random matrix defined by taking empirical copies of the rows of a fixed bounded orthogonal matrix (or by selecting randomly its rows).

Theorem 5.3.1. — *Let $\phi_1, \ldots, \phi_N$ be an orthogonal system in ℓ_2^N such that for a real number K*

$$\forall i \le N,\ |\phi_i|_2 = K \text{ and } |\phi_i|_\infty \le \frac{1}{\sqrt{N}}.$$

Let Y be the random vector defined by $Y = \phi_i$ with probability $1/N$ and $Y_1, \ldots, Y_n$ be independent copies of Y. . If

$$m \le C_1 K^2 \frac{n}{\log N (\log n)^3}$$

then with probability greater than

$$1 - C_2 \exp(-C_3 K^2 n/m)$$

the matrix $\Phi = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$ *is a good reconstruction matrix for sparse signals of size m, that is for every $U \in \Sigma_m$, the basis pursuit algorithm (5.1), $\min_{t\in\mathbb{R}^N} \{|t|_1 : \Phi U = \Phi t\}$, has a unique solution equal to U.*

Remark 5.3.2. — • *By definition of m, the probability of this event is always greater than $1 - C_2 \exp\left(-C_3 \log N (\log n)^3\right)$.*
• *The same result is valid when using the method of selectors.*
• *As we already mentioned, this theorem covers the case of a lot of classical systems like the Fourier system or the Walsh system.*
• *The result is also valid if the orthogonal system $\phi_1, \ldots, \phi_N$ satisfies the weaker condition that for all $i \le N$, $K_1 \le |\phi_i|_2 \le K_2$ and in the statement, K has to be replaced by K_2^2/K_1.*

Proof. — Observe that $\mathbb{E}\langle Y, y\rangle^2 = K^2\,|y|_2^2/N$. We define the class of functions $\mathcal{F}$ in the following way:

$$\mathcal{F} = \left\{ \begin{array}{rcl} f_y : \mathbb{R}^N & \to & \mathbb{R} \\ Y & \mapsto & \langle Y, y\rangle \end{array} , \quad y \in B_1^N \cap \rho S^{N-1} \right\}.$$

Therefore

$$Z = \sup_{f\in\mathcal{F}} \left| \sum_{i=1}^n (f(Y_i)^2 - \mathbb{E} f(Y_i)^2) \right| = \sup_{y \in B_1^N \cap \rho S^{N-1}} \left| \sum_{i=1}^n \langle Y_i, y\rangle^2 - \frac{K^2 n\rho^2}{N} \right|.$$

With the notation of Theorem 5.2.14, we have

$$\sigma_{\mathcal{F}}^2 = \sup_{y\in B_1^N\cap\rho S^{N-1}} \sum_{i=1}^n \mathbb{E}\langle Y_i, y\rangle^2 = \frac{K^2 n \rho^2}{N}. \tag{5.8}$$

Moreover, since $B_1^N \cap \rho S^{N-1} \subset B_1^N$,

$$\gamma_2(B_1^N \cap \rho S^{N-1}, d_{\infty,n}) \le \gamma_2(B_1^N, d_{\infty,n}).$$

It is well known that the γ_2 functional is bounded by the Dudley integral (see (3.7) in Chapter 3):

$$\gamma_2(B_1^N, d_{\infty,n}) \le C \int_0^{+\infty} \sqrt{\log N(B_1^N, \varepsilon, d_{\infty,n})}\; d\varepsilon.$$

However, for every $i \le n$, $|Y_i|_\infty \le 1/\sqrt{N}$ and

$$\sup_{y,\overline{y}\in B_1^N} d_{\infty,n}(y,\overline{y}) = \sup_{y,\overline{y}\in B_1^N} \max_{1\le i\le n} |\langle Y_i, y-\overline{y}\rangle| \le 2 \max_{1\le i\le n} |Y_i|_\infty \le \frac{2}{\sqrt{N}}.$$

The integral is only computed from 0 to $2/\sqrt{N}$ and by the change of variable $t = \varepsilon\sqrt{N}$, we deduce that

$$\int_0^{+\infty} \sqrt{\log N(B_1^N, \varepsilon, d_{\infty,n})} d\varepsilon = \frac{1}{\sqrt{N}} \int_0^2 \sqrt{\log N\left(B_1^N, \frac{t}{\sqrt{N}}, d_{\infty,n}\right)}\; dt.$$

¿From Theorem 1.4.3, since for every $i \le n$, $|Y_i|_\infty \le 1/\sqrt{N}$, we have

$$\sqrt{\log N\left(B_1^N, \frac{t}{\sqrt{N}}, d_{\infty,n}\right)} \le \left\{ \begin{array}{l} \dfrac{C}{t}\; \sqrt{\log n}\sqrt{\log N}\,, \\ C\sqrt{n\log\left(1+\dfrac{3}{t}\right)} \end{array} \right. .$$

We split the integral into two parts, the one when $t \le 1/\sqrt{n}$ and the one when $1/\sqrt{n} \le t \le 2$.

