

# Τριπλό Ολοκλήρωμα

## Απλά Σύνολα στον $\mathbb{R}^3$

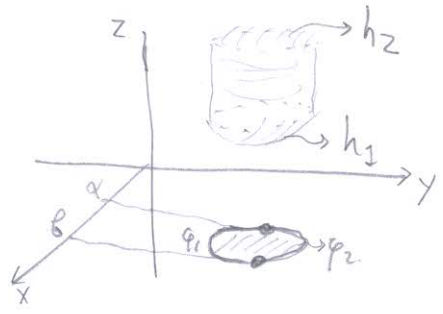
- $xy$ -απλό,  $B = \left\{ (x, y, z) \in \mathbb{R}^3 : a \leq x \leq b, g_1(x) \leq y \leq g_2(x), h_1(x, y) \leq z \leq h_2(x, y) \right\}$
- $g_1, g_2 : [a, b] \rightarrow \mathbb{R}$  *συνεχείς*
- $h_1, h_2 : D = \left\{ (x, y) : a \leq x \leq b, g_1(x) \leq y \leq g_2(x) \right\} \rightarrow \mathbb{R}$
- Δηλαδή, η προβολή του  $B$  στο επίπεδο  $xy$  είναι  $x$ -απλό σύνολο.
- Ανάλογα,  $yx$ -απλό /  $xz, zx, yz, zy$ -απλο.

## Θ. Fubini

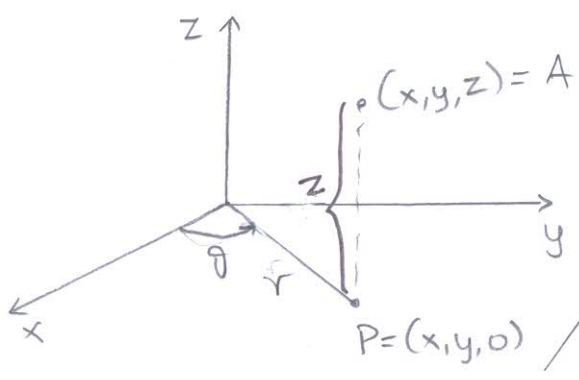
$f: B \rightarrow \mathbb{R}$ ,  $B$   $xy$ -απλό,  $f$  *συνεχής*

Τότε:  $\exists \iiint_B f = \int_a^b \left( \int_{g_1(x)}^{g_2(x)} \left( \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z) dz \right) dy \right) dx$

$V_3(B) = \iiint_B 1$



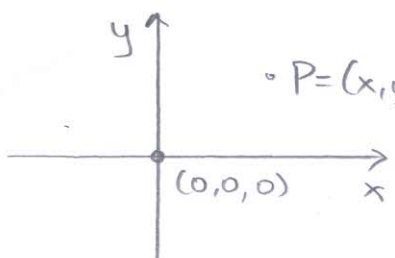
## Κυλινδρικές Συντεταγμένες



$\vec{T}: (\omega, +\infty) \times [0, 2\pi) \times \mathbb{R} \rightarrow \mathbb{R}^3 \setminus \{(0,0,z) : z \in \mathbb{R}\}$

$\vec{T}(r, \theta, z) = (r \cos \theta, r \sin \theta, z)$

$= (x, y, z)$

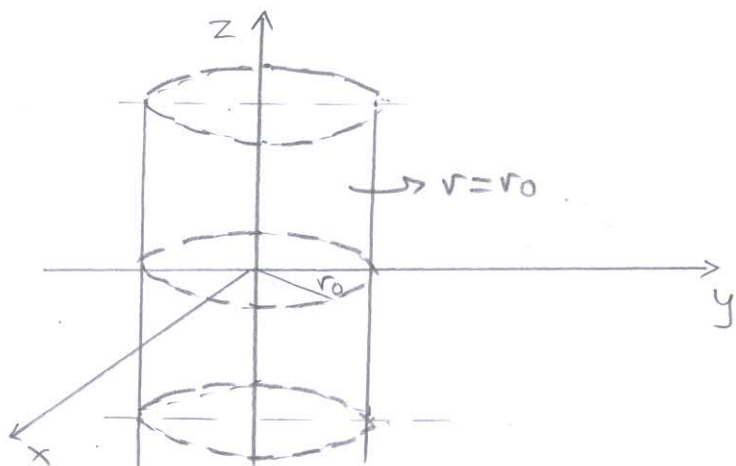


$x = r \cos \theta \quad \left| \begin{array}{l} r \in (\omega, +\infty) \\ \theta \in [0, 2\pi) \end{array} \right. / \quad r = \sqrt{x^2 + y^2} > 0$