$$\begin{aligned} \int_0^{1/\sqrt{n}} \sqrt{n\log\left(1+\frac{3}{t}\right)} dt &= \int_0^1 \sqrt{\log\left(1+\frac{3\sqrt{n}}{u}\right)}\, du \\ &\le \int_0^1 \sqrt{\log n + \log\left(\frac{3}{u}\right)}\, du \le C\; \sqrt{\log n} \end{aligned}$$

and since

$$\int_{1/\sqrt{n}}^{2} \frac{1}{t} dt \leq C \log n,$$

we conclude that

$$\gamma_2(B_1^N \cap \rho S^{N-1}, d_{\infty,n}) \leq \gamma_2(B_1^N, d_{\infty,n}) \leq C \sqrt{\frac{(\log n)^3 \log N}{N}} .$$

Combining this estimate and (5.8) with Theorem 5.2.14, we get that for a real number $C \geq 1$,

$$\mathbb{E}Z \leq C \max\left(\frac{(\log n)^3 \log N}{N}, \; \rho K \sqrt{\frac{n}{N}} \sqrt{\frac{(\log n)^3 \log N}{N}} \right).$$

We choose ρ such that

$$(\log n)^3 \log N \leq \rho K \sqrt{n (\log n)^3 \log N} \leq \frac{1}{3C} K^2 \rho^2 n$$

which means that ρ satisfies

$$K \rho \geq 3 C \sqrt{\frac{(\log n)^3 \log N}{n}}. \tag{5.9}$$

For this choice of ρ, we conclude that

$$\mathbb{E}Z = \mathbb{E} \sup_{y \in B_1^N \cap \rho S^{N-1}} \left| \sum_{i=1}^{n} \langle Y_i, y \rangle^2 - \frac{K^2 n \rho^2}{N} \right| \leq \frac{1}{3} \frac{K^2 n \rho^2}{N}.$$

We use Proposition 5.2.5 to get a deviation inequality for the random variable Z. With the notations of Proposition 5.2.5, we have

$$u = \sup_{y \in B_1^N \cap \rho S^{N-1}} \max_{1 \leq i \leq N} \langle \phi_i, y \rangle^2 \leq \max_{1 \leq i \leq N} |\phi_i|_\infty^2 \leq \frac{1}{N}$$

and

$$\begin{aligned} v &= \sup_{y \in B_1^N \cap \rho S^{N-1}} \sum_{i=1}^{n} \mathbb{E} \left(\langle Y_i, y \rangle^2 - \mathbb{E} \langle Y_i, y \rangle^2 \right)^2 \; + \; 32 \, u \, \mathbb{E}Z \\ &\leq \sup_{y \in B_1^N \cap \rho S^{N-1}} \sum_{i=1}^{n} \mathbb{E} \langle Y_i, y \rangle^4 \; + \frac{C K^2 n \rho^2}{N^2} \leq \frac{C K^2 n \rho^2}{N^2} \end{aligned}$$

since for every $y \in B_1^N$, $\mathbb{E}\langle Y, y \rangle^4 \leq \mathbb{E}\langle Y, y \rangle^2 / N$. Using Proposition 5.2.5 with $t = \frac{1}{3} \frac{K^2 n \rho^2}{N}$, we conclude that

$$\mathbb{P}\left(Z \geq \frac{2}{3} \frac{K^2 n \rho^2}{N} \right) \leq C \exp(-c \, K^2 \, n \, \rho^2).$$

With probability greater than $1 - C \exp(-c \, K^2 \, n \, \rho^2)$, we get that

$$\sup_{y \in B_1^N \cap \rho S^{N-1}} \left| \sum_{i=1}^{n} \langle Y_i, y \rangle^2 - \frac{K^2 n \rho^2}{N} \right| \leq \frac{2}{3} \frac{K^2 n \rho^2}{N}$$

from which it is easy to deduce by Proposition 5.1.6 that

$$\operatorname{diam}\left(\ker\Phi \cap B_1^N\right) < \rho.$$

We choose $m = 1/4\rho^2$ and conclude by Proposition 5.1.1 that with probability greater than $1 - C\exp(-cK^2 n/m)$, the matrix Φ is a good reconstruction matrix for sparse signals of size m, that is for every $U \in \Sigma_m$, the basis pursuit algorithm (5.1) has a unique solution equal to U. The condition on m in Theorem 5.3.1 comes from (5.9). □

Remark 5.3.3. — *In view of Theorem 5.1.2, it is clear that the matrix Φ shares also the property of approximate reconstruction. It is enough to change the choice of m by $m = 1/16\rho^2$. Therefore, if U is any unknown signal and x a solution of*

$$\min_{t\in\mathbb{R}^N} \{|t|_1, \Phi U = \Phi t\}$$

then for any subset I of cardinality less than m,

$$|x - U|_2 \le \frac{|x-U|_1}{4\sqrt{m}} \le \frac{|U_{I^c}|_1}{\sqrt{m}}.$$

5.3.2. Random selection of a coordinate subspace. — In this part, we consider a problem coming from Harmonic Analysis. Let μ be a probability measure and let $(\psi_1, \dots, \psi_N)$ be an orthonormal system of $L_2(\mu)$ bounded in L_∞ i.e. such that for every $i \le N$, $\|\psi_i\|_\infty \le 1$. Typically, we consider a system of characters in $L_2(\mu)$. For a measurable function f and for $p > 0$, we denote its L_p norm and its L_∞ norm by

$$\|f\|_p = \left(\int |f|^p d\mu\right)^{1/p} \quad \text{and} \quad \|f\|_\infty = \sup |f|.$$

In $\mathbb{R}^N$ or $\mathbb{C}^N$, μ is just the counting probability measure so that the L_p-norm of a vector $x = (x_1, \dots, x_N)$ is defined by

$$\|x\|_p = \left(\frac{1}{N}\sum_{i=1}^N |x_i|^p\right)^{1/p}.$$

The spaces ℓ_∞^N and L_∞^N coincide and we observe that if $(\psi_1, \dots, \psi_N)$ is a bounded orthonormal system in L_2^N then $(\psi_1/\sqrt{N}, \dots, \psi_N/\sqrt{N})$ is an orthonormal system of ℓ_2^N such that for every $i \le N$, $|\psi_i/\sqrt{N}|_\infty \le 1/\sqrt{N}$. Therefore the setting is exactly the same as in the previous part except a normalization factor of $\sqrt{N}$.

Of course the notation of the half-diameter of a set T will now be adapted to the $L_2(\mu)$ Euclidean structure. This means that for a set T, its half-diameter is

$$\operatorname{Diam} T = \sup_{t\in T} \|t\|_2.$$

For any $q > 0$, we denote by B_q the unit ball of $L_q(\mu)$ that is

$$B_q = \{\, f \text{ measurable with respect to } \mu, \text{ such that } \|f\|_q \le 1\}$$

and by S_q the unit sphere of $L_q(\mu)$. It is clear that for any subset $I \subset [N]$

$$\forall\, (a_i)_{i\in I}\,, \quad \Big\|\sum_{i\in I} a_i\psi_i\Big\|_1 \le \Big\|\sum_{i\in I} a_i\psi_i\Big\|_2 \le \sqrt{|I|}\, \Big\|\sum_{i\in I} a_i\psi_i\Big\|_1 \,.$$

The Dvoretky's theorem proved by Milman asserts that for any $\varepsilon \in (0,1)$, there exists a subspace $E \subset \mathrm{span}\{\psi_1,\dots,\psi_N\}$ of dimension $\dim E = n = c\,\big(\varepsilon^2/\log(1+2/\varepsilon)\big)\,N$ such that

$$\forall\, (a_i)_{i=1}^N\,, \text{ if } x = \sum_{i=1}^N a_i\psi_i \in E, \text{ then } (1-\varepsilon)\, r\, \|x\|_1 \le \|x\|_2 \le (1+\varepsilon)\, r\, \|x\|_1$$

where r is a number depending on the dimension N which can be bounded from above and below by some numerical constants (independent of the dimension N). Observe that the constant c which appears in the dependance of $\dim E$ is very small therefore this Dvoretzy's theorem does not provide a subspace of say half dimension such that the L_1 norm and the L_2 norm are comparable up to constant factors. This question was solved by Kashin. He proved in fact a very strong result which is called now a Kashin decomposition: there exists a subspace E of dimension $[N/2]$ such that $\forall\, (a_i)_{i=1}^N$,

$$\text{if } x = \sum_{i=1}^N a_i\psi_i \in E \text{ then } \|x\|_1 \le \|x\|_2 \le C\, \|x\|_1\,,$$

$$\text{and if } y = \sum_{i=1}^N a_i\psi_i \in E^\perp \text{ then } \|y\|_1 \le \|y\|_2 \le C\, \|y\|_1$$

where C is a numerical constant. In the setting of Harmonic Analysis, the questions are more related with coordinate subspaces because the questions are related in finding a subset of $\{\psi_1,\dots,\psi_N\}$ which satisfies good comparison properties between the L_r norms. Talagrand, improving a result of Bourgain, showed that there exists a small constant δ_0 such that for any bounded orthonormal system $\{\psi_1,\dots,\psi_N\}$, there exists a subset I of cardinality greater than $\delta_0 N$ such that

$$\forall (a_i)_{i\in I}, \qquad \Big\|\sum_{i\in I} a_i\psi_i\Big\|_2 \le \sqrt{\log N\,(\log\log N)}\, \Big\|\sum_{i\in I} a_i\psi_i\Big\|_1 \,. \tag{5.10}$$

The proof involves the construction of specific majorizing measures. Moreover, it was known from Bourgain that the $\sqrt{\log N}$ is necessary in the estimate. We will now explain why the strategy that we developed in the previous part is adapted to this type of question. For example, we will be able to extend the result (5.10) to a Kashin type setting.