$y = r \sin \theta \quad \left| \right. / \quad \theta = \angle(\vec{Ox}, \vec{OP})$

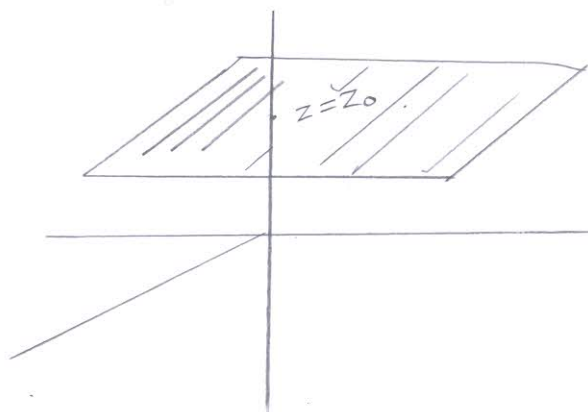
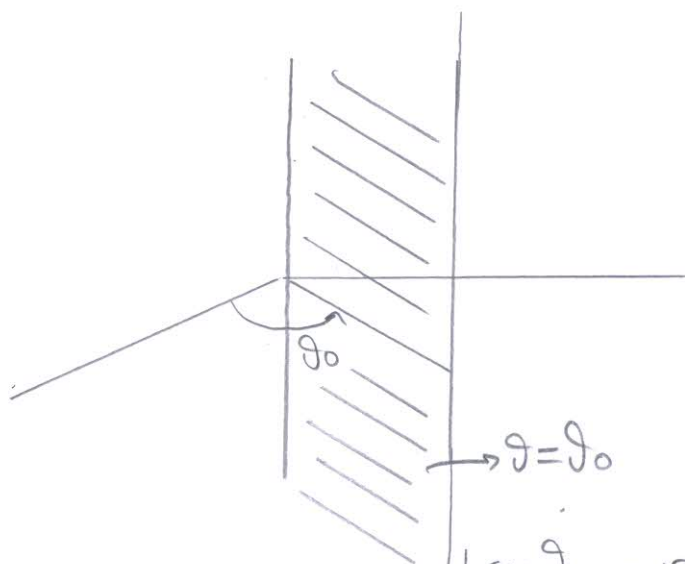
Erzw  $(r_0, \vartheta_0, z_0)$

•  $(x, y, z) \in \mathbb{R}^3 : r = r_0 \quad (\vartheta, z)$   
 $: \sqrt{x^2 + y^2} = r_0, \quad z \in \mathbb{R}$



$$\vartheta = \vartheta_0 \quad (r > 0, z \in \mathbb{R})$$

$$(x, y, z) : \vartheta = \vartheta_0 \quad (r > 0, z \in \mathbb{R})$$

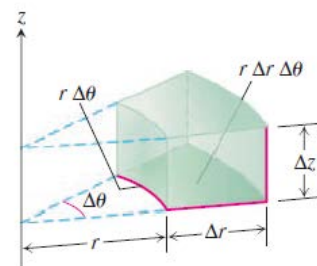


$$\det \vec{T}(r, \vartheta, z) = \begin{vmatrix} r \cos \vartheta & -r \sin \vartheta & 0 \\ \sin \vartheta & r \cos \vartheta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r \cos^2 \vartheta + r \sin^2 \vartheta = r > 0$$

$$\vec{T}(r, \vartheta, z) = (r \cos \vartheta, r \sin \vartheta, z)$$

1:  $B \rightarrow \mathbb{R}^3, \vec{T}(B^*) = B \quad \iiint_B f(x, y, z) \, dx \, dy \, dz =$

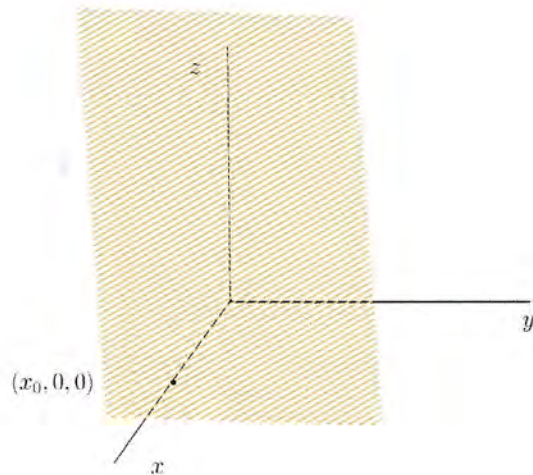
$$= \iiint_{B^*} f(r \cos \vartheta, r \sin \vartheta, z) \cdot r \, dr \, d\vartheta \, dz.$$