We start with the following simple Proposition concerning some properties of the matrix Ψ (that we will later define randomly as in Theorem 5.3.1).

Proposition 5.3.4. — *Let μ be a probability measure and let $(\psi_1,\dots,\psi_N)$ be an orthonormal system of $L_2(\mu)$. Let $Y_1,\dots,Y_n$ be a family of vectors taking values from* *the set of vectors* $\{\psi_1, \ldots, \psi_N\}$*. Let* Ψ *be the matrix* $\Psi = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$*. Then*

(i) $\ker\Psi = \text{span}\,\{\{\psi_1, \ldots, \psi_N\} \setminus \{Y_i\}_{i=1}^n\} = \text{span}\,\{\psi_i\}_{i\in I}$ *where* I *is a subset of cardinality greater than* $N - n$*.*

(ii) $(\ker\Psi)^\perp = \text{span}\,\{\psi_i\}_{i\notin I}$

(iii) If for a star shape body T

$$\sup_{y\in T\cap\rho S_2}\left|\sum_{i=1}^n \langle Y_i, y\rangle^2 - \frac{n\rho^2}{N}\right| \le \frac{1}{3}\frac{n\rho^2}{N} \tag{5.11}$$

then $\text{Diam}\,(\ker\Psi\cap T) < \rho$*.*

(iv) If $n < 3N/4$ *and if* (5.11) *is satisfied then we also have* $\text{Diam}\,((\ker\Psi)^\perp\cap T) < \rho$*.*

Proof. — Since $\{\psi_1, \ldots, \psi_N\}$ is an orthonormal system, the parts (i) and (ii) are obvious. For the proof of (iii), we first remark that if (5.11) holds true then we get from the lower bound that for all $y \in T\cap\rho S_2$,

$$\sum_{i=1}^n \langle Y_i, y\rangle^2 \ge \frac{2}{3}\frac{n\rho^2}{N}$$

and we deduce as in Proposition 5.1.6 that $\text{Diam}\,(\ker\Psi\cap T) < \rho$.
For the proof of (iv), we deduce from the upper bound of (5.11) that for all $y \in T\cap\rho S_2$,

$$\begin{aligned}\sum_{i\in I}\langle\psi_i, y\rangle^2 &= \sum_{i=1}^N\langle\psi_i, y\rangle^2 - \sum_{i=1}^n\langle Y_i, y\rangle^2 = \|y\|_2^2 - \sum_{i=1}^n\langle Y_i, y\rangle^2 \\ &\ge \rho^2 - \frac{4}{3}\frac{n\rho^2}{N} = \rho^2\left(1 - \frac{4n}{3N}\right) > 0 \text{ since } n < 3N/4.\end{aligned}$$

This inequality means that for the matrix $\tilde\Psi$ defined by $\tilde\Psi = \begin{pmatrix} \cdot \\ \psi_i \\ \cdot \end{pmatrix}_{i\in I}$, for every $y \in T\cap\rho S_2$, we have

$$\inf_{y\in T\cap\rho S_2}\|\tilde\Psi y\|_2^2 > 0$$

and we conclude as in Proposition 5.1.6 that $\text{Diam}\,(\ker\tilde\Psi\cap T) < \rho$. Moreover, it is obvious that $\ker\tilde\Psi = (\ker\Psi)^\perp$. □

5.3.2.1. The case of L_2^N*.* — We will now present a result concerning the problem of selection of characters in L_2^N. It is not the most general result but we would like to emphasize the deep similarity between the proofs of this result and the proof of Theorem 5.3.1.

Theorem 5.3.5. — *Let* $(\psi_1, \ldots, \psi_N)$ *be an orthonormal system of* L_2^N *bounded in* L_∞^N *i.e. such that for every* $i \le N$, $\|\psi_i\|_\infty \le 1$.

For any $2 \le n \le N-1$, there exists a subset $I \subset [N]$ of cardinality greater than $N-n$ such that for all $(a_i)_{i\in I}$,

$$\left\|\sum_{i\in I} a_i\psi_i\right\|_2 \le C\sqrt{\frac{N}{n}}\sqrt{\log N}(\log n)^{3/2}\left\|\sum_{i\in I} a_i\psi_i\right\|_1 .$$

Proof. — Let Y be the random vector defined by $Y = \psi_i$ with probability $1/N$ and let $Y_1, \ldots, Y_n$ be independent copies of Y. Observe that $\mathbb{E}\langle Y, y\rangle^2 = \|y\|_2^2/N$ and define

$$Z = \sup_{y\in B_1\cap\rho S_2}\left|\sum_{i=1}^n \langle Y_i, y\rangle^2 - \frac{n\rho^2}{N}\right|.$$