## Συστήματα συντεταγμένων στον $R^3$

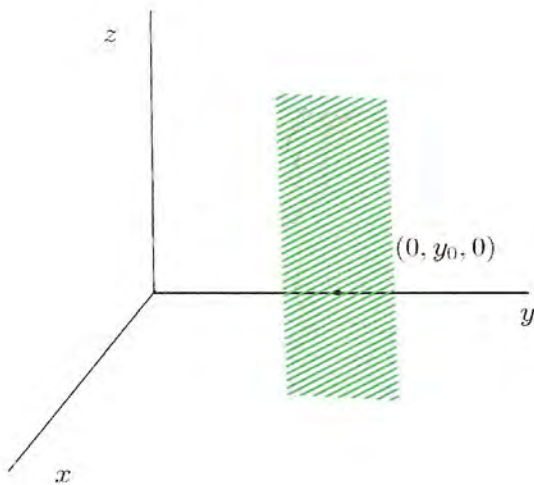
### I. Καρτεσιανές συντεταγμένες $(x, y, z)$

Καρτεσιανές επιφάνειες (επίπεδα)  $x = x_0, y = y_0, z = z_0$ .



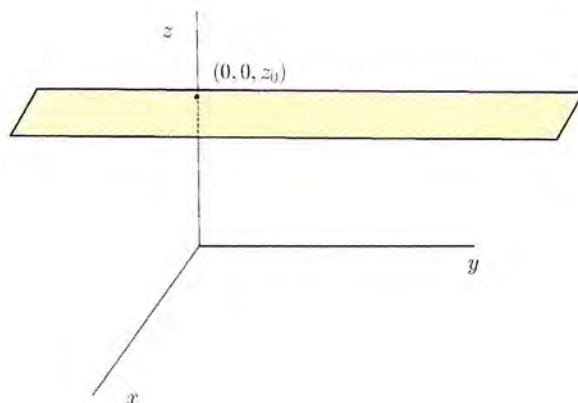
$$S_1: x = x_0, \{(x, y, z) \in R^3: x = x_0, y, z \in R\}$$

$$\{\vec{r}_1(y, z) = (x_0, y, z), y, z \in R\}$$



$$S_2: y = y_0, \{(x, y, z) \in R^3: y = y_0, x, z \in R\}$$

$$\{\vec{r}_2(x, z) = (x, y_0, z), x, z \in R\}$$

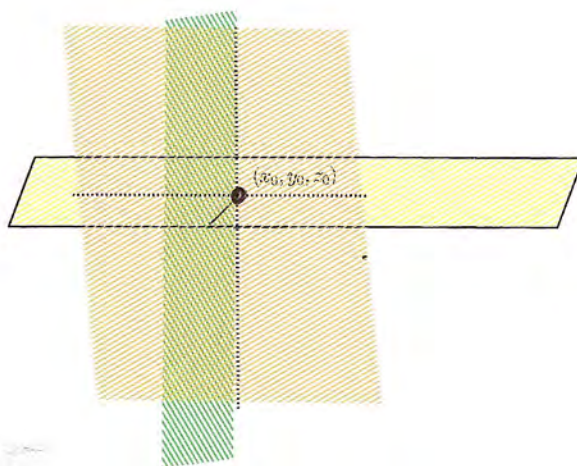


$$S_3: z = z_0, \{(x, y, z) \in R^3: z = z_0, x, y \in R\}$$

$$\{\vec{r}_3(x, y) = (x, y, z_0), x, y \in R\}$$

Το  $(x_0, y_0, z_0)$  είναι το σημείο τομής των επιφανειών

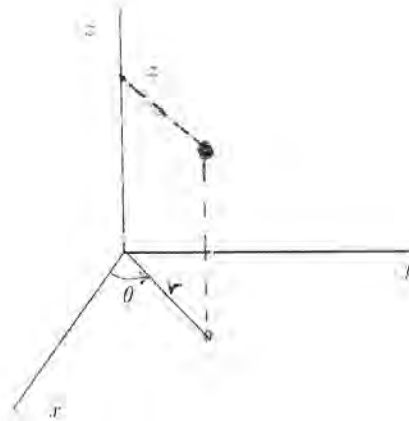
$$S_1: x = x_0, S_2: y = y_0, S_3: z = z_0$$



## II. Κυλινδρικές συντεταγμένες $(r, \theta, z)$

Ο  $\vec{T}: [0, \infty) \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}^3$  (επί), με  $\vec{T}(r, \theta, z) = (r \cos \theta, r \sin \theta, z)$  καλείται κυλινδρικός μετασχηματισμός και τα  $r, \theta, z$  κυλινδρικές συντεταγμένες.