Following the proof of Theorem 5.3.1 (the normalization is different from a factor $\sqrt{N}$), we obtain that if ρ is such that

$$\rho \ge C\sqrt{\frac{N\,(\log n)^3\log N}{n}}$$

then

$$\mathbb{P}\left(Z \ge \frac{1}{3}\frac{n\rho^2}{N}\right) \le C\exp(-c\,\frac{n\,\rho^2}{N}).$$

Therefore there exists a choice of $Y_1, \ldots, Y_n$ (in fact it is with probability greater than $1 - C\exp(-c\,\frac{n\,\rho^2}{N})$) such that

$$\sup_{y\in B_1\cap\rho S_2}\left|\sum_{i=1}^n \langle Y_i, y\rangle^2 - \frac{n\rho^2}{N}\right| \le \frac{1}{3}\frac{n\rho^2}{N}$$

and if I is defined by $\{\psi_i\}_{i\in I} = \{\psi_1, \ldots, \psi_N\}\setminus\{Y_1, \ldots, Y_n\}$ then by Proposition 5.3.4 (*iii*) and (*i*), we conclude that $\mathrm{Diam}\,(\mathrm{span}\,\{\psi_i\}_{i\in I}\cap B_1) \le \rho$ and $|I| \ge N-n$. This means that for every $(a_i)_{i\in I}$,

$$\left\|\sum_{i\in I} a_i\psi_i\right\|_2 \le \rho\left\|\sum_{i\in I} a_i\psi_i\right\|_1 .$$

□

Remark 5.3.6. — *Theorem 5.3.5 implies Theorem 5.3.1. Indeed, if we write the inequality with the classical ℓ_1 and ℓ_2 norms, we get that*

$$\left|\sum_{i\in I} a_i\psi_i\right|_2 \le C\sqrt{\frac{\log N}{n}}(\log n)^{3/2}\left|\sum_{i\in I} a_i\psi_i\right|_1$$

which means that $\mathrm{diam}\,(\ker\Psi\cap B_1^N) \le C\sqrt{\frac{\log N}{n}}(\log n)^{3/2}$. *We conclude about the reconstruction property by using Proposition 5.1.1.*

5.3.2.2. The general case of $L_2(\mu)$. — We can now state a general result about the problem of selection of characters. It is an extension of (5.10) to the existence of a subset of arbitrary size, with a slightly worse dependence in $\log\log N$.

Theorem 5.3.7. — *Let μ be a probability measure and let $(\psi_1, \dots, \psi_N)$ be an orthonormal system of $L_2(\mu)$ bounded in L_∞ i.e. such that for every $i \le N$, $\|\psi_i\|_\infty \le 1$. For any $n \le N-1$, there exists a subset $I \subset [N]$ of cardinality greater than $N-n$ such that for all $(a_i)_{i\in I}$,*

$$\Big\| \sum_{i\in I} a_i\psi_i \Big\|_2 \le C\,\gamma\,(\log\gamma)^{5/2} \Big\| \sum_{i\in I} a_i\psi_i \Big\|_1$$

where $\gamma = \sqrt{\frac{N}{n}}\sqrt{\log n}$.

Remark 5.3.8. — • *If n is chosen to be proportional to N then $\gamma\,(\log\gamma)^{5/2}$ is of the order of $\sqrt{\log N}\,(\log\log N)^{5/2}$. However, if n is chosen to be a power of N then $\gamma\,(\log\gamma)^{5/2}$ is of the order $\sqrt{\frac{N}{n}}\sqrt{\log n}(\log N)^{5/2}$ which is a worse dependence than in Theorem 5.3.5*

• *Exactly as in Theorem 5.3.1 we could assume that $(\psi_1, \dots, \psi_N)$ is an orthogonal system of L_2 such that for every $i \le N$, $\|\psi_i\|_2 = K$ and $\|\psi_i\|_\infty \le 1$ for a fixed real number K.*

The second main result is an extension of (5.10) to a Kashin type setting.

Theorem 5.3.9. — *With the same assumptions as in Theorem 5.3.7, if N is an even natural integer, there exists a subset $I \subset [N]$ with $\frac{N}{2} - c\sqrt{N} \le |I| \le \frac{N}{2} + c\sqrt{N}$ such that for all $(a_i)_{i=1}^N$*

$$\Big\| \sum_{i\in I} a_i\psi_i \Big\|_2 \le C\,\sqrt{\log N}\,(\log\log N)^{5/2} \Big\| \sum_{i\in I} a_i\psi_i \Big\|_1$$

and

$$\Big\| \sum_{i\notin I} a_i\psi_i \Big\|_2 \le C\,\sqrt{\log N}\,(\log\log N)^{5/2} \Big\| \sum_{i\notin I} a_i\psi_i \Big\|_1 .$$

In order to be able to use Theorem 5.2.12 and its Corollary 5.2.13, we would like to replace the unit ball B_1 by a ball which has a good modulus of convexity that is for example B_p for $1 < p \le 2$. We start recalling a classical trick that is used very often when we compare the L_r norms of a measurable functions (for example in the theory of thin sets in Harmonic Analysis).

Lemma 5.3.10. — *Let f be a measurable function with respect to the probability measure μ. For $1 < p < 2$,*

$$\text{if } \|f\|_2 \le A\|f\|_p \text{ then } \|f\|_2 \le A^{\frac{p}{2-p}}\|f\|_1.$$

Proof. — This is just an application of Hölder inequality. Let $\theta \in (0,1)$ such that $1/p = (1-\theta) + \theta/2$ that is $\theta = 2(1-1/p)$. By Hölder,

$$\|f\|_p \leq \|f\|_1^{1-\theta}\|f\|_2^{\theta}.$$

Therefore if $\|f\|_2 \leq A\|f\|_p$ we deduce that $\|f\|_2 \leq A^{\frac{1}{1-\theta}}\|f\|_1$. □

Proposition 5.3.11. — *With the same assumptions as in Theorem 5.3.7, the following holds.*

1) For any $p \in (1,2)$ and any $2 \leq n \leq N-1$ there exists a subset $I \subset \{1,\dots,N\}$ with $|I| \geq N-n$ such that for every $a = (a_i) \in \mathbb{C}^N$,

$$\Big\|\sum_{i\in I} a_i\psi_i\Big\|_2 \leq \frac{C}{(p-1)^{5/2}}\sqrt{N/n}\,\sqrt{\log n}\,\Big\|\sum_{i\in I} a_i\psi_i\Big\|_p .$$

2) Moreover, if N is an even natural integer, there exists a subset $I \subset \{1,\dots,N\}$ with $N/2 - c\sqrt{N} \leq |I| \leq N/2 + c\sqrt{N}$ such that for every $a = (a_i) \in \mathbb{C}^N$,

$$\Big\|\sum_{i\in I} a_i\psi_i\Big\|_2 \leq \frac{C}{(p-1)^{5/2}}\sqrt{N/n}\,\sqrt{\log n}\,\Big\|\sum_{i\in I} a_i\psi_i\Big\|_p$$

and

$$\Big\|\sum_{i\notin I} a_i\varphi_i\Big\|_2 \leq \frac{C}{(p-1)^{5/2}}\sqrt{N/n}\,\sqrt{\log n}\,\Big\|\sum_{i\notin I} a_i\psi_i\Big\|_p .$$

Combining the first part of Proposition 5.3.11 with Lemma 5.3.10, it is easy to make the proof of Theorem 5.3.7. Indeed, let $\gamma = \sqrt{N/n}\,\sqrt{\log n}$ and choose $p = 1 + 1/\log\gamma$. Using Proposition 5.3.11, there is a subset I of cardinality greater than $N-n$ for which

$$\forall (a_i)_{i\in I},\quad \Big\|\sum_{i\in I} a_i\psi_i\Big\|_2 \leq C_p\,\gamma\,\Big\|\sum_{i\in I} a_i\psi_i\Big\|_p$$

where $C_p = C/(p-1)^{5/2}$. By the choice of p and Lemma 5.3.10,

$$\Big\|\sum_{i\in I} a_i\psi_i\Big\|_2 \leq \gamma\, C_p^{p/(2-p)}\gamma^{2(p-1)/(2-p)}\Big\|\sum_{i\in I} a_i\psi_i\Big\|_1 \leq C\,\gamma\,h(\gamma)\Big\|\sum_{i\in I} a_i\psi_i\Big\|_1$$

where $h(\gamma) = (\log\gamma)^{5/2}$.

The same argument works for the Theorem 5.3.9 using the second part of Proposition 5.3.11.

It remains to make the proof of Proposition 5.3.11.

Proof. — Let Y be the random vector defined by $Y = \psi_i$ with probability $1/N$ and let $Y_1,\dots,Y_n$ be independent copies of Y. Observe that $\mathbb{E}\langle Y, y\rangle^2 = \|y\|_2^2/N$, let $E = \operatorname{span}\{\psi_1,\dots,\psi_N\}$ and for $\rho > 0$ let E_p be the vectorial space E endowed with the norm defined by

$$\|y\| = \left(\frac{\|y\|_p^2 + \rho^{-2}\|y\|_2^2}{2}\right)^{1/2}.$$

We restrict our study to the vectorial space E and it is clear that

$$(B_p \cap \rho B_2) \subset B_{E_p} \subset \sqrt{2}(B_p \cap \rho B_2) \tag{5.12}$$

where B_{E_p} is the unit ball of E_p. Moreover, the Clarkson inequality tells that for any $f, g \in L_p$,

$$\left\| \frac{f+g}{2} \right\|_p^2 + \frac{p(p-1)}{8} \left\| \frac{f-g}{2} \right\|_p^2 \leq \frac{1}{2}(\|f\|_p^2 + \|g\|_p^2).$$

It is therefore easy to deduce that E_p is a Banach space with modulus of convexity of power type 2 with constant λ such that $\lambda^{-2} = p(p-1)/8$.