Ο περιορισμός του  $\vec{T}: (0, \infty) \times [0, 2\pi) \times \mathbf{R} \rightarrow \mathbf{R}^3 \setminus \{(0, 0, z), z \in \mathbf{R}\}$  είναι 1-1 και επί.

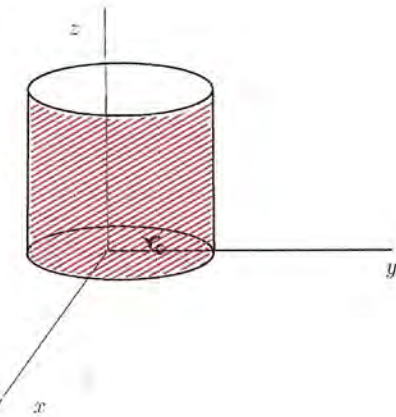


### Σχέση Καρτεσιανών Κυλινδρικών συντεταγμένων.

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \\ z = z \end{cases} \quad (r, \theta, z) \in [0, \infty) \times \mathbf{R} \times \mathbf{R}$$

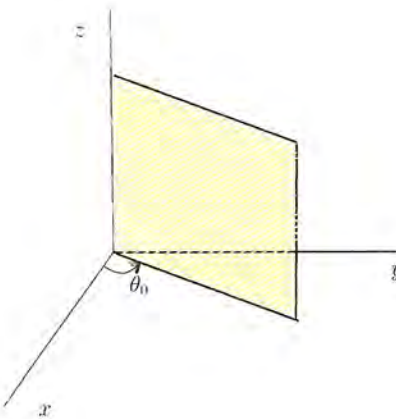
$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \cos \theta = \frac{x}{r} \text{ αν } x \neq 0. \text{ Αν } x = 0: \theta = \frac{\pi}{2} \text{ για } y > 0, \theta = \frac{3\pi}{2} \text{ για } y < 0 \\ z = z \end{cases} \quad (x, y, z) \in \mathbf{R}^3 \setminus \{(0, 0, z)\}$$

Κυλινδρικές επιφάνειες  $r = r_0 (> 0)$ ,  $\theta = \theta_0$ ,  $z = z_0$  (στο καρτεσιανό σύστημα)



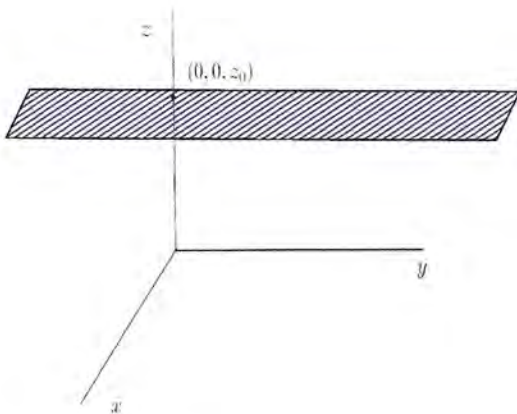
$$S_1: r = r_0, \text{ κύλινδρος } \{(x, y, z) \in \mathbf{R}^3: x^2 + y^2 = r_0^2\}$$

$$\{\vec{r}_1(\theta, z) = (r_0 \cos \theta, r_0 \sin \theta, z), \theta \in [0, 2\pi], z \in \mathbf{R}\}$$



$$S_2: \theta = \theta_0, \text{ ημιεπίπεδο}$$

$$\{\vec{r}_2(r, z) = (r \cos \theta_0, r \sin \theta_0, z), r \geq 0, z \in \mathbf{R}\}$$



$$S_3: z = z_0, \text{ επίπεδο } \{(x, y, z) \in \mathbf{R}^3: z = z_0, (x, y) \in \mathbf{R}^2\}$$

$$\{\vec{r}_3(r, \theta) = (r \cos \theta, r \sin \theta, z_0), r > 0, \theta \in [0, 2\pi]\}$$

Το  $(x_0, y_0, z_0) = (r_0 \eta \mu \theta_0, r_0 \sigma \upsilon \nu \theta_0, z_0)$  είναι το σημείο τομής των επιφανειών

$$S_1 : r = r_0, S_2 : \theta = \theta_0, S_3 : z = z_0$$