We define the random variable

$$Z = \sup_{y \in B_p \cap \rho S_2} \left| \sum_{i=1}^n \langle Y_i, y \rangle^2 - \frac{n\rho^2}{N} \right|$$

and we deduce from (5.12) that

$$\mathbb{E}Z \leq \mathbb{E} \sup_{y \in B_{E_p}} \left| \sum_{i=1}^n \langle Y_i, y \rangle^2 - \mathbb{E}\langle Y_i, y \rangle^2 \right|.$$

We use Corollary 5.2.13. We deduce from (5.12) that $\sigma^2 = \sup_{y \in B_{E_p}} n\|y\|_2^2/N \leq 2n\rho^2/N$ and that for every $i \leq N$, $\|\psi_i\|_{E_p^\star} \leq \sqrt{2}\|\psi_i\|_\infty \leq \sqrt{2}$. By Corollary 5.2.13, we get

$$\mathbb{E} \sup_{y \in B_{E_p}} \left| \sum_{i=1}^n \langle Y_i, y \rangle^2 - \mathbb{E}\langle Y_i, y \rangle^2 \right| \leq C \max\left(\lambda^{10} \log n, \rho\lambda^5 \sqrt{\frac{n \log n}{N}} \right).$$

We conclude that

$$\text{if} \quad \rho \geq C\lambda^5 \sqrt{\frac{N \log n}{n}} \quad \text{then} \quad \mathbb{E}Z \leq \frac{1}{3}\frac{n\rho^2}{N}$$

and using Proposition 5.1.6 we get that

$$\operatorname{Diam}(\ker \Psi \cap B_p) < \rho$$

where $\Psi = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}$. We choose $\rho = C\lambda^5 \sqrt{\frac{N \log n}{n}}$ and deduce from Proposition 5.3.4 (iii) and (i) that for I defined by $\{\psi_i\}_{i \in I} = \{\psi_1, \dots, \psi_N\} \setminus \{Y_1, \dots, Y_n\}$ we have

$$\forall (a_i)_{i \in I}, \quad \left\| \sum_{i \in I} a_i \psi_i \right\|_2 \leq \rho \left\| \sum_{i \in I} a_i \psi_i \right\|_p$$

which ends the proof of the first part of Proposition 5.3.11.

For the proof of the second part, we add the following observation. By a combinatorial argument, it is not difficult to prove that if $n = [\delta N]$ with $\delta = \log 2 < 3/4$ then with probability greater than $3/4$,

$$N/2 - c\sqrt{N} \leq |I| = N - |\{Y_1, \dots, Y_n\}| \leq N/2 + c\sqrt{N},$$

for some absolute constant $c > 0$. Hence $n < 3N/4$ and we can also use part (iv) of Proposition 5.3.4 which proves that

$$\mathrm{Diam}\,(\ker\Psi \cap B_p) \leq \rho \quad \text{and} \quad \mathrm{Diam}\,((\ker\Psi)^{\perp} \cap B_p) \leq \rho.$$

Since $\ker\Psi = \mathrm{span}\ \{\psi_i\}_{i\in I}$ and $(\ker\Psi)^{\perp} = \mathrm{span}\ \{\psi_i\}_{i\notin I}$, this ends the proof of the Proposition. □

5.4. Comments

Concerning the Gelfand numbers of the operator $id : \ell_1^N \to \ell_2^N$, a major breakthrough was made by Kashin in [**Kaš77**]. The statement of Theorem 5.1.4 is a result due to Garnaev and Gluskin [**GG84**] and to Gluskin [**Glu83**]. The study of the Gelfand and Kolmogorov numbers was deeply developed during the eighties and we refer also to [**PTJ86, CP88, PTJ89, PTJ90**]. The Proposition 5.1.6 comes from [**PTJ86**] (Proposition 2) and was at the heart of several improvements about these approximation numbers.

For the study of the supremum of an empirical process and the connection with Rademacher averages, we already referred to chapter 4 of [**LT91**]. Theorem 5.2.1 is due to Talagrand and can be found in theorem 4.12 in [**LT91**]. Theorem 5.2.2 is often called a "symmetrization principle". This strategy is already used by Kahane in [**Kah68**] for studying random series on Banach spaces. It was pushed forward by Giné and Zinn in [**GZ84**] for studying limit theorem for empirical processes. The concentration inequality, Theorem 5.2.4, is due to Talagrand [**Tal96b**]. Several improvements and simplifications are known, in particular in the case of independent identically distributed random variables. We refer to [**Rio02, Bou03, Kle02, KR05**] for more precise results. The Proposition 5.2.5 is taken from [**Mas00**].

Theorem 5.2.6 is due to Rudelson [**Rud99**]. The proof that we presented was suggested by Pisier to Rudelson. It used a refined version of non-commutative Khinchine inequality that can be found in [**LP86, LPP91, Pis98**]. However, it is based on an expression related to some operator norms and we have seen that we need some estimates of the supremum of some empirical processes that can not be expressed in terms of operator norms. The original proof of Rudelson can be found in [**Rud96**] and used the majorizing measure theory. Some improvements of this result are proved in [**GR07**] and in [**GMPTJ08**]. The proof of Theorem 5.2.12 can be found in [**GMPTJ08**] and it is based on the same type of construction of majorizing measures than in [**GR07**] and on deep results about the duality of covering numbers [**BPSTJ89**]. The notions of type and cotype of a Banach space are important in this study and we refer the interested reader to [**Mau03**].

Theorem 5.2.14 comes from [**GMPTJ07**]. It was used to prove some results about the problem of selection of characters like Theorem 5.3.5. As we have seen, the proof is very similar to the proof of Theorem 5.3.1 and this result is due to Rudelson and Vershynin [**RV08c**]. They improved a result due to Candès and Tao [**CT05**] and the strategy of their proofs was to study the RIP condition instead of the size of the half-diameter of the section by B_1^N. Moreover, the probabilistic estimate is slightly better than in [**RV08c**] and it was shown to us by Holger Rauhut [**Rau10**]. We refer to [**Rau10, FR10**] for a deeper presentation of the problem of compressed sensing and for several different points of view. We refer also to [**CDD09, KT07**] for the study of approximate reconstruction. Theorem 5.1.2 comes from [**KT07**] where connections between the Compressed Sensing problem and the problem of estimating the Kolmogorov widhts are discussed.

For the classical study of the local theory of Banach spaces, we refer to [**MS86**] and to [**Pis89**]. The study of the Euclidean sections or projections of a convex body is studied in detail in [**FLM77**] and the Kashin decomposition can be found in [**Kaš77**]. About the question of selection of characters, we refer the interested reader to the paper of Bourgain [**Bou89**] where he proved for $p > 2$ the existence of $\Lambda(p)$ sets which are not $\Lambda(r)$ for $r > p$. This problem was related to the theory of majorizing measure in [**Tal95**]. The existence of a subset of a bounded orthonormal system satisfying the inequality (5.10) is proved by Talagrand in [**Tal98**]. Theorems 5.3.7 and 5.3.9 are proved in [**GMPTJ08**]. We refer also to that paper for a proof of the fact that the factor $\sqrt{\log N}$ is necessary in the estimate.

NOTATIONS

- The sets of numbers are $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$
- For all $x \in \mathbb{R}^N$ and $p > 0$,
$$|x|_p = (|x_1|^p + \cdots + |x_N|^p)^{1/p} \quad \text{and} \quad |x|_\infty = \max_{1 \leqslant i \leqslant N} |x_i|$$
- $B_p^N = \{x \in \mathbb{R}^N \, : \, |x|_p \leq 1\}$
- Scalar product $x \cdot y$ or $\langle x, y \rangle$ and $x \perp y$ means $x \cdot y = 0$
- $A^* = \bar{A}^\top$ is the conjugate transpose of the matrix A
- $s_1(A) \geqslant \cdots \geqslant s_n(A)$ are the singular values of the $n \times N$ matrix A where $n \leqslant N$
- $\|A\|_{2\to 2}$ is the operator norm of A ($\ell^2 \to \ell^2$)
- $\|A\|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm of A
- $e_1, \ldots, e_n$ is the canonical basis of $\mathbb{R}^n$
- $\stackrel{d}{=}$ stands for the equality in distribution
- $\stackrel{d}{\to}$ stands for the convergence in distribution
- $\stackrel{w}{\to}$ stands for weak convergence of measures
- $\mathcal{M}_{m,n}(K)$ are the $m \times n$ matrices with entries in K, and $\mathcal{M}_n(K) = \mathcal{M}_{n,n}(K)$
- I is the identity matrix
- $x \wedge y = \min(x, y)$ and $x \vee y = \max(x, y)$
- $|S|$ cardinal of the set S
- $\mathrm{dist}_2(x, E) = \inf_{y \in E} |x - y|_2$
- supp$x$ is the subset of non-zero coordinate of x
- The vector x is said to be m-sparse if $|\mathrm{supp}\, x| \leq m$.
- $\Sigma_m = \Sigma_m(\mathbb{R}^N)$ m-sparse vectors
- $S_p(\Sigma_m) = \{x \in \mathbb{R}^N \, : \, |x|_p = 1, |\mathrm{supp}\, x| \leq m\}$
- $B_p(\Sigma_m) = \{x \in \mathbb{R}^N \, : \, |x|_p \leq 1, |\mathrm{supp}\, x| \leq m\}$
- $\mathrm{conv}(E)$ is the convex hull of E
- $\mathrm{diam}(F, \|\cdot\|) = \sup\{\|x\| : x \in F\}$
- For a random variable Z and any $\alpha \geqslant 1$, $\|Z\|_{\psi_\alpha} = \inf\left\{s > 0 \, ; \, \mathbb{E} \exp\left(|Z|/s\right)^\alpha \leqslant e\right\}$
- $\ell_*(T) = \mathbb{E} \sup_{t \in T} |\sum_{i=1}^N g_i t_i|_2$

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